数理解析研究所講究録
第2042巻 2017年 52-71

On some downward transfer properties in Foreman-Laver model

Sakaé Fuchino

Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp

André Ottenbreit Maschio Rodrigues

Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
ottenbreit@stu.kobe-u.ac.jp

Hiroshi Sakai

Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
hsakai@people.kobe-u.ac.jp

Abstract

In [FL88], Foreman and Laver proved, assuming the existence of a huge cardinal the consistency of the transfer property of maximal chromatic number from $\aleph_2$ to $\aleph_1$. We combine this result with another result by Baumgartner [Bau84] to prove, also assuming the existence of a huge cardinal, that such a transfer property from $\aleph_2$ to $\aleph_1$ does not follow from the transfer property from $\aleph_3$ to $\aleph_1$.

1 Introduction

By using a technique created by Kunen [Kun78], Foreman and Laver constructed in [FL88] a model of set theory with some nice reflection properties on $\aleph_2$. One of those properties is related with the maximality of the chromatic number of graphs.

Recall the definition of the chromatic number of a graph:

Definition 1.1. *Given a graph $\mathcal{G} = \langle X, E\rangle$ (X is the set of vertices and $E \subset [X]^2$ is the set of edges), we say that a function f with domain X is a* good coloring *of $\mathcal{G}$ if for every edge $\{a, b\} \in E$ we have $f(a) \neq f(b)$.*

The chromatic number *of the graph $\mathcal{G}$, denoted by* $\mathrm{Chr}(\mathcal{G})$, *is the minimal cardinal ρ such that there exists a good coloring $f : X \longrightarrow \rho$.*

A feature of the model by Foreman and Laver is that every graph of size (i.e. number of vertices) and chromatic number $\aleph_2$ has a subgraph of size and chromatic number $\aleph_1$ in the model. Since the chromatic number of a graph is at most the size of the same graph, this property can be interpreted as the maximality of the chromatic number of graphs is always transferred from graphs of size $\aleph_2$ to a subgraph of size $\aleph_1$. In more general terms, we shall refer to such property as follows:

Definition 1.2. *Given two cardinals $\gamma < \delta$, we denote by $\mathrm{Tr}_{\mathrm{Chr}}(\delta, \gamma)$ the following statement: "Any graph G of size and chromatic number δ has a subgraph of size and chromatic number γ".*

With this notation, the result by Foreman and Laver can be stated as: if the existence of a huge cardinal is consistent, then so is $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$. As claimed in the original article, this result is easily generalizable by changing $\aleph_2$ and $\aleph_1$ for $\aleph_m$ and $\aleph_n$ respectively, with $n < m < \omega$.

By definition 1.2, it is clear that, $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_2)$ together with $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$ implies $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1)$. From this fact arises the question of whether the converse holds. In this paper, we shall prove that, if the existence of a huge cardinal is consistent, the answer to such question is negative. We accomplish this by combining the previous result by Foreman and Laver with a result by Baumgartner in [Bau84] and obtaining the following theorem:

Theorem 1.3. *Suppose there exists a huge cardinal. Then there exists a forcing extension of V which satisfies simultaneously $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1)$ and $\neg\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$.*

The goal of this paper is to present a proof of theorem 1.3. We also present a slight generalization of it (theorem 4.2).

1.1 Definitions and preliminaries

Definition 1.4. *For regular cardinals $\gamma < \delta$, define the* Silver collapse $\mathbb{S}(\gamma, \delta)$ *as the poset consisting of all partial functions p from $\gamma \times \delta$ to δ satisfying the following: $|p| \leq \gamma$; $\exists \eta < \gamma(\mathrm{dom}(p) \subset \eta \times \delta)$; and $\forall \langle \alpha, \beta \rangle \in \mathrm{dom}(p)(p(\alpha, \beta) = 0 \vee p(\alpha, \beta) \in \beta)$.*

Definition 1.5. *Given a poset $\mathbb{Q}$ and a set of conditions $S \subseteq \mathbb{Q}$, we define the quotient*

$$\mathbb{Q}/S = \{q \in \mathbb{Q} : q \text{ is compatible with every } s \in S\}$$

ordered by $\leq_{\mathbb{Q}/S} := \leq_{\mathbb{Q}} \cap (\mathbb{Q}/S \times \mathbb{Q}/S)$.

Fact 1.6. *If $\mathbb{Q} \lessdot \mathbb{P}$ and $\underset{\sim}{G}_{\mathbb{Q}}$ is the usual $\mathbb{Q}$-name for a generic, then $\mathbb{P} \approx \mathbb{Q} * (\mathbb{P}/\underset{\sim}{G}_{\mathbb{Q}})$.*

We now want to define the concept of a termspace. For that, we need the following remark.

Remark 1.7. *We say that a $\mathbb{Q}$-name $\underset{\sim}{t}$ has minimal rank if for any $\mathbb{Q}$-name $\underset{\sim}{t'}$ which is forcing equivalent to $\underset{\sim}{t}$, i.e. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{t} = \underset{\sim}{t'}$", we have $\mathrm{rank}(t) \leq \mathrm{rank}(t')$. From the well known fact 1.8, it is standard to show that for any $\mathbb{Q}$-name $\underset{\sim}{X}$ we have that*

$$\{t \in V^{\mathbb{Q}} : \Vdash_{\mathbb{Q}} \text{“} t \in \underset{\sim}{X} \text{”} \text{ and } t \text{ has minimal rank}\}$$

is a set.

Fact 1.8. *Let $\mathbb{Q}$ be a poset. Suppose $\underset{\sim}{x}$ is a $\mathbb{Q}$-name and χ is a large enough regular cardinal such that $\Vdash$ "$\underset{\sim}{x} \in H(\check{\chi})$". Then there exists a $\mathbb{Q}$-name $\underset{\sim}{x}' \in H(\chi)$ which is forcing equivalent to $\underset{\sim}{x}$.*

We can now define the termspace of a name for a poset.

Definition 1.9. *Suppose $\mathbb{Q}$ is a poset and $\underset{\sim}{\mathbb{T}}$ is a $\mathbb{Q}$-name for a poset. We define $\overline{\mathbb{T}}^{\mathbb{Q}}$ the* termspace *for $\underset{\sim}{\mathbb{T}}$ (with respect to $\mathbb{Q}$) as the quotient by the forcing equivalence relation of the set*

$$\{t \in V^{\mathbb{Q}} : \Vdash_{\mathbb{Q}} \text{“} t \in \underset{\sim}{\mathbb{T}} \text{” and } t \text{ has minimal rank}\}$$

ordered by

$$t_1 \leq_{\mathbb{Q}} t_2 \iff \Vdash_{\mathbb{Q}} \text{“} \underset{\sim}{t_1} \leq_{\underset{\sim}{\mathbb{T}}} \underset{\sim}{t_2} \text{”}$$

where $\underset{\sim}{t_1}$ and $\underset{\sim}{t_2}$ are representatives of the classes t_1 and t_2 respectively.

Notice that we shall omit the superscript $\mathbb{Q}$ when it is clear by the context. Also, we shall make no difference of between the class and the representative when the distinction is clear by the context.

Proposition 1.10. *Suppose $\mathbb{Q}$ is a poset of regular size δ and $\underset{\sim}{\mathbb{T}}$ is a $\mathbb{Q}$-name for a poset such that $\Vdash_{\mathbb{Q}}$ "$|\underset{\sim}{\mathbb{T}}| = |\check{\kappa}|$", for some regular cardinal κ. Then the termspace $\overline{\mathbb{T}}^{\mathbb{Q}}$ has size $\leq \kappa^{\delta}$.*

Definition 1.11. *Let δ be a cardinal. We say that a poset $\mathbb{Q}$ is δ-centered if there exists a function $f : \mathbb{Q} \longrightarrow \delta$ such that for every $\alpha < \delta$, each finite subset of $f^{-1}(\{\alpha\})$ has a common extension in $\mathbb{Q}$ (we call such function f a δ-centering of $\mathbb{Q}$). Alternatively, if $\mathbb{Q}$ can be partitioned into $\leq \delta$ many centered (in $\mathbb{Q}$) subsets.*

Fact 1.12. *Suppose $j : V \longrightarrow M$ is an elementary embedding and $\mathbb{Q}$ a forcing poset in V and G a $(V, \mathbb{Q})$-generic filter. If there exists some $(M, j(\mathbb{Q}))$-generic filter $\widehat{G}$ such that $j''G \subset \widehat{G}$, then we can extend j to an elementary embedding $j' : V[G] \longrightarrow M[\widehat{G}]$.*

Notation 1.13. *Given two names $\underset{\sim}{a}, \underset{\sim}{b}$, we denote by $\mathrm{op}(\underset{\sim}{a}, \underset{\sim}{b})$ the canonical name such that $\Vdash$ "$\mathrm{op}(\underset{\sim}{a}, \underset{\sim}{b}) = \langle \underset{\sim}{a}, \underset{\sim}{b} \rangle$".*

2 The partial orders

We shall construct 3 posets: $\mathbb{P}$, $\mathbb{R}(\kappa^+, \lambda)$ and $\mathbb{B}$. Our final poset, which shall force the main result (theorem 1.3), will be the two step iteration $\mathbb{P} * \left(\underset{\sim}{\mathbb{R}}(\kappa^+, \lambda) \times \underset{\sim}{\mathbb{B}}\right)$.

2.1 Baumgartner's Poset and the $\mathbb{R}$ collapse

Let $\mathbb{B}$ be the poset defined in [Bau84]. For a transitive model W of ZFC, we denote by $\mathbb{B}^W$ the poset constructed in W with the relative definition. We can summarize the features of $\mathbb{B}$ in the following theorem (also from [Bau84]):

Theorem 2.1 (Baumgartner). *Assume CH. Then there exists a partial order $\mathbb{B}$ which forces $\neg\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$ and has the following properties:*

- *$|\mathbb{B}| = \aleph_2$, $\mathbb{B}$ is σ-closed and preserves cardinals;*
- *If W_1, W_2 are transitive models of ZFC such that ω_1, ω_2 and On^ω are absolute between them, then $\mathbb{B}^{W_1} = \mathbb{B}^{W_2}$.*

Now we present the collapse $\mathbb{R}(\gamma, \delta)$, a modification of the Silver collapse (definition 1.4) introduced by Foreman and Laver in [FL88]. Notice that the notation used here is just slightly different from the original.

Definition 2.2. *For any regular γ, δ with $\gamma < \delta$, we define recursively the collapse $\mathbb{R}(\gamma, \delta)$*

- *First, let $\mathbb{R}^0(\gamma, \delta) = \mathbb{S}(\gamma, \delta)$.*
- *Assuming $\mathbb{R}^n(\beta, \delta)$ constructed for some $n \geq 0$ and for all regular β with $\gamma \leq \beta < \delta$, let*

$$\mathbb{R}^{n+1}(\gamma, \delta) = \prod_{\beta \in [\gamma, \delta) \cap \mathrm{REG}}^{\leq \gamma} \mathbb{R}^n(\beta, \delta)$$

  where the superscript $\leq \gamma$ indicates the size of the support and REG is the class of all regular cardinals.

- *Finally, define*

$$\mathbb{R}(\gamma, \delta) = \prod_{n \in \omega} \mathbb{R}^n(\gamma, \delta)$$

Lemma 2.3. *For regular γ, δ, there is a meet operation $\bigwedge$ in $\mathbb{R}(\gamma, \delta)$ which assigns a greatest lower bound to any collection of $< \gamma$ many pairwise compatible conditions in $\mathbb{R}(\gamma, \delta)$.*

Proof. We define recursively a function $\bigwedge$ on subsets of the collapse $\mathbb{R}^n(\gamma, \delta)$, for any suitable γ, δ.

- For $n = 0$, for $S \in [\mathbb{R}^0(\gamma, \delta)]^{<\gamma}$, define $\bigwedge S = \bigcup S$.

  Clearly, if S is pairwise compatible, $\bigwedge S$ is indeed a condition in $\mathbb{R}^0(\gamma, \delta)$ and it is a greatest lower bound of S.

- suppose $\bigwedge$ defined on $\mathbb{R}^n(\beta, \delta)$ for every suitable $\gamma \leq \beta \leq \delta$. For $S \in [\mathbb{R}^{n+1}(\gamma, \delta)]^{<\gamma}$ pairwise compatible, define

$$\bigwedge S = \langle \bigwedge S_\beta \ : \ \gamma \leq \beta < \delta, \ \beta \text{ regular} \rangle$$

  where $S_\beta := \{s(\beta) \ : \ s \in S\}$.

By induction hypothesis, we have that each $\bigwedge S_\beta \in \mathbb{R}^n(\beta, \gamma)$ is a greatest lower bound of S_β. Since $\operatorname{supp}(\bigwedge S) = \bigcup_{s \in S} \operatorname{supp}(s)$ and $|S| < \gamma$, clearly $|\operatorname{supp}(\bigwedge S)| \leq \gamma$, so indeed $\bigwedge S \in \mathbb{R}^{n+1}(\gamma, \delta)$. It is also clear that $\bigwedge S$ is a lower bound of S.

Furthermore, suppose $q \in \mathbb{R}^{n+1}(\gamma, \delta)$ is a lower bound of S. Clearly for each β we have that $q(\beta)$ is a lower bound for S_β, so by induction hypothesis, $q(\beta) \leq \bigwedge S_\beta$, hence $q \leq \bigwedge S$.

- finally, for $S \in [\mathbb{R}(\gamma, \delta)]^{<\gamma}$ pairwise compatible, define

$$\bigwedge S = \langle \bigwedge S_n \ : \ n \in \omega \rangle$$

where $S_n := \{s(n) \ : \ s \in S\}$. By the same argument on the previous item (with no need to care about the support), we have that $\bigwedge S \in \mathbb{R}(\gamma, \beta)$ is a greatest lower bound of S.

□

We will use a generalized version of the Δ-system lemma, whose proof can be found for example in [Kun11].

Lemma 2.4 (Δ-system lemma). *Let $\gamma < \delta$ be infinite regular cardinals such that $\alpha^{<\gamma} < \delta$ for all $\alpha < \delta$. Then for any family $\langle A_\alpha \ : \ \alpha < \delta \rangle$ with $|A_\alpha| < \gamma$ $(\alpha < \delta)$ there is some $B \subset \delta$ such that B is stationary (hence $|B| = \delta$) such that $\langle A_\alpha \ : \ \alpha \in B \rangle$ forms a Δ-system, i.e., there is a set R such that $A_\alpha \cap A_\beta = R$ for any distinct $\alpha, \beta \in B$.*

The proof of the following result is by Philipp Lücke [Lï6]:

Lemma 2.5. *Let δ be a weakly compact cardinal and $\gamma < \delta$. For each $\alpha < \delta$, let P_α be a δ-cc forcing poset. Then the $\leq \gamma$-support product $P := \prod_{\alpha<\delta}^{\leq\gamma} P_\alpha$ is δ-Knaster.*

Proof. Let $\langle p_\xi \ : \ \xi < \delta \rangle$ be a sequence of conditions in P. Since δ is inaccessible, γ^+ and δ satisfy the hypothesis of lemma 2.4, so we can assume w.l.o.g. that $\langle \operatorname{dom}(p_\xi) \ : \ \xi < \delta \rangle$ forms a Δ-system with root $R \subset \delta$, $|R| \leq \gamma$.

Define $f : [\gamma]^2 \longrightarrow R \cup \{\delta\}$ by:

$$f(\{\xi, \xi'\}) = \begin{cases} \min\{\alpha \in R \ : \ p_\xi(\alpha) \perp_{P_\alpha} p_{\xi'}(\alpha)\} & , \quad \text{if such set is not empty} \\ \delta & , \quad \text{otherwise} \end{cases}$$

Since δ is a weakly compact cardinal, using the arrow notation from Ramsey theory, we have $\delta \longrightarrow (\delta)^2_{|R|}$. Therefore, there is a set $H \in [\delta]^\delta$ such that $f''[H]^2 = \{\beta\}$, for some $\beta \in R \cup \{\delta\}$. Since each P_α is δ-cc, we must have $\beta = \delta$. Notice that this implies that $\langle p_\xi \ : \ \xi \in H \rangle$ is pairwise compatible, because if there were $\xi, \xi' \in H$ with p_ξ and $p_{\xi'}$ incompatible, there would be some α in the root R witnessing it. □

Lemma 2.6. *Let $\gamma < \delta$, both regular cardinals. Then $\mathbb{R}(\gamma, \delta)$ is $< \gamma$-closed. Also, if δ is weakly compact, then it is δ-Knaster.*

Proof. Lemma 2.3 clearly imply that $\mathbb{R}(\gamma, \delta)$ is $< \gamma$-closed.

Suppose δ is weakly compact. The δ-Knaster property follows from lemma 2.5 and the following claim:

Claim 2.7. *For every $n \in \omega$ and every regular $\gamma < \delta$, $\mathbb{R}^n(\gamma, \delta)$ is δ-Knaster.*

Proof of claim 2.7. We proceed by induction on n. For $n = 0$, we have $\mathbb{R}^0(\gamma, \delta) = \mathbb{S}(\gamma, \delta)$. Notice that $\mathbb{S}(\gamma, \delta)$ can alternatively described as the product of $\langle \mathrm{Col}(\beta, \{\delta\}) : \gamma \leq \beta < \delta \rangle$ with bounded support of size $\leq \gamma$. Since $\mathrm{Col}(\beta, \{\delta\})$ has size $\beta < \gamma$, it is δ-Knaster. Notice also that the proof of lemma 2.5 also holds in this case with the bounded support, so $\mathbb{R}^0(\gamma, \delta)$ is δ-Knaster.

The successor step follows directely from the induction hypothesis together with lemma 2.5. □

□

2.2 Kunen's universal collapse $\mathbb{P}$

Now, we are going to construct $\mathbb{P}$ using the notation by Cox in [Cox15]. This kind of construction was originally done by Kunen in [Kun78], where he refers to posets with such kind of property as *universal collapses.*

We will define recursively a sequence $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ of posets, and our $\mathbb{P}$ will be $\mathbb{P}_\kappa$. This is a slight modification of the usual concept of iteration; Cox calls it a (finite support) universal Kunen iteration.

Define:

- $\mathbb{P}_0 = \mathbb{S}(\omega, \kappa)$.
- Suppose $\mathbb{P}_\alpha$ constructed. Suppose $B_\alpha := \mathbb{P}_\alpha \cap V_\alpha$ is a regular suborder of $\mathbb{P}_\alpha$, $\Vdash_{B_\alpha}$ "$\alpha = \omega_1$" and α is an inaccessible cardinal. In this case, we call α an *active stage*. Let $\underset{\sim}{\mathbb{Q}}_\alpha$ be a B_α*-name* for $\mathbb{R}(\alpha^+, \kappa) \times \mathbb{B}$ and define $\mathbb{P}_{\alpha+1}$ as the set of all partial functions f on $\alpha + 1$ such that:
  - $f{\restriction}\alpha \in \mathbb{P}_\alpha$
  - if $\alpha \in \mathrm{dom}(f)$, then $f(\alpha)$ is a B_α*-name* of minimal rank such that
  $$\Vdash_{B_\alpha} \text{“}f(\alpha) \in \underset{\sim}{\mathbb{Q}}_\alpha\text{”}.$$

  and the order on $\mathbb{P}_{\alpha+1}$ is given by: $f_1 \leq f_2$ if and only if
  $$f_1{\restriction}\alpha \leq f_2{\restriction}\alpha \wedge \alpha \in \mathrm{dom}(f_2) \rightarrow (\alpha \in \mathrm{dom}(f_1) \wedge$$
  $$f_1{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“}V[\underset{\sim}{G}_{\mathbb{P}_\alpha} \cap V_\alpha] \models f_1(\alpha)[\underset{\sim}{G}_{\mathbb{P}_\alpha} \cap V_\alpha] \leq f_2(\alpha)[\underset{\sim}{G}_{\mathbb{P}_\alpha} \cap V_\alpha]\text{”}),$$

  where $\underset{\sim}{G}_{\mathbb{P}_\alpha}$ denotes the canonical $\mathbb{P}_\alpha$-name for a generic;
- otherwise, we call α a *passive stage* and let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha$;
- for limit ordinals $\delta \leq \kappa$, let $\mathbb{P}_\delta = \bigcup_{\alpha<\delta} \mathbb{P}_\alpha$ with the order induced by the $\mathbb{P}_\alpha$'s.

Remark 2.8. *Notice that the conditions on each step have finite support. Also, as in 1.7, the requirement that each $f(\alpha)$ is chosen with minimal rank guarantees that $\mathbb{P}_{\alpha+1}$ is indeed a set.*

Lemma 2.9. *Suppose κ is a regular cardinal and $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$, $\mathbb{P} := \mathbb{P}_\kappa$, is a universal collapse like constructed above. Then:*

(1) $\mathbb{P} \subset V_\kappa$;

(2) $\mathbb{P}$ *is* κ*-cc;*

(3) $|\mathbb{P}| \leq \kappa$;

(4) if $j : V \longrightarrow M$ *is an elementary embedding with critical point* κ*, then there exists a regular embedding* $h : \mathbb{P} * \left(\underset{\sim}{\mathbb{R}}(\kappa^+, \lambda) \times \underset{\sim}{\mathbb{B}} \right) \longrightarrow j(\mathbb{P})$ *extending* $j{\restriction}_{\mathbb{P}}$.

Proof. We start by proving item (1). We show by induction on $\alpha \leq \kappa$ that $\mathbb{P}_\alpha \subset V_\kappa$. The limit step clearly holds, so we just need to prove the successor step. Suppose that $\mathbb{P}_\alpha \subset V_\kappa$ for some active stage $\alpha < \kappa$ and let $f \in \mathbb{P}_{\alpha+1}$. By the induction hypothesis, we have $f{\restriction}_\alpha \in \mathbb{P}_\alpha \subset V_\kappa$, so it is enough to show that $f(\alpha) \in V_\kappa$.

Notice that, under CH, since $|\mathbb{B}| = \aleph_2$ we can assume wlog that $\mathbb{B} \subset V_{\aleph_2}$. Therefore, by a direct calculation, we have that $\mathbb{R}(\aleph_2, \kappa) \times \mathbb{B} \subset V_\kappa$. Thus, we have

$$\Vdash_{B_\alpha} \text{“} \underset{\sim}{\mathbb{Q}}_\alpha \subset V_{\check{\kappa}} \text{”}.$$

So by fact 1.8 and the requirement of the minimum rank on the definition of $\mathbb{P}_{\alpha+1}$, we have that $f(\alpha) \in V_\kappa$.

For item (2), we prove by induction on $\alpha \leq \kappa$ that $\mathbb{P}_\alpha$ is κ-cc. Since we are working with a finite support iteration, at limit steps we can use a simple delta system argument (the same used for proving that finite support iteration preserves κ-cc). So we just need to prove the result for each active stage. In order to prove that, we use the following claim:

Claim 2.10. *At each active stage* $\alpha < \kappa$, $\mathbb{P}_{\alpha+1} \approx B_\alpha * (\underset{\sim}{\mathbb{Q}}_\alpha \times \check{\mathbb{P}}_\alpha / \underset{\sim}{G}_{B_\alpha})$.

The proof of claim 2.10 is straightforward and we omit it. It follows from the claim that it is enough to show that B_α is κ-cc and that the trivial condition on B_α forces $\underset{\sim}{\mathbb{Q}}_\alpha \times \check{\mathbb{P}}_\alpha / \underset{\sim}{G}_{B_\alpha}$ to be κ-cc. By the induction hypothesis and since $B_\alpha \lessdot \mathbb{P}_\alpha$, we have that both B_α and any quotient $\mathbb{P}_\alpha / G_{B_\alpha}$ are κ-cc. So it is enough to show that $\underset{\sim}{\mathbb{Q}}_\alpha$ is forced to be κ-Knaster.

By definition, we have $\Vdash_{B_\alpha} \text{“} \underset{\sim}{\mathbb{Q}}_\alpha = \underset{\sim}{\mathbb{R}}(\alpha^+, \kappa) \times \underset{\sim}{\mathbb{B}} \text{”}$. By lemma 2.6, we have that $\mathbb{R}(\alpha^+, \kappa)$ is κ-Knaster. Also, since α is an active step, we have that $\Vdash_{B_\alpha} \text{“} \alpha = \omega_1 \text{”}$ and $|\mathbb{B}| = \aleph_2$, and hence we have

$$\Vdash_{B_\alpha} \text{“} |\underset{\sim}{\mathbb{B}}| = \aleph_2 = \alpha^+ < \kappa \text{”}$$

Therefore, $\underset{\sim}{\mathbb{B}}$ is forced to be vacuously κ-Knaster, thus concluding the proof of item (2).

For item (3), notice that since κ is inaccessible, we have $|V_\kappa| = \kappa$, thus item (3) follows from item (1).

For item (4), since $\mathbb{P}$ is κ-cc (item (2)) and $\mathbb{P} \subset V_\kappa$ (item (1)), we have that $j{\restriction}_{\mathbb{P}} = \mathrm{id}_{\mathbb{P}}$ is a regular embedding from $\mathbb{P}$ into $j(\mathbb{P})$. Notice also that $j(\mathbb{P}) \cap V_\kappa = \mathbb{P}$. By elementarity, we have that $j(\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle)$ is a finite support Kunen universal iteration of length λ, denoted by $\langle j(\mathbb{P})_\beta : \beta \leq \lambda \rangle$, and that $j(\mathbb{P})_\alpha = j(\mathbb{P}_\alpha)$ for $\alpha \leq \kappa$. So we can conclude that κ is an active stage of the iteration $\langle j(\mathbb{P})_\beta : \beta \leq \lambda \rangle$. Then, by the definition of the iteration, we have a natural correspondence $h : \mathbb{P} * (\underset{\sim}{\mathbb{R}}(\kappa^+, \lambda) \times \underset{\sim}{\mathbb{B}}) \longrightarrow j(\mathbb{P})_{\kappa+1}$. Similarly to an usual iteration, we can straightforwardly show that $j(\mathbb{P})_{\kappa+1} \lessdot j(\mathbb{P})$, so we conclude that h is the desired regular embedding. $\square$

Remark 2.11. *Notice also that each $\mathbb{P}_\alpha$ for $\alpha \leq \kappa$ and also $\mathbb{P} * \underset{\sim}{\mathbb{R}}(\kappa^+, \lambda)$ force GCH. This is clear by their respective sizes and chain conditions.*

3 Main result

Now that we constructed our forcing posets, we restate theorem 1.3 more precisely:

Theorem 3.1. *Suppose $j : V \longrightarrow M$ is a huge embedding with $\kappa := \mathrm{crit}(j)$, and $\lambda := j(\kappa)$ ($M^\lambda \subseteq M$). Then the poset $\mathbb{P} * \left(\underset{\sim}{\mathbb{R}}(\kappa^+, \lambda) \times \underset{\sim}{\mathbb{B}}\right)$ forces $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1)$ and $\neg\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$.*

For the rest of this section, we fix κ, λ, j and M as in theorem 3.1. We also fix a $\mathbb{P} * (\underset{\sim}{\mathbb{R}}(\kappa^+, \lambda) \times \underset{\sim}{\mathbb{B}})$-generic $G * H$ over V. Notice that, by the construction of the posets $\mathbb{P}$, $\mathbb{R}$ and $\mathbb{B}$, we have that $V[G * H] \models$ "$\kappa = \omega_1, (\kappa^+)^V = \omega_2$ and $\lambda = \omega_3$".

3.1 Extending the elementary embedding

The first step in order to prove theorem 3.1 is to extend the embedding j to an elementary embedding $\widehat{j} : V[G * H] \longrightarrow M[\widehat{G} * \widehat{H}]$, for some appropriate $\widehat{G} * \widehat{H}$. To simplify the notation, we denote such extensions also by j, making no distinction with the original embedding. We shall extend j in 2 stages.

First, by lemma 2.9(4) and fact 1.6 , we can find $\widehat{G}$ a $(V, j(\mathbb{P}))$-generic with $h''G * H \subset \widehat{G}$. Thus, by fact 1.12, we can extend j to $j : V[G] \longrightarrow M[\widehat{G}]$. Notice that, by elementarity, $j(\mathbb{P})$ collapses all the cardinals between ω and λ, thus ${\aleph_1}^{V[\widehat{G}]} = \lambda$. Notice also that, since $j(\mathbb{P})$ is λ-cc and $M^\lambda \cap V \subseteq M$, the extension $M[\widehat{G}]$ still retains some closure from M; more explicitly:

Fact 3.2. $M[\widehat{G}]^\lambda \cap V[\widehat{G}] \subset M[\widehat{G}]$.

The second stage is to further extend j to an elementary embedding with domain $V[G * H]$. In order to do so, we use the following lemma.

Lemma 3.3. *In $M[\widehat{G}]$, there is a master condition m for H over j, i.e. there exists $m \in j(\mathbb{R}(\kappa^+, \lambda)^{V[G]} \times \mathbb{B}^{V[G]})$ such that for any $q \in H$ we have $m \leq j(q)$.*

Proof. We work in $V[\widehat{G}]$. Since $G * H \subset \widehat{G}$, we have $H \in V[\widehat{G}]$. Let H_1, H_2 be respectively $\mathbb{R}(\kappa^+, \lambda)^{V[G]}$ and $\mathbb{B}^{V[G]}$ generic filters such that $H_1 \times H_2 = H$. It is enough to construct $m = \langle m_1, m_2 \rangle \in M[\widehat{G}]$ such that m_i is a master condition for H_i ($i = 1, 2$).

We start by constructing m_1. By elementarity, we have

$$j\left(\mathbb{R}(\kappa^+, \lambda)^{V[G]}\right) = \mathbb{R}(\lambda^+, j(\lambda))^{M[\widehat{G}]}.$$

Also, we have $|\mathbb{R}(\kappa^+,\lambda)^{V[G]}| = \lambda$, so H_1 also has size λ. By fact 3.2, since $H_1 \in V[\widehat{G}]$, we have that $j''H_1 \in M[\widehat{G}]$. By lemma 2.3, we can define $m_1 = \bigwedge j''H_1 \in \mathbb{R}(\lambda^+, j(\lambda))^{M[\widehat{G}]}$, which clearly satisfies the master condition property.

Now we construct m_2. Once again, by elementarity we have

$$j\left(\mathbb{B}^{V[G]}\right) = \mathbb{B}^{M[\widehat{G}]}.$$

Notice that $\mathbb{B}^{V[\widehat{G}]}$ is countable in $V[\widehat{G}]$ (since ${\aleph_1}^{V[\widehat{G}]} = \lambda$). So in $V[\widehat{G}]$ we can construct recursively a decreasing sequence $\langle q_n : n \in \omega\rangle$ generating H_2. By fact 3.2, we have that $\langle j(q_n) : n \in \omega\rangle \in M[\widehat{G}]$. By the σ-closure of $\mathbb{B}^{M[\widehat{G}]}$, we can find $m_2 \in \mathbb{B}^{M[\widehat{G}]}$ a lower bound for $\langle j(q_n) : n \in \omega\rangle$, which clearly satisfies the master condition property. $\square$

By fact 1.12, if we take $\widehat{H}$ an $(M[\widehat{G}], j(\underset{\sim}{\mathbb{R}}(\kappa^+,\lambda) \times \underset{\sim}{\mathbb{B}})[\widehat{G}])$-generic filter containing the condition m from lemma 3.3, we can extend j to $j : V[G * H] \longrightarrow M[\widehat{G} * \widehat{H}]$.

3.2 κ^+-centeredness

The next theorem states a key property used to show that $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1)$ holds in $V[G * H]$.

Theorem 3.4. *In $V[G * H]$, the poset $j(\mathbb{P})/h''(G * H)$ is κ^+-centered.*

Notice that by $j(\mathbb{P})/h''(G * H)$ we mean $j(\mathbb{P})^V/h''(G * H)$. The proof of theorem 3.4 is basically the original proof on [FL88], plus the argument that $\mathbb{B}$ is forced to have size $< \kappa^+$. We are going to need some lemmata:

Lemma 3.5. *Suppose δ is a regular cardinal such that $2^{<\delta} = \delta$. Then, given $\gamma < \delta$ and a family $\langle D_\alpha : \alpha < 2^\delta\rangle$ of δ-centered posets, we have that $\prod^{\leq\gamma}_{\alpha<2^\delta} D_\alpha$ is δ-centered.*

Lemma 3.5 is a generalization of the fact that Tychonoff product of continuum many separable spaces is separable. A proof for it can be found in [FL88].

For the next lemma, we shall fix some notations. By elementarity, we have that $j(\mathbb{P})$ is the limit of a finite support universal Kunen iteration of length λ. For every active stage $\alpha < \lambda$, let $B'_\alpha := j(\mathbb{P})_\alpha \cap V_\alpha$. Let $\overline{\mathbb{Q}'_\alpha}$ the termspace for the B'_α-name $\underset{\sim}{\mathbb{Q}'_\alpha} := \underset{\sim}{\mathbb{R}}(\alpha^+,\lambda) \times \underset{\sim}{\mathbb{B}}$ (definition 1.9).

Lemma 3.6. *In $V[G*H]$, the product $P := (\mathbb{S}(\omega,\lambda))^V \times \prod^{<\omega}_{\text{active } \alpha<\lambda}(\overline{\mathbb{Q}'_\alpha})^V$ is κ^+-centered.*

Proof. We work in $V[G * H]$. By remark 2.11, the conditions needed for lemma 3.5 do hold, so it is enough to prove that each factor of P is κ^+-centered.

Foreman and Laver proved that $(\mathbb{S}(\omega,\lambda))^V$ is κ^+-centered in [FL88]. We shall now prove that each $(\overline{\mathbb{Q}'_\alpha})^V$ is also κ^+-centered.

We fix $\alpha < \lambda$ an active stage. By definition 2.2, we have that

$$(\overline{\mathbb{Q}'_\alpha})^V \approx \left(\left(\prod_{n\in\omega} \overline{\mathbb{R}^n(\alpha^+,\lambda)}\right) \times \overline{\mathbb{B}}\right)^V. \tag{3.2.1}$$

We shall first show that $(\overline{\mathbb{B}})^V$ and each $\left(\overline{\mathbb{R}^n(\alpha^+,\lambda)}\right)^V$ are κ^+-centered, and then show that this implies the κ^+-centeredness of $(\overline{\mathbb{Q}'_\alpha})^V$. The proof that each $\left(\overline{\mathbb{R}^n(\alpha^+,\lambda)}\right)^V$ is κ^+-centered is the same as in [FL88]. Actually, the proof of regarding $(\mathbb{S}(\omega,\lambda))^V$ is a particular case of the one regarding $(\overline{\mathbb{R}^n(\alpha^+,\lambda)})^V$.

Now we work in V. Since α an active stage, it is inaccessible. Thus, we have $|V_\alpha| = \alpha$, so $|B'_\alpha| \leq \alpha$. Furthermore, by theorem 2.1 we have:

$$V \models \Vdash_{B'_\alpha} \text{“} |\underset{\sim}{\mathbb{B}}| = \aleph_2 = \alpha^+ \text{”}.$$

By proposition 1.10, we have $|\overline{\mathbb{B}}| \leq (\alpha^+)^\alpha = \alpha^+$.Therefore, back to $V[G*H]$, we have $|(\overline{\mathbb{B}})^V| \leq \kappa^+$, so $(\overline{\mathbb{B}})^V$ is indeed κ^+-centred.

Now we shall prove that $(\overline{\mathbb{Q}'_\alpha})^V$ is κ^+-centered in $V[G*H]$. Since $(\kappa^+)^{\aleph_0} = \kappa^+$, we can construct in $V[G*H]$ a κ^+-centering C for $\left(\prod_{n\in\omega}\left(\overline{\mathbb{R}^n(\alpha^+,\lambda)}\right)^V\right) \times (\overline{\mathbb{B}})^V$. We claim that the restriction of C to V is a κ^+-centering of $\left(\left(\prod_{n\in\omega}\overline{\mathbb{R}^n(\alpha^+,\lambda)}\right) \times \overline{\mathbb{B}}\right)^V$ - thus, by (3.2.1), $(\overline{\mathbb{Q}'_\alpha})^V$ is κ^+-centered.

Let $s_1,\ldots,s_m \in \left(\left(\prod_{n\in\omega}\overline{\mathbb{R}^n(\alpha^+,\lambda)}\right) \times \overline{\mathbb{B}}\right)^V$ such that $C(s_1) = \ldots = C(s_m)$. By the κ^+-centering, we have $t \in V[G*H]$ a common extension of $s_1,\ldots,s_m$, but it may be the case that $t \notin V$. However, for each $n \in \omega$, $t(n) \in V$ witnesses that $s_1(n),\ldots,s_m(n)$ are pairwise compatible in $\left(\overline{\mathbb{R}^n(\alpha^+,\lambda)}\right)^V$. By lemma 2.3, we can construct $t' \in V$ defined by $t'(\omega) = t(\omega)$ and for each $n \in \omega$ $V \models \Vdash_{B'_\alpha} \text{“}t'(n) = \bigwedge_{1\leq i\leq n} s_i(n)\text{”}$. Clearly $t' \leq s_1,\ldots,s_m$, so this completes the proof. $\square$

Proof of theorem 3.4. We work in $V[G*H]$. Notice that there is a natural projection from the poset $j(\mathbb{P})$ onto the product P from lemma 3.6, where each $p(\alpha)$ is sent to the correspondent equivalence class in the termspace. Therefore, given $C : P \longrightarrow \kappa^+$ witnessing that P is κ^+ centered, we can w.l.o.g. consider the same function over $j(\mathbb{P})$.

Let $p_1,\ldots,p_n \in j(\mathbb{P})/h''(G*H)$ such that $C(p_1) = \ldots = C(p_n)$. We want to show that those conditions have a common extension in $j(\mathbb{P})/h''(G*H)$. Notice that for each $\alpha \in \bigcup_{1\leq i\leq n}\mathrm{dom}(p_i)$ we have

$$\Vdash_{B'_\alpha} \text{“}p_1(\alpha),\ldots,p_n(\alpha) \text{ have a common extention”}.$$

For each $\alpha \in \bigcup_{1\leq i\leq n}\mathrm{dom}(p_i)$ and each $i = 1,\ldots,n$ we can assume that $p_i(\alpha)$ is of the form $p_i(\alpha) = \mathrm{op}(\underset{\sim}{a}_{i,\alpha}, \underset{\sim}{b}_{i,\alpha})$ (op as in 1.13), where $\underset{\sim}{a}_{i,\alpha}$ and $\underset{\sim}{b}_{i,\alpha}$ are B'_α-names for conditions in $\underset{\sim}{\mathbb{R}}(\alpha^+,\lambda)$ and $\underset{\sim}{\mathbb{B}}$ respectively.

By the size of the termspace $\overline{B}^{B'_\alpha}$, we can assume that

$$\Vdash_{B'_\alpha} \text{“} \underset{\sim}{b}_{1,\alpha} = \ldots = \underset{\sim}{b}_{n,\alpha} \text{”}. \tag{3.2.2}$$

Hence we can further assume $\underset{\sim}{b}_{1,\alpha} = \ldots = \underset{\sim}{b}_{n,\alpha}$, so denote such name by $\underset{\sim}{b}_\alpha$.

We then define $q \in j(\mathbb{P})$ with domain $\bigcup_{1\leq i\leq n}\mathrm{dom}(p_i)$ such that

$$\Vdash_{B'_\alpha} \text{“}q(\alpha) = \langle \bigwedge_{1\leq i\leq n} \underset{\sim}{a}_{i,\alpha}, \underset{\sim}{b}_\alpha\rangle\text{”} \tag{3.2.3}$$

for every $\alpha \in \mathrm{dom}(q)$. Clearly q is a common extension for $p_1, \dots, p_n$ in $j(\mathbb{P})$, so it is enough to show that $q \in j(\mathbb{P})/h''(G * H)$. Let $x \in h''(G * H)$. We shall show that x and q are compatible in $j(\mathbb{P})$. Since $p_1, \dots, p_n \in j(\mathbb{P})/h''(G \cap H)$, we can take $y_1, \dots, y_n \in j(\mathbb{P})$ such that $y_i \leq_{j(\mathbb{P})} x, p_i$ for $i = 1, \dots, n$.

Like before, we can assume that, for $\alpha \in \mathrm{dom}(x)$, $x(\alpha) = \mathrm{op}(\underset{\sim}{x}_{1,\alpha}, \underset{\sim}{x}_{2,\alpha})$. Likewise, for $i = 1, \dots, n$ and $\alpha \in \mathrm{dom}(y_i)$, we assume $y_i(\alpha) = \mathrm{op}(\underset{\sim}{u}_{i,\alpha}, \underset{\sim}{v}_\alpha)$, where $\underset{\sim}{x}_{1,\alpha}, \underset{\sim}{u}_{i,\alpha}$ and $\underset{\sim}{x}_{2,\alpha}, \underset{\sim}{v}_\alpha$ are B'_α-names for conditions in $\underset{\sim}{\mathbb{R}}(\alpha^+, \lambda)$ and $\underset{\sim}{\mathbb{B}}$ respectively. Notice that $\underset{\sim}{v}_\alpha$ does not deppend on i, by the same argument used in (3.2.2). We can further assume that

$$y_{i,\alpha} \restriction \alpha \Vdash_{j(\mathbb{P})_\alpha} \text{“} M[\underset{\sim}{G}_{j(\mathbb{P})_\alpha} \cap (V_\alpha)^M] \models \underset{\sim}{u}_{i,\alpha} = \underset{\sim}{x}_{1,\alpha} \wedge \underset{\sim}{a}_{i,\alpha} \text{”} \tag{3.2.4}$$

We shall construct $y \in j(\mathbb{P})$ such that $y \leq_{j(\mathbb{P})} y_1, \dots y_n$. By induction on $\alpha \leq \lambda$, we construct y such that $\mathrm{dom}(y) = \bigcup_{1 \leq i \leq n} \mathrm{dom}(y_i)$ and

$$y \restriction \alpha \leq_{j(\mathbb{P})_\alpha} y_1 \restriction \alpha, \dots, y_n \restriction \alpha \text{ for all } \alpha < \lambda. \tag{3.2.5}$$

If α is a limit ordinal, since the support of $j(\mathbb{P})_\alpha$ is finite, it is enough to take $y \restriction \alpha = \bigcup_{\beta < \alpha} y \restriction \beta$. So we assume that the induction hypothesis hold for some active $\alpha < \lambda$ and construct $y(\alpha)$. Since $y_i \restriction \alpha \leq_{j(\mathbb{P})_\alpha} x \restriction \alpha, p_i \restriction \alpha$, from 3.2.5 we have:

$$y \restriction \alpha \Vdash_{j(\mathbb{P})_\alpha} \text{“} M[\underset{\sim}{G}_{j(\mathbb{P})_\alpha} \cap (V_\alpha)^M] \models \forall i = 1, \dots, n, \underset{\sim}{u}_{i,\alpha} \leq \underset{\sim}{x}_{1,\alpha}, \underset{\sim}{a}_{i,\alpha} \text{ and } \underset{\sim}{v}_\alpha \leq \underset{\sim}{x}_{2,\alpha}, \underset{\sim}{b}_\alpha \text{”} \tag{3.2.6}$$

By (3.2.3), we have

$$\Vdash_{B'_\alpha} \text{“} \underset{\sim}{a}_{1,\alpha}, \dots, \underset{\sim}{a}_{n,\alpha} \text{ are pairwise compatible in } \underset{\sim}{\mathbb{R}}(\alpha^+, \lambda) \text{”} \tag{3.2.7}$$

Since $B'_\alpha \lessdot j(\mathbb{P})_\alpha$, (3.2.7) together with (3.2.6) implies

$$y \restriction \alpha \Vdash_{j(\mathbb{P})_\alpha} \text{“} M[\underset{\sim}{G}_{j(\mathbb{P})_\alpha} \cap (V_\alpha)^M] \models \underset{\sim}{a}_{1,\alpha}, \dots, \underset{\sim}{a}_{n,\alpha}, \underset{\sim}{x}_{1,\alpha} \text{ are pairwise compatible in } \underset{\sim}{\mathbb{R}}(\alpha^+, \lambda) \text{”}$$

By lemma 2.3, we can then find a B'_α name $y(\alpha)$ such that:

$$y \restriction \alpha \Vdash_{j(\mathbb{P})_\alpha} \text{“} M[\underset{\sim}{G}_{j(\mathbb{P})_\alpha} \cap (V_\alpha)^M] \models y(\alpha) = \langle \bigwedge \{\underset{\sim}{a}_{1,\alpha}, \dots, \underset{\sim}{a}_{n,\alpha}, \underset{\sim}{x}_{1,\alpha}\}, \underset{\sim}{v}_\alpha \rangle \text{”}$$

By (3.2.4), we have that $y \restriction \alpha + 1 \leq_{j(\mathbb{P})_{\alpha+1}} y_1 \restriction \alpha + 1, \dots y_n \restriction \alpha + 1$, so this concludes the construction of y. From 3.2.6 and 3.2.3, we have that $y \leq_{j(\mathbb{P})} x, q$. □

3.3 Proof of the main result

Notice that, in $V[G]$, since $\omega_1 = \kappa$ and $\mathbb{R}(\kappa^+, \lambda)$ does not add any new ω-sequence of $\aleph_2$, by the last property of $\mathbb{B}$ in theorem 2.1 we have $\mathbb{R}(\kappa^+, \lambda) * \underset{\sim}{\mathbb{B}} \approx \mathbb{R}(\kappa^+, \lambda) \times \mathbb{B}$. Therefore, we have:

$$V[G * H] \models \neg \mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1).$$

The following theorem completes the proof of theorem 3.1.

Theorem 3.7. *In $V[G * H]$, every graph of size and chromatic number $\aleph_3$ has a subgraph of size and chromatic number $\aleph_1$, i.e. $V[G * H] \models \mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1)$.*

The proof for theorem 3.7 is almost the same proof as in [FL88]. It follows from the two similar lemmata below:

Lemma 3.8. *Let $\mathcal{G} = \langle \lambda, E\rangle$ be a graph and D a $< \lambda$-closed poset. Suppose there exists a countable good coloring of $\mathcal{G}$ (i.e. a good color $f : \lambda \longrightarrow \omega$) in some generic extension by D. Then there is some countable good coloring of $\mathcal{G}$ in the ground model.*

Proof. Assume there are a D-name $\underset{\sim}{f}$ and a condition $d_0 \in D$ such that $d_0 \Vdash_D$ "$\underset{\sim}{f}$: is an ω-good coloring of $\check{\mathcal{G}}$". Since D is $< \lambda$-closed, we can construct a decreasing sequence $\langle d_\alpha : \alpha < \lambda\rangle$ such that for each $\alpha < \lambda$, d_α decides $\underset{\sim}{f}(\alpha)$.

We can define a function $f' : \lambda \longrightarrow \omega$ in the ground model by $f'(\alpha) = n$, where $n \in \omega$ is such that $d_\alpha \Vdash_D$ "$\underset{\sim}{f}(\check{\alpha}) = n$". Since $\langle d_\alpha : \alpha < \lambda\rangle$ is decreasing, it is easy to see that f' is a countable good coloring of $\mathcal{G}$. ☐

Lemma 3.9. *Let $\mathcal{G} = \langle \lambda, E\rangle$ be a graph and C a δ-centered poset ($\aleph_0 \leq \delta < \lambda$). Suppose there exists a countable good coloring of $\mathcal{G}$ in some generic extension by C. Then there is some good coloring for $\mathcal{G}$ in the ground model with δ many colors .*

Proof. Assume there are a C-name $\underset{\sim}{f}$ and a condition $c \in C$ such that c forces $\underset{\sim}{f}$ to be an ω-good coloring of $\mathcal{G}$. For each $\alpha \in \lambda$, chose $c_\alpha \leq c$ and $n_\alpha \in \omega$ such that $c_\alpha \Vdash_C$ "$\underset{\sim}{f'}(\check{e_\alpha}) = n_\alpha$".

Let $g : C \longrightarrow \delta$ witness that D is δ-centered. We can define a function $f' : \lambda \longrightarrow \omega \times \delta$ in the ground model by $f'(\alpha) = \langle n_\alpha, g(c_\alpha)\rangle$. Since g is a centering, it is clear that f' is a good coloring of $\mathcal{G}$. Also, since $|\omega \times \delta| = \delta$, f' indeed has δ many colors. ☐

Proof of theorem 3.7. In $V[G*H]$, let $\mathcal{G}$ be a graph of size and chromatic number $\aleph_3$. We can assume w.l.o.g. that $\mathcal{G} = \langle \lambda, E\rangle$. By elementarity, it is enough to show that

$$M[\widehat{G} * \widehat{H}] \models \text{“}j(\mathcal{G}) \text{ has a subgraph of size and chromatic number } j(\kappa) = \lambda\text{”}$$

We claim that $\langle j''\lambda, j''E\rangle$ is a witness for the previous statement. Notice that we indeed have $\langle j''\lambda, j''E\rangle \in M[\widehat{G} * \widehat{H}]$, since $j''\lambda \in M \subseteq M[\widehat{G} * \widehat{H}]$ by the closure of M and $j''E = j(E) \cap j''([\lambda]^2) \in M[\widehat{G} * \widehat{H}]$ by elementarity.

Assume, towards a contradiction, that there is some countable good coloring f for $\mathcal{G}$ in $M[\widehat{G} * \widehat{H}]$. Since being a good coloring is upwards absolute, we have that f is still a good coloring in $V[\widehat{G} * \widehat{H}]$. Working in $V[\widehat{G} * \widehat{H}]$, we have that $\langle j''\lambda, j''E\rangle$ is isomorphic to $\mathcal{G}$, and hence we have $f' : \lambda \longrightarrow \omega$ a good coloring $\mathcal{G}$ induced by f.

Now, notice that $V[\widehat{G} * \widehat{H}] = V[\widehat{G}][\widehat{H}]$ is a generic extension of $V[\widehat{G}]$ by the poset $j(\underset{\sim}{\mathbb{R}}(\kappa^+, \lambda) \times \underset{\sim}{\mathbb{B}})[\widehat{G}]$, which is $< \lambda$-closed in $V[\widehat{G}]$ (by elementarity, fact 3.2, lemma 2.6 and theorem 2.1). Next, we have that $V[\widehat{G}]$ is a generic extension of $V[G * H]$ by the poset $j(\mathbb{P})/h''(G*H)$, which is κ^+-centered in $V[G*H]$ (theorem 3.4). Therefore, we can apply lemma 3.8 and then lemma 3.9 to construct a good coloring $f'' : \lambda \longrightarrow \kappa^+$ of $\mathcal{G}$ in $V[G * H]$. However, we have

$$V[G*H] \models \mathrm{Chr}(\mathcal{G}) = \aleph_3 = \lambda > \kappa^+ = \aleph_2$$

So f'' witnesses a contradiction. ☐

From theorem 3.1, we have:

Corollary 3.10. *If the existence of a huge cardinal is consistent, then* $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1)$ *does not imply* $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$.

Question 1. *Is it consistent (modulo some large cardinal assumption) that*

$$\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1) \wedge \neg \mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_2)?$$

Notice that the same technique used here would not work to solve question 1, since we heavily used the small size of the poset $\mathbb{B}$.

4 Generalization

We can also generalize theorem 3.1 for other cardinalities. In order to do so, we generalize 2.1.

Theorem 4.1. *Let κ be a regular cardinal and assume $2^\kappa = \kappa^+$. Then there exists a poset $\mathbb{B}(\kappa)$ of size κ^{++} which is κ-closed, κ^{++}-cc and forces $\neg \mathrm{Tr}_{\mathrm{Chr}}(\kappa^{++}, \kappa^+)$.*

More specifically, $\mathbb{B}(\kappa)$ forces the existence of a graph of size and chromatic number κ^{++} for which all subgraphs of size $\leq \kappa^+$ have chromatic number $\leq \kappa$.

We can use theorem 4.1 to generalize theorem 3.1 and obtain:

Corollary 4.2. *Suppose κ is a huge cardinal with target λ. Let α be an ordinal such that $(\aleph_\alpha)^V < \kappa$ is regular. Let also $n + 2 < m < \omega$. Then, there exists W a generic extension of V such that $(\aleph_{\alpha+1})^W = \kappa$, $(\aleph_{\alpha+m})^W = \lambda$ and*

$$W \models \text{“}\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\alpha+m}, \aleph_{\alpha+1}) \wedge \neg \mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\alpha+n+2}, \aleph_{\alpha+n+1})\text{”}$$

Proof. We proceed similarly to the proof of theorem 3.1, but use theorem 4.1 instead of theorem 2.1. This way we construct the poset $\mathbb{P}' * \underset{\sim}{\mathbb{Q}}$, where $\mathbb{P}'$ is the Kunen universal collapse with respect to $\mathbb{Q}$, $\mathbb{P}'_0 = \mathbb{S}(\aleph_\alpha, \kappa)$ and

$$\underset{\sim}{\mathbb{Q}} = \left(\underset{\sim}{\mathbb{R}} \left(\kappa^{+(m-2)}, \lambda\right) \times \underset{\sim}{\mathbb{B}} \left({\aleph_{\alpha+n}}^{V^{\mathbb{P}'}} \right) \right).$$

The proof that this poset is the desired one is analogous to the original proof of theorem 3.1. $\square$

In the rest of this section, we shall construct the poset $\mathbb{B}(\kappa)$ and prove theorem 4.1.

4.1 The poset

The poset defined here and the following proof are a straightforward generalization of Baumgartner's construction in [Bau84].

We denote $W := E_\kappa^{\kappa^{++}} = \{\alpha \in \kappa^{++} : \mathrm{cf}(\alpha) = \kappa\}$ and fix a family of functions $\{f_\xi : W \longrightarrow \kappa^{++} : \xi < \kappa\}$ such that for each $\alpha \in W$ $\langle f_\xi(\alpha) : \xi \in \kappa \rangle$ is an increasing sequence cofinal in α. Such a family is called a *ladder system*.

The poset $\mathbb{B}(\kappa)$ consists of conditions p of the form:

$$p = \langle a, e, \langle g_\beta : \beta \in a\rangle, \langle A(\alpha,\beta) : \alpha,\beta \in a\rangle, \langle B(\alpha,\beta,\xi) : \alpha,\beta \in a, \xi \in \kappa\rangle\rangle$$

satisfying the following properties:

(1) $a \in [W]^\kappa$;

(2) $e \subset [a]^2$;

(3) $g_\beta : \beta \cap a \longrightarrow \kappa$ is a good coloring of $\langle \beta \cap a, e \cap [\beta \cap a]^2\rangle$;

(4) $A(\alpha,\beta) \subset \kappa$ is unbounded and counbounded;

(5) $B(\alpha,\beta,\xi) \subset A(\alpha,\beta)$ is unbounded in κ and $B(\alpha,\beta,\xi) \cap B(\alpha,\beta,\xi') = \emptyset$ for $\xi \neq \xi'$;

(6) If $\gamma < \alpha,\beta$, $\{\alpha,\gamma\} \in e$ and $\beta \in a$, then $g_\beta(\gamma) \in A(\alpha,\beta)$;

(7) If $\gamma < \alpha,\beta$, $\{\alpha,\gamma\} \in e$, $\beta \in a$ and $g_\beta(\gamma) \in B(\alpha,\beta,\xi)$, then $\gamma \leq f_\xi(\alpha)$;

Notation 4.3. *For $p \in \mathbb{B}(\kappa)$ like above, we denote $a^p := a$, $e^p := e$, $g^p_\beta := g_\beta$, $A^p(\alpha,\beta) := A(\alpha,\beta)$ and $B^p(\alpha,\beta,\xi) := B(\alpha,\beta,\xi)$. Furthermore, we denote $a^{p_1} := a^1$, $a^{p_2} := a^2$, $e^{p_1} := e^1$, and so on.*

The order on $\mathbb{B}(\kappa)$ is defined by: given $p^1, p^2 \in \mathbb{B}$, $p^1 \leq p^2$ iff $a^1 \supseteq a^2$, $e^1 \cap [a^2]^2 = e^2$, $g^1_\beta \restriction a^2 = g^2_\beta$, $A^1(\alpha,\beta) = A^2(\alpha,\beta)$, $B^1(\alpha,\beta,\xi) = B^2(\alpha,\beta,\xi)$ for all $\alpha,\beta \in a^2$ and $\xi \in \kappa$.

Notice that each p contains the following information: a graph $\langle a^p, e^p\rangle$ of size κ approximating the generic graph; a function g^p_β approximating a good coloring of a small subgraph of the generic graph; a family $A^p(\alpha,\beta)$ consisting of the colors still available for the colorings g^q_β, for $q \leq p$; and a family $B^p(\alpha,\beta,\xi)$ consisting of colors which should be avoided by g^q_β for $q \leq p$ (so that $B^p(\alpha,\beta,\xi)$ controls a "cofinal growth" of g^q_β).

Proposition 4.4. *For κ regular, $\mathbb{B}(\kappa)$ is κ-closed. Moreover, any decreasing sequence of length κ of conditions in $\mathbb{B}$ has a greatest lower bound.*

Proof. Just notice that since κ is regular, the coordinatewise union of a decreasing sequence of length κ of conditions in $\mathbb{B}(\kappa)$ is itself a condition in $\mathbb{B}(\kappa)$. $\square$

The next proposition will be useful to construct conditions in $\mathbb{B}(\kappa)$.

Proposition 4.5. *Suppose that for some $X \subset a \times a$ we have*

$$\langle a, e, \langle g_\beta : \beta \in a\rangle, \langle A(\alpha,\beta) : \langle\alpha,\beta\rangle \in X\rangle, \langle B(\alpha,\beta,\xi) : \langle\alpha,\beta\rangle \in X, \xi \in \kappa\rangle\rangle$$

satisfying properties (1)-(7) for $A(\alpha,\beta)$, $B(\alpha,\beta,\xi)$ defined only for $\langle\alpha,\beta\rangle \in X$. If for all $\langle\alpha,\beta\rangle \in a \times a \setminus X$ we have

$$\{g_\beta(\gamma) : \{\gamma,\alpha\} \in e, \gamma < \alpha,\beta\} \text{ is counbounded in } \kappa \tag{4.1.1}$$

then we can define $A(\alpha,\beta)$ and $B(\alpha,\beta,\xi)$ for $\langle\alpha,\beta\rangle \in a \times a \setminus X$ so that the resulting sequence is an condition in $\mathbb{B}(\kappa)$.

Proof. Fix $\langle \alpha, \beta \rangle \in a \times a \setminus X$. Define $C := \kappa \setminus \{g_\beta(\gamma) : \{\gamma, \alpha\} \in e, \gamma < \alpha, \beta\}$. By (4.1.1), C is unbounded in κ. Therefore, we can partition C into $\langle C_\xi : \xi < \kappa \rangle$ such that each C_ξ is unbounded in κ. We define $A(\alpha, \beta) = \kappa \setminus C_0$ and $B(\alpha, \beta, \xi) = C_{\xi+1}$. Clearly, conditions (4) and (5) are satisfied. Since $\{g_\beta(\gamma) : \{\gamma, \alpha\} \in e, \gamma < \alpha, \beta\} \subset A(\alpha, \beta)$, condition (6) is satisfied. Condition (7) holds vacuously, since $\{g_\beta(\gamma) : \{\gamma, \alpha\} \in e, \gamma < \alpha, \beta\} \cap B(\alpha, \beta, \xi) = \emptyset$ for each $\xi \in \kappa$. $\square$

Using proposition 4.5, it is straightforward to prove:

Lemma 4.6. *For every $\alpha \in W$, the set $D_\alpha = \{p \in \mathbb{B}(\kappa) : \alpha \in a^p\}$ is dense in $\mathbb{B}(\kappa)$.*

4.2 Amalgamation of conditions

We shall fix the following framework: For each $\delta \in [\kappa^+, \kappa^{++})$, we fix a bijection $h_\delta : \kappa^+ \longrightarrow \delta$. Let θ be a large enough regular cardinal and $\lhd$ a well ordering of $H(\theta)$. Consider $\mathfrak{A}$ the structure with underlying set $H(\theta)$, fixed relations $\in$ and $\lhd$ and fixed functions $\langle h_\delta : \delta \in [\kappa^+, \kappa^{++}) \rangle$, $\langle h_\delta^{-1} : \delta \in [\kappa^+, \kappa^{++}) \rangle$ and $\langle f_\xi : \xi \in \kappa \rangle$.

Now we can define:

Definition 4.7. *Let $x \in [\kappa^{++}]^\kappa$. We say that x is* strongly closed *if it is a substructure of $\mathfrak{A}$. For any $x \in [\kappa^{++}]^\kappa$, we define the* strong closure *of x, denoted by* $\mathrm{scl}(x)$, *as the smallest strongly closed set containing x. Also, we say that a condition $p \in \mathbb{B}(\kappa)$ is* closed *if $a^p = \mathrm{scl}(a^p) \cap W$.*

Proposition 4.8. *The set of all closed conditions of $\mathbb{B}(\kappa)$ is dense.*

Proof. It follows from proposition 4.4 and lemma 4.6. $\square$

Lemma 4.9. *Suppose $x, y \in [\kappa^{++}]^\kappa$ are strongly closed and $x \cap \kappa^+ = y \cap \kappa^+$. Then $x \cap y$ is an initial segment of both x and y.*

Proof. If $x \cap y \subset \kappa^+$, the result clearly holds. Thus, let $\delta \in x \cap y$. Since x is closed under h_δ and h_δ^{-1}, we have that $x \cap \delta = h_\delta''(x \cap \kappa^+)$ Similarly, we have $y \cap \delta = h_\delta''(y \cap \kappa^+)$. Since we assumed $x \cap \kappa^+ = y \cap \kappa^+$, we have $x \cap \delta = y \cap \delta$, thus $x \cap y$ is indeed an initial segment of x and of y. $\square$

Now we can define the concept of isomorphism.

Definition 4.10. *We shall say that two closed condition $p^1, p^2 \in \mathbb{B}(\kappa)$ are* isomorphic *if $a^1 \cap \kappa^+ = a^2 \cap \kappa^+$ and there exists an order preserving bijection $\varphi : \mathrm{scl}(a^1) \longrightarrow \mathrm{scl}(a^2)$ which preserves the side-conditions and the ladder system $\langle f_\xi : \xi \in \kappa \rangle$, i.e.:*

- $a^2 = \varphi'' a^1$, $e^2 = \{\{\varphi(\alpha), \varphi(\beta)\} : \{\alpha, \beta\} \in e^1\}$;
- $g^2_{\varphi(\beta)}(\varphi(\alpha)) = g^1_\beta(\alpha)$;
- $A^2(\varphi(\alpha), \varphi(\beta)) = A^1(\alpha, \beta)$, $B^2(\varphi(\alpha), \varphi(\beta), \xi) = B^1(\alpha, \beta, \xi)$;
- $\varphi(f_\xi(\alpha)) = f_\xi(\varphi(\alpha))$.

for all $\alpha, \beta \in a^1, \xi \in \kappa$.

As usual, such a φ is called an isomorphism. Now we can prove our main lemma.

Lemma 4.11.

(a) If $p^1, p^2 \in \mathbb{B}(\kappa)$ are closed isomorphic conditions, they are compatible;

(b) if, furthermore, there exist $\alpha^* \in a^1$ and $\xi^* \in \kappa$ such that

$$a^1 \cap a^2 \subseteq f_{\xi^*}(\varphi(\alpha^*)) \leq \alpha^* < \varphi(\alpha^*) \tag{4.2.1}$$

where $\varphi : \mathrm{scl}(a^1) \longrightarrow \mathrm{scl}(a^2)$ is the isomorphism between p^1 and p^2, then there exists a condition $p^3 \leq p_1, p_2$ such that $\{\alpha^*, \varphi(\alpha^*)\} \in e^3$.

Proof. Since the proof of item (b) contains the proof of item (a), we shall prove both simultaneously, calling attention just to the parts where they differ.

Let p^1, p^2 be closed isomorphic conditions in $\mathbb{B}(\kappa)$. Begin the construction of $p^3 \leq p^1, p^2$ by setting $a^3 = a^1 \cup a^2$, $A^3(\alpha, \beta) = A^i(\alpha, \beta)$ and $B^3(\alpha, \beta, \xi) = B^i(\alpha, \beta, \xi)$ for $\alpha, \beta \in a^i$, for $i = 1, 2$. In case (a), define $e^3 = e^1 \cup e^2$, while on case (b) $e^3 = e^1 \cup e^2 \cup \{\{\alpha^*, \varphi(\alpha^*)\}\}$.

We need to construct the functions g^3_β for $\beta \in a^3$ and also construct suitable $A^3(\alpha, \beta)$ and $B^3(\alpha, \beta, \xi)$ for the remaining pairs $\{\alpha, \beta\} \in (a^3)^2 \setminus ((a^1)^2 \cup (a^2)^2)$.

For simplicity , define $\Delta := a^1 \cap a^2$, $b_1 := a^1 \setminus \Delta$ and $b_2 := a^2 \setminus \Delta$. We shall construct g^3_β. First, assume $\beta \in \Delta$. In this case we define $g^3_\beta = g^1_\beta \cup g^2_\beta$. By lemma 4.9, we have that Δ is an initial segment of a^1 and a^2. This implies that $\varphi \restriction \Delta$ is the identity function, thus by the the isomorphism we have that g^1_β and g^2_β are compatible functions. In case (a), we clearly have that g^3_β satisfies (3), (6) and (7). In case (b), by assumption (4.2.1) we have $\beta < \alpha^*$, thus α^*, β^* do not belong to the domain of g^3_β, therefore the satisfaction of (3), (6) and (7) follow from the same argument as in case (a).

Now, we construct g^3_β for $\beta \in b_1$. We shall construct it recursively so that it satisfies the following properties:

(i) $g^3_\beta \restriction a^1 = g^1_\beta$;

(ii) $g^3_\beta \restriction b_2$ is 1-1;

(iii) $\forall \gamma \in b_2 \cap \beta, g^3_\beta(\gamma) \in \kappa \setminus A^2(\gamma, \varphi(\beta))$;

(iv) $\forall \alpha \in b_2, A^2(\alpha, \varphi(\beta)) \cup g^3_\beta{}''(b_2 \cap \beta)$ is counbounded in κ;

(v) In case (b), we also want: $\varphi(\alpha^*) < \beta \Rightarrow g^3_\beta(\varphi(\alpha^*)) \neq g^1_\beta(\alpha^*)$.

Before constructing it, we enumerate $b_2 \cap \beta := \{\gamma(\xi) : \xi \in \kappa\}$. Separately we also enumerate $b_2 := \{\alpha(\xi) : \xi \in \kappa\}$, so that each element of b_2 reappears cofinally many times on the enumeration. Then we shall define $g^3_\beta(\gamma(\xi))$ recursively on ξ. Simultaneously, we construct a sequence $\langle x_\xi : \xi \in \kappa \rangle \subset \kappa$ consisting of "colors that g^3_β must avoid".

In case (b), if $\varphi(\alpha^*) < \beta$ we fix $x_0 = g^1_\beta(\alpha^*)$ and choose arbitrarily $g^3_\beta(\gamma(0)) \in \kappa \setminus (\{x_0\} \cup A^2(\gamma(0), \varphi(\beta)))$, thus satisfying (v). Otherwise, we define x_0 and $g^3_\beta(\gamma(0))$ as in the following general case.

Given $\xi < \kappa$, suppose $g^3_\beta(\gamma(\zeta))$ and x_ξ have already been defined for all $\zeta < \xi$.

Since by property (4) we have that $A^2(\alpha(\xi), \varphi(\beta))$ is counbounded in κ, we can choose

$$x_\xi \in \kappa \setminus (\xi \cup A^2(\alpha(\xi), \varphi(\beta)) \cup \{g^3_\beta(\gamma(\zeta)) : \zeta < \xi\}).$$

Since $A^2(\gamma(\xi), \varphi(\beta))$ and X_ξ are counbounded in κ, we can take

$$g^3_\beta(\gamma(\xi)) \in \kappa \setminus \left(A^2(\gamma(\xi), \varphi(\beta)) \cup \{x_\zeta : \zeta < \xi\} \cup \{g^3_\beta(\gamma(\zeta)) : \zeta < \xi\}\right).$$

Clearly such $g^3_\beta(\gamma(\xi))$ satisfies (ii) and (iii). Also, notice that $\langle x_\xi : \xi < \kappa\rangle$ is an unbounded sequence witnessing the satisfaction of (iv), so this concludes the construction.

Now we construct g^3_β for $\beta \in b_2$. This is basically the dual of the construction above, except for one extra property regarding $g^3_\beta(\alpha^*)$ in case (b). More explicitly, we construct g^3_β so that:

(i') $g^3_\beta \restriction a^2 = g^2_\beta$;

(ii') $g^3_\beta \restriction b_1$ is 1-1;

(iii') $\forall \gamma \in (b_1 \cap \beta) \setminus \{\alpha^*\}, g^3_\beta(\gamma) \in \kappa \setminus A^1(\gamma, \varphi^{-1}(\beta))$;

(iv') $\forall \alpha \in b_1, A^1(\alpha, \varphi^{-1}(\beta)) \cup g^3_\beta{}''(b_1 \cap \beta)$ is counbounded in κ;

(v') in case (b), we want: $\varphi(\alpha^*) < \beta \Rightarrow g^3_\beta(\alpha^*) \neq g^2_\beta(\varphi(\alpha^*))$;

(vi') also in case (b), we want: $\alpha^* < \beta \Rightarrow g^3_\beta(\alpha^*) \in B^2(\varphi(\alpha^*), \beta, \xi^*)$.

Like before, we enumerate $b_1 \cap \beta := \{\gamma(\xi) : \xi \in \kappa\}$ and separately enumerate $b_1 := \{\alpha(\xi) : \xi \in \kappa\}$ like previously. The difference is that this time, in the case b, if $\alpha^* < \beta$, we fix $\gamma(0) := \alpha^*$. In this case, we begin the induction by choosing x_0 (like before, let $x_0 = g^2_\beta(\varphi(\alpha^*))$ if $\varphi(\alpha^*) < \beta$, otherwise let x_0 be arbitrary), then choose $g^3_\beta(\gamma(0)) \in B^2(\varphi(\alpha^*), \beta, \xi^*) \setminus \{x_0\}$, if $\alpha^* < \beta$. Thus, item (vi') holds. The rest of the induction is dual to the previous case.

We now shall prove that such g^3_β adequate. The first step towards this end is proving the following:

Claim 4.12. *The function g^3_β constructed above is a good coloring of $\langle a^3 \cap \beta, e^3 \cap [\beta]^2\rangle$.*

Proof of claim 4.12. Suppose $\alpha < \gamma < \beta$ and $\{\alpha, \gamma\} \in e^3$. We divide the proof in the following cases:

- If $\{\gamma, \alpha, \beta\} \subset a^i$, for $i = 1, 2$, we have $g^3_\beta(\alpha) = g^i_\beta(\alpha) \neq g^i_\beta(\gamma) = g^3_\beta(\gamma)$, by either (i) or (i');
- If $\gamma \in \Delta$, since Δ is an initial segment of a^1 and of a^2 (by lemma 4.9), we have $\alpha \in \Delta$. Therefore we fall in the previous case;
- If $\beta \in a^1$ and $\alpha, \gamma \in b_2$ (or dually $\beta \in a^2$ and $\alpha, \gamma \in b_1$), we have that $g^3_\beta(\alpha) \neq g^3_\beta(\gamma)$ by item (ii) (or dually by item (ii'));
- if $\beta \in a^1$, $\alpha \in \Delta$ and $\gamma \in b_2$, we have $\{\alpha, \gamma\} \in e^2$, so by (6) we have $g^2_{\varphi(\beta)}(\alpha) \in A^2(\gamma, \varphi(\beta))$. Also, since $\alpha \in \Delta$, we have $g^2_{\varphi(\beta)}(\alpha) = g^1_\beta(\alpha) = g^3_\beta(\alpha)$, so $g^3_\beta(\alpha) \in A^2(\gamma, \varphi(\beta))$. But by (iii) we have $g^3_\beta(\gamma) \notin A^2(\gamma, \varphi(\beta))$, hence $g^3_\beta(\alpha) \neq g^3_\beta(\gamma)$
- if $\beta \in a^2$, $\alpha \in \Delta$ and $\gamma \in b_1$, the proof is dual to the above one;

These are all the possible cases for case (a). For case (b), we have some extra possibilities:

- If $\beta \in a^2$, $\alpha = \alpha^*$ and $\gamma \in a^1$, by (4.2.1) we have $\alpha^* \in b_1$, so $\gamma \in b_1$. Once again we have $g^3_\beta(\alpha) \neq g^3_\beta(\gamma)$ by item (ii');
- If $\beta \in a^2$, $\alpha \in \Delta$ and $\gamma = \alpha^*$, we have $\{\alpha, \varphi(\alpha^*)\} \in e^2$. By (4.2.1), we have $\Delta \subset f_{\xi^*}(\varphi(\alpha^*))$, so $\alpha < f_{\xi^*}(\varphi(\alpha^*))$. By (7), we have $g^2_\beta(\alpha) \notin B^2(\varphi(\alpha^*), \beta, \xi^*)$. But by item (vi'), we have $g^3_\beta(\alpha^*) \in B^2(\varphi(\alpha^*), \beta, \xi^*)$, so $g^3_\beta(\alpha) = g^2_\beta(\alpha) \neq g^3_\beta(\alpha^*)$;
- If $\{\alpha, \gamma\} = \{\alpha^*, \varphi(\alpha^*)\}$, it is clear by either item (v) or (v');

□

Now we need to check that the g_β we constructed is compatible with the sets $A^3(\alpha, \beta)$, $B^3(\alpha, \beta, \xi)$ already constructed.

Claim 4.13. *Conditions (6) and (7) holds for all* $\langle \alpha, \beta \rangle \in (a^1)^2 \cup (a^2)^2$.

Proof of claim 4.13. Suppose $\gamma < \alpha, \beta$ and $\{\alpha, \gamma\} \in e^3$. In case (a) by symmetry we can assume w.l.o.g. that $\beta \in b_1$. Thus we have $\alpha \in a^1$. We have the following cases:

- If $\{\gamma, \alpha, \beta\} \subset a^i$, for $i = 1, 2$, we have $g^3_\beta(\alpha) = g^i_\beta(\alpha)$ and $g^3_\beta(\gamma) = g^i_\beta(\gamma)$, by either (i) or (i'), so we are done because (6) and (7) holds for p^1 and p^2;
- If $\alpha, \beta \in a^1$ and $\{\gamma, \alpha\} \in e^2$, we have $\alpha \in \Delta$. Since Δ is an initial segment of a^1, we have that $\gamma \in a^1$, hence we fall in the previous case;
- the case $\alpha, \beta \in a^2$ and $\{\gamma, \alpha\} \in e^1$ is the dual of the previous case.

In the case (b), we have also the following possibilities:

- If $\beta \in a^2$, $\gamma = \alpha^*$ and $\alpha = \varphi(\alpha)$, we have $g^3_\beta(\alpha^*) \in B^3(\varphi(\alpha^*), \beta, \xi^*) \subset B^3(\varphi(\alpha^*), \beta)$ by item (vi'). Since, by (4.2.1), $f_{\xi^*}(\varphi(\alpha^*)) \leq \alpha^*$ and the B^3 sets are pairwise disjoint, (6) and (7) are satisfied;
- If $\beta \in a^1$, $\gamma = \alpha^*$ and $\alpha = \varphi(\alpha^*)$ we have $\langle \alpha, \beta \rangle \notin (a^1)^2 \cup (a^2)^2$, so $A^3(\alpha, \beta)$, $B^3(\alpha, \beta, \xi)$ have not been constructed yet;
- if $\beta \in a^1$ and $\varphi(\alpha^*) \in \{\alpha, \gamma\} \in e^2$, we have $\alpha \in b_2$ by (4.2.1) . Like above we have $\langle \alpha, \beta \rangle \notin (a^1)^2 \cup (a^2)^2$;
- the case where $\beta \in a^2$ and $\alpha^* \in \{\alpha, \gamma\} \in e^1$ is dual to the previous case.

□

The only remaining part is to show that (4.1.1) holds so we can apply proposition 4.5 and finish the construction of p^3.

Claim 4.14. *Let* $X = (a^1)^2 \cup (a^2)^2$. *Then condition (4.1.1) holds for all* $\langle \alpha, \beta \rangle \in (a^3)^2 \backslash X$.

Proof of claim 4.14. It is enough to prove for the case (a), since case (b) adds only one extra edge. By symmetry, w.l.o.g we assume $\beta \in a^1$ and $\alpha \in b_2$. Let $\gamma < \alpha, \beta$ be such that $\{\gamma, \alpha\} \in e^3$. Since $\alpha \in b_2$, we have $\gamma \in a^2$, we have either $\gamma \in \Delta$ or $\gamma \in b^2$. If $\gamma \in \Delta$, we have $g^3_\beta(\gamma) = g^2_\beta(\gamma) \in A^2(\alpha, \varphi(\gamma))$ by (i) and (6). Therefore

$$\{g^3_\beta(\gamma) \,:\, \{\gamma, \alpha\} \in e^3, \gamma < \alpha, \beta\} \subset A^2(\alpha, \varphi(\beta)) \cup g^3_\beta{}''(b_2 \cap \beta)$$

By (iv), the set above is counbounded in κ. □

□

4.3 Proof of the generalization

Now we can check what kind of chain condition $\mathbb{B}(\kappa)$ satisfies

Lemma 4.15. *If $2^\kappa = \kappa^+$, then $\mathbb{B}(\kappa)$ is κ^{++}-Knaster.*

Proof. Let $A \in [\mathbb{B}(\kappa)]^{\kappa^{++}}$. Since the set of closed conditions is dense, we may assume w.l.o.g. that all the conditions in A are closed. Assuming $2^\kappa = \kappa^+$, there are only κ^+ many possible isomorphism types, there is $A' \in [A]^{\kappa^{++}}$ which is pairwise isomorphic.

By lemma 4.11a, we have that A' is pairwise compatible. □

So by propositon 4.4 combined with lemma 4.15, we have that (under, for example, GCH) $\mathbb{B}(\kappa)$ preserves cardinals.

Let $\underset{\sim}{E}$ be a $\mathbb{B}(\kappa)$-name such that $\Vdash_{\mathbb{B}\kappa}$ "$\underset{\sim}{E} = \bigcup\{e^p : p \in \underset{\sim}{G}_{\mathbb{B}(\kappa)}\}$". By the construction of the side conditions g_β and the assumption that λ is regular, it is easy to see that:

Lemma 4.16. *Assuming $2^\kappa = \kappa^+$, we have*

$$\Vdash \text{" If } \mathcal{G} \text{ is a subgraph of } \langle \check{W}, \underset{\sim}{E} \rangle \text{ of size} \leq \kappa^+ \text{, then } \mathrm{Chr}(\mathcal{G}) \leq \kappa \text{"}$$

Finally, we use lemma 4.11 to prove the last piece of the result.

Lemma 4.17. *Assuming $2^\kappa = \kappa^+$, we have*

$$\Vdash \text{" } \langle \check{W}, \underset{\sim}{E} \rangle \text{ has chromatic number } \kappa^{++} \text{"}$$

Proof. Let $p \in \mathbb{B}(\kappa)$ be such that $p \Vdash$ "$\underset{\sim}{g} : \check{W} \longrightarrow \kappa^+$". We shall find a condition $q \leq p$ which forces $\underset{\sim}{g}$ not to be a good coloring of $\langle \check{W}, \underset{\sim}{E} \rangle$.

For each $\alpha \in W$, we can choose some closed condition $p^\alpha \leq p$, with $\alpha \in a^\alpha$, and some $\zeta_\alpha < \kappa^+$ such that $p^\alpha \Vdash$ "$\underset{\sim}{g}(\alpha) = \zeta_\alpha$". Assuming $2^\kappa = \kappa^+$, there are only κ^+ many isomorphism types, since W is stationary in κ^{++}, there is a stationary $S \subset W$ such that:

- $\{p^\alpha : \alpha \in S\}$ is pairwise isomorphic;
- $\exists \zeta < \kappa^+, \forall \alpha \in S, \zeta_\alpha = \zeta$;
- for each $\alpha, \beta \in S$, if $\varphi : \mathrm{scl}(a^\alpha) \longrightarrow \mathrm{scl}(a^\beta)$ is an isomorphism, then $\varphi(\alpha) = \beta$.

Consider all the functions f_ξ restricted to S. Since they are regressive and S is stationary, for each $\xi \in \kappa$ there is some $\alpha_\xi \in \lambda$ such that $f_\xi^{-1}(\{\alpha_\xi\})$ is stationary. Notice that we cannot for all $\xi < \kappa$ choose such an α_ξ uniquely. The reason for it is that if we could do so, it would be possible to construct a nonstationary set N such that $\forall \alpha \in S \setminus N, \forall \xi < \kappa, f_\xi(\alpha) = \alpha_\xi$, which contradicts the definition of f_ξ. Therefore, there is some $\xi^* < \kappa$ and distinct $\gamma, \gamma' < \kappa^+$ such that $f_{\xi^*}^{-1}(\{\gamma\})$ and $f_{\xi^*}^{-1}(\{\gamma'\})$ are stationary subsets of S.

Choose $\alpha^* \in f_{\xi^*}^{-1}(\{\gamma\})$ and $\beta^* \in f_{\xi^*}^{-1}(\{\gamma'\})$ such that $\alpha^* < \beta^*$. Fix an isomorphism $\varphi : \mathrm{scl}(a^{\alpha^*}) \longrightarrow \mathrm{scl}(a^{\beta^*})$. Notice that $\gamma \in \mathrm{dom}(\varphi)$ because p^{α^*} is a closed condition. Recall that, by lemma 4.9, $\mathrm{scl}(a^{\alpha^*}) \cap \mathrm{scl}(a^{\beta^*})$ is an initial segment of both $\mathrm{scl}(a^{\alpha^*})$ and $\mathrm{scl}(a^{\beta^*})$, thus $\varphi\restriction \mathrm{scl}(a^{\alpha^*}) \cap \mathrm{scl}(a^{\beta^*})$ is the identity function. Since $\varphi(\alpha^*) = \beta^*$ and φ preserves f_{ξ^*} (definition 4.10), we have

$$\varphi(\gamma) = \varphi(f_{\xi^*}(\alpha^*)) = f_{\xi^*}(\varphi(\alpha^*)) = f_{\xi^*}(\beta^*) = \gamma' \neq \gamma$$

hence $\gamma \notin \mathrm{scl}(a^{\alpha^*}) \cap \mathrm{scl}(a^{\beta^*})$. Therefore, since $\mathrm{scl}(a^{\alpha^*}) \cap \mathrm{scl}(a^{\beta^*})$ is an initial segment of $\mathrm{scl}(a^{\alpha^*})$, we have

$$a^{\alpha^*} \cap a^{\beta^*} \subset \gamma = f_{\xi^*}(\alpha^*) < \alpha^* < \beta^* = \varphi(\alpha^*)$$

Therefore, p^{α^*} and p^{β^*} satisfy (4.2.1), so by lemma 4.11, there is a condition $q \leq p^{\alpha^*}, p^{\beta^*}$ such that $\{\alpha^*, \beta^*\} \in e^q$. Thus

$$q \Vdash \text{“} \underset{\sim}{g}(\alpha^*) = \underset{\sim}{g}(\beta^*) \text{ and } \{\alpha^*, \beta^*\} \in \underset{\sim}{E} \text{”}$$

so q is the desired condition. $\square$

This concludes the proof of theorem 4.1. However, the proof of theorem 4.1 arose the question of whether it is possible to further generalize the result, i.e.:

Question 2. *For any regular cardinals κ and λ, with $\lambda > \kappa^+$, is it consistent the existence of a graph of size and chromatic number λ such that all subgraphs of size $\leq \kappa^+$ have chromatic number $\leq \kappa$?*

References

[Bau84] James E. Baumgartner. Generic graph construction. *J. Symbolic Logic*, 49(1):234–240, 1984.

[Cox15] S. D. Cox. Layered posets and Kunen's universal collapse. *ArXiv e-prints*, May 2015.

[FL88] Matthew Foreman and Richard Laver. Some downwards transfer properties for $\aleph_2$. *Adv. in Math.*, 67(2):230–238, 1988.

[Kun78] Kenneth Kunen. Saturated ideals. *J. Symbolic Logic*, 43(1):65–76, 1978.

[Kun11] Kenneth Kunen. *Set theory.* London: College Publications, 2011.

[L̈16] Philipp Lücke. The infinite productivity of knaster properties. Talk at Logic Colloquium 2016, Oct. 2 2016.