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ON HOROWITZ AND SHELAH'S BOREL MAXIMAL EVENTUALLY DIFFERENT FAMILY

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Abstract. We give an exposition of Horowitz and Shelah's proof that there exists an effectively Borel maximal eventually different family (working in ZF or less) and announce two related theorems.

1. Introduction

A. Two functions f_0, f_1 on $\mathbb{N}$ are called *eventually different* if and only if

$$\{n \in \mathbb{N} \mid f_0(n) = f_1(n)\}$$

is finite. A set $\mathcal{E}$ is called an *eventually different family* (of functions from $\mathbb{N}$ to $\mathbb{N}$) if and only if $\mathcal{E} \subseteq {}^{\mathbb{N}}\mathbb{N}$ and any two distinct $f_0, f_1 \in \mathcal{E}$ are eventually different; such a family is called *maximal* if and only if it is maximal with respect to inclusion among eventually different families (we abbreviate *maximal eventually different family* by *medf*).

In [2] Horowitz and Shelah prove the following (working in ZF).

Theorem 1.1 ([2]). *There is a Δ^1_1 maximal eventually different family.*

This was surprising as the analogous statement is false in many seemingly similar situations: e.g., infinite so-called *mad* families cannot be analytic [5] (see also [9]). In a more recent, related result [1] Horowitz and Shelah obtain a Δ^1_1 *maximal cofinitary group.*

In this note we present a short and elementary proof of their first result, i.e., that there is a Δ^1_1 maximal eventually different family. We also take the opportunity to announce the following improvement of Theorem 1.1:

Theorem 1.2 ([8, 7]). *There is a Π^0_1 maximal eventually different family.*

The following question was asked by Asger Törnquist [10]: For $F\colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$ such that $\liminf_{n\to\infty} F(n) = \infty$ does there exist a Borel or even a compact *medf* in the space $\mathcal{N}_F$? Here the space $\mathcal{N}_F$ is defined to be the closed subspace $\mathcal{N}_F = \{g \in {}^{\mathbb{N}}\mathbb{N} \mid (\forall n \in \mathbb{N})\, g(n) < F(n)\}$ of Baire space ${}^{\mathbb{N}}\mathbb{N}$.

To make the question entirely precise, we give the definition of (maximal) eventually different families a broader context:

Definition 1.3. Let a function $F\colon \mathbb{N} \to \omega + 1$ be given. A set $\mathcal{E}$ is an *eventually different family in $\mathcal{N}_F$* if and only if $\mathcal{E} \subseteq \mathcal{N}_F$ and any two distinct $g_0, g_1 \in \mathcal{E}$ are eventually different; such a family is called *maximal* (or short: a *medf*) if and only if it is maximal among such families under inclusion.

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We also announce the following results (see [7] for proofs).

Theorem 1.4 ([7]). *Suppose* $F\colon \mathbb{N} \to (\omega+1)\setminus\{0\}$ *is such that* $\lim_{n\to\infty} F(n) = \omega$. *There is a perfect* $\Pi^0_1(F)$ *maximal eventually different family in* $\mathcal{N}_F$.

Corollary 1.5 ([7]). *With* F *as above, there is a compact* $\Pi^0_1(F)$ medf *in* $\mathcal{N}_F$ *if and only if* $F(n) < \omega$ *for infinitely many* $n \in \mathbb{N}$. *Moreover we have two cases:*

(1) $\liminf_{n\to\infty} F(n) < \omega$. *In this case every* medf *is finite and there is a finite* medf *consisting of constant functions.*

(2) $\lim_{n\to\infty} F(n) = \omega$. *In this case there is a perfect* $\Pi^0_1(F)$ medf *but no countable* medf.

We ask $(\forall n \in \mathbb{N})\ F(n) > 0$ to preclude the trivial case of the empty space. When $\liminf_{n\to\infty} F(n) < \omega$ there is m such that $\{n \in \mathbb{N} \mid F(n) = m\}$ is infinite and the set $\{c_k \mid k < m\}$ where c_k is the constant function with value k constitutes a *medf.* So the question posed by Törnquist is only interesting if $\lim_{n\to\infty} F(n) = \omega$ holds.

Note: This note is an abridged version of [8] in which we show that there exists a Π^0_1 (i.e., effectively closed) maximal eventually different family. In the related [7] we present a further simplification of the construction, as well as provide an answer to Törnquist's question.

B. We fix some notation and terminology (generally, our reference for notation is [3]). '$\exists^\infty$' means 'there are infinitely many...,' ${}^{\mathbb{N}}\mathbb{N}$ means the set of functions from $\mathbb{N}$ to $\mathbb{N}$ and ${}^{<\mathbb{N}}\mathbb{N}$ means the set of finite sequences from $\mathbb{N}$; we write $\mathrm{lh}(s)$ for the length of s when $s \in {}^{<\mathbb{N}}\mathbb{N}$. For $s, t \in {}^{n}\mathbb{N}$, $s ^\frown t$ is the *concatenation* of s and t, i.e., the unique $u \in {}^{\mathrm{lh}(s)+\mathrm{lh}(t)}\mathbb{N}$ such that $s \subseteq u$ and $(\forall k < \mathrm{lh}(t))\ u(\mathrm{lh}(s)+k) = t(k)$.

We write $f_0 =^\infty f_1$ to mean that f_0 and f_1 are *not* eventually different (they are infinitely equal). Two sets $A, B \subseteq \mathbb{N}$ are called *almost disjoint* if and only if $A \cap B$ is finite, and an *almost disjoint family* is a set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ any two elements of which are almost disjoint.

It makes sense to talk about $\Delta^0_1(F)$ for $F\colon \omega \to \omega+1$ as above because we may identify F in an obvious way with a subset of $\mathbf{H}(\omega)$ (the set of hereditarily finite sets). Consult [6, 4, 3] for more on the (effective) Borel and projective hierarchies, i.e., on Π^0_1, $\Pi^0_1(F)$, Δ^1_1, ... sets.

In this paper we work in the theory ZF (or in fact, in a not so strong subsystem of second order arithmetic).

C. This note is organized as follows. In Section 2 we make some motivating observations, leading to Lemma 2.5 which gives an abstract recipe for creating maximal eventually different families. We take the opportunity to give a rough sketch of the proof of Theorem 1.1 as given in [2].

We then give a simpler construction instantiating the recipe from Lemma 2.5 and yielding a *medf* which is $\Sigma^0_3 \vee \Pi^0_3$ in Section 3.

We close in Section 4 with some open questions.

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2. The recipe

Definition 2.1. Fix a computable (i.e., Δ^0_1) bijection $n \mapsto s_n$ of $\mathbb{N}$ with ${}^{<\mathbb{N}}\mathbb{N}$ and write $s \mapsto \# s$ for its inverse. Given $f\colon \mathbb{N} \to \mathbb{N}$, let $e(f)\colon \mathbb{N} \to \mathbb{N}$ be the function defined by

$$\mathrm{e}(f)(n) = \# f \restriction n.$$

Clearly $\{\mathrm{e}(f) \mid f \in {}^{\mathbb{N}}\mathbb{N}\}$ is an eventually different family. At first sight, it may seem a naive strategy to make it also maximal by varying the definition of $\mathrm{e}(f)$ so that it leaves f intact on some infinite set. But this is just how [2] succeeds.

Definition 2.2. Let $f\colon \mathbb{N} \to \mathbb{N}$.

A. Let $\mathrm{B}(f) = \{2n+1 \mid s_n \subseteq f\}$.
B. For a set $B \subseteq \mathbb{N}$, let $\ddot{\mathrm{e}}(f, B)\colon \mathbb{N} \to \mathbb{N}$ be the function defined by

$$\ddot{\mathrm{e}}(f,B)(n) = \begin{cases} f(n) & \text{if } n \in B, \\ \# f \restriction n & \text{if } n \notin B. \end{cases}$$

Remark 2.3. Note for later that f is recursive in $\ddot{\mathrm{e}}(f, \mathrm{B}(f))$ as $\ddot{\mathrm{e}}(f, \mathrm{B}(f)) \restriction 2\mathbb{N} = \mathrm{e}(f) \restriction 2\mathbb{N}$.

The family $\mathcal{E}_0 = \{\ddot{\mathrm{e}}(f, \mathrm{B}(f)) \mid f \in {}^{\mathbb{N}}\mathbb{N}\}$ is obviously maximal in the sense that $(\forall h \in {}^{\mathbb{N}}\mathbb{N})(\exists g \in \mathcal{F})\ h =^\infty g$. Interestingly, $\mathcal{E}_0$ is also in some sense close to being eventually different: For if $\ddot{\mathrm{e}}(f, \mathrm{B}(f))(n) = \ddot{\mathrm{e}}(f', \mathrm{B}(f'))(n)$ for infinitely many n, almost all of these n must lie in $\mathrm{B}(f) \cup \mathrm{B}(f')$ and hence as $\{\mathrm{B}(f) \mid f \in {}^{\mathbb{N}}\mathbb{N}\}$ is an almost disjoint family,

$$(\exists^\infty n \in \mathrm{B}(f))\ f(n) = \mathrm{e}(f')(n)$$

or the same holds with f and f' switched.

The brilliant idea of Horowitz and Shelah is the following: Ensure maximality with respect to f which look like $\mathrm{e}(f')$ on an infinite set using $\mathrm{e}(f')$; restrict the use of $\ddot{\mathrm{e}}$ to f which *don't* look like they arise from e on some infinite subset of $\mathrm{B}(f)$ to avoid the situation described above. We make these ideas precise in the following definition and in Lemma 2.5 below.

Definition 2.4. Let a function $f\colon \mathbb{N} \to \mathbb{N}$ and $X \subseteq \mathbb{N}$ be given. We say f *is ∞-coherent on* X if and only if there is $f' \in {}^{\mathbb{N}}\mathbb{N}$ and infinite $X' \subseteq X$ such that $f \restriction X' = e(f') \restriction X'$.

We can now give a general recipe for constructing a *medf.*

Lemma 2.5. *Suppose that* $\mathcal{T} \subseteq {}^{\mathbb{N}}\mathbb{N}$ *and* $\mathrm{C}\colon {}^{\mathbb{N}}\mathbb{N} \to \mathcal{P}(\mathbb{N})$ *is a function such that*

(A) If $f \notin \mathcal{T}$, there is an infinite set $X' \subseteq \mathrm{C}(f)$ and $f' \in {}^{\mathbb{N}}\mathbb{N}$ such that $f \restriction X' = \mathrm{e}(f') \restriction X'$; i.e., f is ∞-coherent on $\mathrm{C}(f)$.
(B) If $f \in \mathcal{T}$, for no $f' \in {}^{\mathbb{N}}\mathbb{N}$ does f agree with $e(f')$ on infinitely many points in $\mathrm{C}(f)$; i.e., f is not ∞-coherent on $\mathrm{C}(f)$.
(C) $\{\mathrm{C}(f) \mid f \in \mathcal{T}\}$ is an almost disjoint family.

Then

$$\mathcal{E} = \{\ddot{\mathrm{e}}(f, \mathrm{C}(f)) \mid f \in \mathcal{T}\} \cup \{\mathrm{e}(f) \mid f \notin \mathcal{T}\}$$

is a maximal eventually different family.

Of course the challenge here is to define C and $\mathcal{T}$ so that $\mathcal{E}$ is Δ^1_1; before we discuss this aspect, we prove the lemma.

For the sake of this proof it will be convenient to define the map $\dot{\mathrm{e}}\colon {}^{\mathbb{N}}\mathbb{N} \to {}^{\mathbb{N}}\mathbb{N}$ as follows: For $f \in {}^{\mathbb{N}}\mathbb{N}$ let $\dot{\mathrm{e}}(f)$ be the function defined by

$$\dot{\mathrm{e}}(f) = \begin{cases} \ddot{\mathrm{e}}(f, \mathrm{C}(f)) & \text{if } f \in \mathcal{T}, \\ \mathrm{e}(f) & \text{otherwise.} \end{cases} \tag{1}$$

Clearly $\mathcal{E} = \{\dot{\mathrm{e}}(f) \mid f \in {}^{\mathbb{N}}\mathbb{N}\}$.

Proof of Lemma 2.5. To show $\mathcal{E}$ consists of pairwise eventually different functions, fix distinct g_0 and g_1 from $\mathcal{E}$ and suppose $g_i = \dot{\mathrm{e}}(f_i)$ for each $i \in \{0,1\}$. Clearly we can disregard

$$N = \{n \in \mathbb{N} \mid g_0(n) = \mathrm{e}(f_0)(n) \text{ and } g_1(n) = \mathrm{e}(f_1)(n)\}$$

as g_0 and g_1 can only agree on finitely many such n.

If $n \notin N$ then it must be the case that for some $i \in \{0,1\}$, $f_i \in \mathcal{T}$ and $n \in \mathrm{C}(f_i)$; suppose $i = 0$ for simplicity. By (C) we may restrict our attention to $\mathrm{C}(f_0) \setminus \mathrm{C}(f_1)$ where g_0 agrees with f_0 and g_1 agrees with $\mathrm{e}(f_1)$. But f_0 and $\mathrm{e}(f_1)$ can't agree on an infinite subset of $\mathrm{C}(f_0) \setminus \mathrm{C}(f_1)$ by (B).

It remains to show maximality. So let $f\colon \mathbb{N} \to \mathbb{N}$ be given. If $f \in \mathcal{T}$ we have $\dot{\mathrm{e}}(f) \restriction \mathrm{C}(f) = f \restriction \mathrm{C}(f)$ and $\dot{\mathrm{e}}(f) \in \mathcal{E}$ by definition.

If on the other hand $f \notin \mathcal{T}$ there is $f' \in {}^{\mathbb{N}}\mathbb{N}$ such that $\mathrm{e}(f')$ agrees with f on an infinite subset of $\mathrm{C}(f)$. As $\dot{\mathrm{e}}(f') \in \mathcal{E}$ it suffices to show $f =^\infty \dot{\mathrm{e}}(f')$.

If $f' \notin \mathcal{T}$ as well this is clear as $\dot{\mathrm{e}}(f') = \mathrm{e}(f')$. If on the contrary $f' \in \mathcal{T}$, we have $f \neq f'$ and so $\mathrm{C}(f) \cap \mathrm{C}(f')$ is finite by (C). So $\dot{\mathrm{e}}(f')$ agrees with $\mathrm{e}(f')$ for all but finitely many points in $\mathrm{C}(f)$ and hence agrees with f on infinitely many points. □

Note that letting $\mathcal{T} = \{f \in {}^{\mathbb{N}}\mathbb{N} \mid f \text{ is not } \infty\text{-coherent on } \mathrm{B}(f)\}$ and $\mathrm{C}(f) = \mathrm{B}(f)$ the requirements of Lemma 2.5 are trivially satisfied; but the resulting $\mathcal{E}$ will not be Borel (only $\Pi^1_1 \vee \Sigma^1_1$). On the other hand if $\mathcal{T}$ is Δ^1_1 and $\mathrm{C}\colon {}^{\mathbb{N}}\mathbb{N} \to \mathcal{P}(\mathbb{N})$ is Σ^1_1, then $\mathcal{E}$ is clearly Σ^1_1, and in fact it follows that $\mathcal{E}$ is Δ^1_1 in this case because it is a *medf* and so

$$h \notin \mathcal{E} \iff (\exists g \in {}^{\mathbb{N}}\mathbb{N})\, h \neq g \wedge h =^\infty g \wedge g \in \mathcal{E}.$$

(Of course the function $\mathrm{C}\colon {}^{\mathbb{N}}\mathbb{N} \to \mathcal{P}(\mathbb{N})$ is also automatically Δ^1_1.) We may view the task at hand to be: find a reasonably effective process producing from a function f either a subset of $\mathrm{B}(f)$ where f agrees with some $\mathrm{e}(f')$ or a set $\mathrm{C}(f) \subseteq \mathrm{B}(f)$ on which f can be seen effectively to not be ∞-coherent.

From this we can sketch what is arguably the core of Horowitz and Shelah's construction from [2]. The present author has not verified whether their construction yields an arithmetic family.

Proof of Theorem 1.1. Given $f\colon \mathbb{N} \to \mathbb{N}$ define a coloring of unordered pairs from $\mathbb{N}$ as follows (supposing without loss of generality that $k < k'$):

$$c(\{k, k'\}) = \begin{cases} 0 & \text{if } s_{f(k)} \subsetneq s_{f(k')}, \\ 1 & \text{if } \neg\big(s_{f(k)} \subsetneq s_{f(k')}\big). \end{cases}$$

Let $\mathcal{T}$ consist of those $f \in {}^{\mathbb{N}}\mathbb{N}$ such that there is an infinite set $X \subseteq \mathrm{B}(f)$ which is 1-homogeneous, i.e., c assigns the color 1 to every unordered pair from X. By the Infinite Ramsey Theorem $f \notin \mathcal{T}$ if and only if there is an infinite 0-homogeneous $X \subseteq \mathrm{B}(f)$, whence $\mathcal{T}$ is Δ^1_1 and (A) holds. For $f \in \mathcal{T}$ let $\mathrm{C}(f)$ pick some infinite

1-homogeneous $X \subseteq \mathrm{B}(f)$; for $f \notin \mathcal{T}$ let $\mathrm{C}(f) = \mathrm{B}(f)$. Then (B) and (C) hold by definition and by Lemma 2.5, $\mathcal{E}$ is a *medf.*

By the proof of the Infinite Ramsey Theorem, the function $\mathrm{C}\colon {}^{\mathbb{N}}\mathbb{N} \to \mathcal{P}(\mathrm{B}(f))$ can be chosen to be Σ^1_1. Thus $\mathcal{E}$ as defined in Lemma 2.5 is Δ^1_1. □

In the next section, we essentially replace the appeal to the Infinite Ramsey Theorem by a simple instance of the law of excluded middle.

3. A maximal eventually different family with a simple definition

We now give a simpler construction of a family satisfying the requirements of Lemma 2.5.

Definition 3.1 (The *medf* $\mathcal{E}$).

A. Let $f\colon \mathbb{N} \to \mathbb{N}$. Define a binary relation $\prec^f$ on $\mathbb{N}$ by

$$m \prec^f m' \iff \Big[\big(\mathrm{lh}(s_{f(m)}) = m\big) \wedge \big(\mathrm{lh}(s_{f(m')}) = m'\big) \wedge \big(s_{f(m)} \subsetneq s_{f(m')}\big)\Big]$$

B. Let $\mathcal{T}$ be the set of $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$(\forall n \in \mathrm{B}(f))(\exists m \in \mathrm{B}(f) \setminus n)(\forall m' \in \mathrm{B}(f) \setminus m)\, \neg(m \prec^f m') \tag{2}$$

We also say f *is tangled* to mean $f \in \mathcal{T}$.

C. For $f \notin \mathcal{T}$, define $\mathrm{C}(f)$ to be $\mathrm{B}(f)$ and for $f \in \mathcal{T}$ define

$$\mathrm{C}(f) = \{m \in \mathrm{B}(f) \mid (\forall m' \in \mathrm{B}(f) \setminus m)\, \neg(m \prec^f m')\}.$$

D. Let $\mathcal{E}$ be defined from $\mathcal{T}$ and C as in Lemma 2.5, i.e.,

$$\mathcal{E} = \{\, \dot{\mathrm{e}}(f) \mid f \in {}^{\mathbb{N}}\mathbb{N}\}$$

where $\dot{\mathrm{e}}(f)$ is the function defined as in (1):

$$\dot{\mathrm{e}}(f) = \begin{cases} \ddot{\mathrm{e}}(f, \mathrm{C}(f)) & \text{if } f \in \mathcal{T}, \\ \mathrm{e}(f) & \text{otherwise.} \end{cases}$$

We want to call the following to the readers attention:

(i) $\{\mathrm{C}(f) \mid f \in {}^{\mathbb{N}}\mathbb{N}\}$ is an almost disjoint family (as $\mathrm{C}(f) \subseteq \mathrm{B}(f)$ by definition).

(ii) When f is tangled, $\mathrm{C}(f)$ is an infinite set by (2) and for no $f' \in {}^{\mathbb{N}}\mathbb{N}$ does f agree with $\mathrm{e}(f')$ on infinitely many (or in fact, just two) points in $\mathrm{C}(f)$—i.e., f is not ∞-coherent on $\mathrm{C}(f)$.

Lemma 3.2. *The set $\mathcal{E}$ is a maximal eventually different family.*

Proof. We show that Lemma 2.5 can be applied. Requirements (C) and (B) hold by (i) and (ii) above. For (A), suppose f is not tangled, i.e.,

$$(\exists n \in \mathrm{B}(f))(\forall m \in \mathrm{B}(f) \setminus n)(\exists m' \in \mathrm{B}(f) \setminus m)\; m \prec^f m'.$$

Let m_0 be the least witness to the leading existential quantifier above; by recursion let m_{i+1} be the least m' in $\mathrm{B}(f)$ above m_i such that $m_i \prec^f m'$. Letting $f' = \bigcup\{s_{f(m_i)} \mid i \in \mathbb{N}\}$ yields a well-defined function in ${}^{\mathbb{N}}\mathbb{N}$ such that $f =^\infty \mathrm{e}(f')$, i.e., f is ∞-coherent on $\mathrm{C}(f)$. □

It is obvious that $\mathcal{E}$ is Δ^1_1. We now show a stronger result.

Lemma 3.3. *The set $\mathcal{E}$ is in the Boolean algebra generated by the Σ^0_3 sets in $\mathbb{N}^{\mathbb{N}}$.*

Proof. By construction $g \in \mathcal{E}$ if and only if the following holds of g (see Remark 2.3):

(I) $(\forall n \in \mathbb{N})\ \mathrm{lh}(s_{g(2n)}) = 2n$, and
(II) $(\forall n \in \mathbb{N})(\forall m \leq n)\ s_{g(2m)} \subseteq s_{g(2n)}$, and letting $f = \bigcup_{n\in 2\mathbb{N}} s_{g(2n)}$,
(III) either the following three requirements hold:
   (a) f is tangled and
   (b) $(\forall n \in \mathbb{N})\ n \in \mathrm{C}(f) \Rightarrow g(n) = f(n)$ and
   (c) $(\forall n \in \mathbb{N})\ n \notin \mathrm{C}(f) \Rightarrow g(n) = \mathrm{e}(f)(n)$;
(IV) or both of the following hold:
   (a) f is not tangled and
   (b) $(\forall n \in \mathbb{N})\ g(n) = \mathrm{e}(f)(n)$.

As $\mathrm{C}(f)$ is $\Pi^0_1(f)$ for $f \in \mathcal{T}$ and (IIIa) is $\Pi^0_3(f)$, clearly (III) is $\Pi^0_3(g, f)$. Likewise (IV) is $\Sigma^0_3(g, f)$. As f is recursive in g, (III) can be expressed by a $\Pi^0_3(g)$ formula and (IV) can be expressed by a $\Sigma^0_3(g)$ formula (substitute each expression of the form $f(n) = m$ by $s_{g(2n+2)}(n) = m$ and $f \restriction n$ by $s_{g(2n)} \restriction n$). ☐

4. Questions

1. Is it the case that for some $F\colon \mathbb{N} \to \mathbb{N}$ there is a compact $\Pi^0_1(F)$ maximal cofinitary group in $\mathcal{N}_F$?
2. For which F is the answer to the previous question 'yes' (if any)?
3. Is there a natural, minimal fragment of second order arithmetic which proves there is a Π^0_1 eventually different family (or, respectively, a Π^0_1 maximal cofinitary group)?
4. For any set X let $X^{[\infty]}$ denote the set of infinite subsets of X. Given any $F\colon \mathbb{N} \to \{\mathbb{N}\} \cup \mathbb{N}$ and a *medf* $\mathcal{E}$ on $\mathcal{N}_F$ consider the co-ideal
$$\mathcal{C}_{\mathcal{E}} = \{X \in \mathcal{P}(\mathbb{N}) \mid \{g \restriction X \mid g \in \mathcal{E}\} \text{ is a } \textit{medf} \text{ in } \prod_{n\in E} F(n)\}.$$
Is there a closed *medf* $\mathcal{E}$ in $\mathcal{N}$ or $\mathcal{N}_F$ (under some assumption on F) such that $\mathcal{C}_{\mathcal{E}} = \mathbb{N}^{[\infty]}$?

To make Question 4 precise, we make the following (slightly artificial) definition:

Definition 4.1. Any two functions g_0, g_1 with a countably infinite domain X are called *eventually different* if and only if $\{x \in X \mid g_0(x) = g_1(x)\}$ is finite.

Given $E \subseteq \omega$ and $F\colon E \to \omega + 1$, $\prod_{n\in E} F(n)$ of course means
$$\{f\colon E \to \mathbb{N} \mid (\forall n \in E)\ f(n) < F(n)\}.$$
A set $\mathcal{E}$ is an *eventually different family in* $\prod_{n\in E} F(n)$ if and only if $\mathcal{E} \subseteq \prod_{n\in E} F(n)$ and any two distinct $g_0, g_1 \in \mathcal{E}$ are eventually different; such a family is called *maximal* (or short: a *medf*) in $\prod_{n\in E} F(n)$ if and only if it is maximal among such families under inclusion.

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