数理解析研究所講究録
第2043巻 2017年 121-127

Asymptotic stability in a two-species chemotaxis-competition system

Masaaki Mizukami

Department of Mathematics
Tokyo University of Science

1. Introduction

We consider the two-species chemotaxis system

$$(1.1)\qquad \begin{cases} u_t = d_1\Delta u - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - \nabla\cdot(v\chi_2(w)\nabla w) + \mu_2 v(1-a_2u-v), & x\in\Omega,\ t>0,\\ w_t = d_3\Delta w + h(u,v,w), & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = \nabla w\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n\in\mathbb{N}$) with smooth boundary $\partial\Omega$ and ν is the outward normal vector to $\partial\Omega$. The initial data u_0, v_0 and w_0 are assumed to be nonnegative functions. The unknown functions $u(x,t)$ and $v(x,t)$ represent the population densities of two species and $w(x,t)$ shows the concentration of the substance at place x and time t.

The problem (1.1) consists of the influence of chemotaxis, diffusion, and the Lotka–Volterra kinetics. In mathematical view, global existence and behavior of solutions are fundamental theme. In the case $\chi_i(w)=\chi_i$ and $h(u,v,w)=\alpha u+\beta v-\gamma w$, Bai–Winkler [**1**] considered asymptotic behavior of solutions to (1.1). When $a_1,a_2\in(0,1)$, they proved that the solution (u,v,w) satisfies $u(t)\to u^*$, $v(t)\to v^*$, $w(t)\to\frac{\alpha u^*+\beta v^*}{\gamma}$ in $L^\infty(\Omega)$ as $t\to\infty$, where $u^*=\frac{1-a_1}{1-a_1a_2}$ $v^*=\frac{1-a_2}{1-a_1a_2}$, under the conditions

$$(1.2)\qquad \mu_1 > \frac{d_2\chi_1^2u^*}{\frac{4a_1\gamma(1-a_1a_2)d_1d_2d_3}{(a_1\alpha^2+a_2\beta^2-2a_1a_2\alpha\beta)} - \frac{d_1a_1\chi_2^2v^*}{4\mu_2a_2}},\quad \mu_2 > \frac{\chi_2^2v^*(a_1\alpha^2+a_2\beta^2-2a_1a_2\alpha\beta)}{16d_2d_3a_2\gamma(1-a_1a_2)}.$$

These conditions are not natural because they are not symmetric.

The purpose of the present report is to improve the method in [**1**] for obtaining asymptotic stability of solutions to (1.1) under a more general and sharp condition for the sensitivity function $\chi_i(w)$. We shall suppose throughout this report that h, χ_i ($i=1,2$) satisfy the following conditions:

$$(1.3)\qquad \chi_i\in C^{1+\theta}([0,\infty))\cap L^1(0,\infty)\ (0<\exists\,\theta<1),\quad \chi_i>0\quad (i=1,2),$$

$$(1.4)\qquad h\in C^1([0,\infty)\times[0,\infty)\times[0,\infty)),\quad h(0,0,0)\ge 0,$$

$$(1.5)\qquad \exists\,\gamma>0;\ \frac{\partial h}{\partial u}(u,v,w)\ge 0,\quad \frac{\partial h}{\partial v}(u,v,w)\ge 0,\quad \frac{\partial h}{\partial w}(u,v,w)\le -\gamma,$$

$$(1.6)\qquad \exists\,\delta>0,\ \exists\,M>0;\ |h(u,v,w)+\delta w|\le M(u+v+1),$$

$$(1.7)\qquad \exists\,k_i>0;\ -\chi_i(w)h(0,0,w)\le k_i\quad (i=1,2).$$

We also assume that

(1.8)
$$\exists p > n;\ 2d_i d_3 \chi_i'(w) + \left((d_3 - d_i)p + \sqrt{(d_3-d_i)^2p^2 + 4d_i d_3 p}\right)[\chi_i(w)]^2 \le 0 \quad (i = 1, 2).$$

The above conditions cover the prototypical example $\chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}}$ $(K_i > 0,\ \sigma_i > 1)$, $h(u, v, w) = u + v - w$. We assume that the initial data u_0, v_0, w_0 satisfy

$$0 \le u_0 \in C(\overline{\Omega}) \setminus \{0\},\ 0 \le v_0 \in C(\overline{\Omega}) \setminus \{0\},\ 0 \le w_0 \in W^{1,q}(\Omega)\ (\exists q > n). \tag{1.9}$$

The following result which is concerned with global existence and boundedness in (1.1) was established in [**2**].

Theorem 1.1. *Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 \ge 0$. Assume that h, χ_1, χ_2 satisfy* (1.3)–(1.8). *Then for any u_0, v_0, w_0 satisfying* (1.9) *for some $q > n$, there exists an exactly one pair (u, v, w) of nonnegative functions*

$$u,\ v,\ w \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),$$

which satisfy (1.1). *Moreover, the solutions u, v, w are uniformly bounded, i.e., there exists a constant $C_1 > 0$ such that*

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \le C_1 \quad \text{for all} t \ge 0,$$

and the solutions u, v, w are the Hölder continuous functions, i.e., there exist $\alpha \in (0, 1)$ and $C_2 > 0$ such that

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[1,t])} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[1,t])} + \|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[1,t])} \le C_2 \quad \text{for all } t \ge 1.$$

Since Theorem 1.1 guarantees that u, v and w exist globally and are bounded and nonnegative, it is possible to define nonnegative numbers α_1, α_2, β_1, β_2 by

$$\begin{aligned} \alpha_1 &:= \min_{(u,v,w)\in I} h_u(u, v, w), & \alpha_2 &:= \max_{(u,v,w)\in I} h_u(u, v, w), \\ \beta_1 &:= \min_{(u,v,w)\in I} h_v(u, v, w), & \beta_2 &:= \max_{(u,v,w)\in I} h_v(u, v, w), \end{aligned} \tag{1.10}$$

where $I = (0, C_1)^3$ and C_1 is defined in Theorem 1.1.

In the case $a_1, a_2 \in (0, 1)$ asymptotic behavior of solutions to (1.1) will be discussed under the following additional conditions: there exists $\delta_1 > 0$ such that

$$4\delta_1 - a_1 a_2 (1 + \delta_1)^2 > 0 \tag{1.11}$$

and

$$\mu_1 > \frac{\chi_1(0)^2 u^*(1+\delta_1)(\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1\beta_1 a_1 a_2(1+\delta_1))}{4a_1 d_1 d_3 \gamma (4\delta_1 - a_1 a_2(1+\delta_1)^2)}, \tag{1.12}$$

$$\mu_2 > \frac{\chi_2(0)^2 v^*(1+\delta_1)(\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1\beta_1 a_1 a_2(1+\delta_1))}{4a_2 d_2 d_3 \gamma (4\delta_1 - a_1 a_2(1+\delta_1)^2)}. \tag{1.13}$$

Now the main result reads as follows. The main theorem is concerned with asymptotic stability in (1.1) in the case $a_1, a_2 \in (0, 1)$.

Theorem 1.2. *Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$ and $a_1, a_2 \in (0,1)$. Under the conditions* (1.3)–(1.9) *and* (1.11)–(1.13), *the unique global solution (u, v, w) of* (1.1) *has the following asymptotic behavior*:

$$\|u(t) - u^*\|_{L^\infty(\Omega)} \to 0, \quad \|v(t) - v^*\|_{L^\infty(\Omega)} \to 0 \quad \|w(t) - w^*\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty).$$

where

$$u^* := \frac{1 - a_1}{1 - a_1 a_2}, \quad v^* := \frac{1 - a_2}{1 - a_1 a_2}$$

and $w^* \geq 0$ such that $h(u^*, v^*, w^*) = 0$.

Remark 1.1. Theorem 1.2 can be applied to the case $\chi_i(w) = \chi_i$ and $h(u, v, w) = \alpha u + \beta v - \gamma w$. Then the conditions (1.11)–(1.13) have symmetry and relax the condition (1.2) assumed in [**1**]. Indeed, the conditions (1.2) are stronger than (1.11)–(1.13) when $\delta_1 = 1$. Moreover, in view of considering the function

$$f(x) = \frac{a_1(\alpha^2 - \alpha\beta a_2)x^2 + (\beta^2 a_2 - \alpha^2 a_1)x}{-a_1 a_2 x^2 + 4x - 4}$$

(we put $x = 1 + \delta_1$), $x = 2$ ($\delta_1 = 1$) is not a minimizer of the right-hand sides of (1.12) and (1.13) except the case $\beta^2 a_2 = \alpha^2 a_1$. Thus the conditions (1.11)–(1.13) relax (1.2).

Remark 1.2. In Theorem 1.2 we can find $w^* \geq 0$ satisfying $h(u^*, v^*, w^*) = 0$. Indeed, from (1.4)–(1.6) for every $a, b \geq 0$ there exists $\overline{w}$ such that $h(a, b, \overline{w}) = 0$. Indeed, if we choose $w_1 \geq \frac{M(a+b+1)}{\delta}$, then (1.6) yields $h(a, b, w_1) \leq M(a+b+1) - \delta w_1 \leq 0$. On the other hand, (1.4) and (1.5) imply that $h(a, b, 0) \geq h(0, 0, 0) \geq 0$. Hence, by the intermediate value theorem there exists $\overline{w} \geq 0$ such that $h(a, b, \overline{w}) = 0$.

The strategy for the proof of Theorem 1.2 is to modify an argument in [**1**]. The key for this strategy is to construct the following energy estimate:

$$\frac{d}{dt}E(t) \leq -\varepsilon \left(\int_\Omega (u - \overline{u})^2 + \int_\Omega (v - \overline{v})^2 + \int_\Omega (w - \overline{w})^2 + \int_\Omega |\nabla w|^2 \right)$$

with some function $E(t) \geq 0$ and some $\varepsilon > 0$, where $(\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^3$ is a solution of (1.1). For finding the above inequality we apply more "suitable" estimates for

$$\int_\Omega \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w \qquad \text{and} \qquad \int_\Omega \frac{\chi_1(w)}{v} \nabla v \cdot \nabla w.$$

These enable us to improve the condition (1.2).

2. Proof of the main result

In this section we will establish asymptotic stability of solutions to (1.1) in the case $a_1, a_2 \in (0,1)$. For the proof of Theorem 1.2, we shall prepare some elementary results.

Lemma 2.1 (see [**1**, Lemma 3.1]). *Suppose $f : (1, \infty) \to \mathbb{R}$ is a uniformly continuous nonnegative function satisfying $\int_1^\infty f(t)\,dt < \infty$. Then $f(t) \to 0$ as $t \to \infty$.*

Lemma 2.2. *Let $a, b, c, d, e, f \in \mathbb{R}$. Suppose that*

$$a > 0, \quad d - \frac{b^2}{4a} > 0, \quad f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} > 0. \tag{2.1}$$

Then

$$ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \geq 0 \tag{2.2}$$

holds for all $x, y, z \in \mathbb{R}$.

Proof. From straightforward calculations we obtain

$$\begin{aligned} &ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \\ &= a\left(x + \frac{by + cz}{2a}\right)^2 + \left(d - \frac{b^2}{4a}\right)\left(y + \frac{2ae - bc}{4ad - b^2}\right)^2 + \left(f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)}\right)z^2. \end{aligned}$$

In view of the above equation, (2.1) leads to (2.2). □

Now we will prove the key estimate for the proof of Theorem 1.2.

Lemma 2.3. *Let $a_1, a_2 \in (0, 1)$ and (u, v, w) a solution to (1.1). Under the conditions (1.3)–(1.9) and (1.11)–(1.13), there exist $\delta_1, \delta_2 > 0$ and $\varepsilon > 0$ such that the nonnegative functions E_1 and F_1 defined by*

$$E_1(t) := \int_\Omega \left(u - u^* - u^* \log \frac{u}{u^*}\right) + \delta_1 \frac{a_1\mu_1}{a_2\mu_2} \int_\Omega \left(v - v^* - v^* \log \frac{v}{v^*}\right) + \frac{\delta_2}{2} \int_\Omega (w - w^*)^2$$

and

$$F_1(t) := \int_\Omega (u - u^*)^2 + \int_\Omega (v - v^*)^2 + \int_\Omega (w - w^*)^2 + \int_\Omega |\nabla w|^2$$

satisfy

$$\frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \qquad (t > 0). \tag{2.3}$$

Proof. Thanks to (1.11)–(1.13) we can choose $\delta_1 > 0$ defined in (1.11)–(1.13) and $\delta_2 > 0$ satisfying

$$\frac{\chi_1(0)^2 u^*(1 + \delta_1)}{4d_1 d_3} < \delta_2 < \frac{a_1\mu_1\gamma(4\delta_1 - a_1a_2(1 + \delta_1)^2)}{\alpha_2^2 a_1\delta_1 + \beta_2^2 a_2 - \alpha_1\beta_1 a_1 a_2(1 + \delta_1)}$$

and

$$\frac{a_1\mu_1\chi_2(0)^2 v^*(1 + \delta_1)}{4a_2\mu_2 d_2 d_3} < \delta_2 < \frac{a_1\mu_1\gamma(4\delta_1 - a_1a_2(1 + \delta_1)^2)}{\alpha_2^2 a_1\delta_1 + \beta_2^2 a_2 - \alpha_1\beta_1 a_1 a_2(1 + \delta_1)}.$$

We denote by $A_1(t)$, $B_1(t)$, $C_1(t)$ the functions defined as

$$A_1(t) := \int_\Omega \left(u - u^* - u^* \log \frac{u}{u^*}\right), \quad B_1(t) = \int_\Omega \left(v - v^* - v^* \log \frac{v}{v^*}\right),$$
$$C_1(t) := \frac{1}{2}\int_\Omega (w - w^*)^2,$$

and we write as

$$E_1(t) = A_1(t) + \delta_1 \frac{a_1\mu_1}{a_2\mu_2} B_1(t) + \delta_2 C_1(t).$$

The Taylor formula applied to $H(s) = s - u^* \log s$ $(s \geq 0)$ yields $A_1(t) = \int_\Omega (H(u) - H(u^*))$ is a nonnegative function for $t > 0$ (more detail, see [**1**, Lemma 3.2]). Similarly, we have that $B_1(t)$ is a positive function. By the straightforward calculations we infer

$$\frac{d}{dt}A_1(t) = -\mu_1 \int_\Omega (u - u^*)^2 - a_1\mu_1 \int_\Omega (u-u^*)(v-v^*) - d_1 u^* \int_\Omega \frac{|\nabla u|^2}{u^2}$$
$$+ u^* \int_\Omega \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w,$$
$$\frac{d}{dt}B_1(t) = -\mu_2 \int_\Omega (v - v^*)^2 - a_2\mu_2 \int_\Omega (u-u^*)(v-v^*) - d_2 v^* \int_\Omega \frac{|\nabla v|^2}{v^2}$$
$$+ v^* \int_\Omega \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w,$$
$$\frac{d}{dt}C_1(t) = \int_\Omega h_u (u-u^*)(w-w^*) + \int_\Omega h_v (v-v^*)(w-w^*) + \int_\Omega h_w (w-w^*)^2$$
$$- d_3 \int_\Omega |\nabla w|^2$$

with some derivatives h_u, h_v and h_w. Hence we have

$$\frac{d}{dt}E_1(t) = I_3(t) + I_4(t), \tag{2.4}$$

where

$$I_3(t) := -\mu_1 \int_\Omega (u-u^*)^2 - a_1\mu_1(1+\delta_1)\int_\Omega (u-u^*)(v-v^*) - \delta_1 \frac{a_1\mu_1}{a_2}\int_\Omega (v-v^*)^2$$
$$+ \delta_2 \int_\Omega h_u(u-u^*)(w-w^*) + \delta_2 \int_\Omega h_v (v-v^*)(w-w^*) + \delta_2 \int_\Omega h_w (w-w^*)^2$$

and

$$I_4(t) := -d_1 u^* \int_\Omega \frac{|\nabla u|^2}{u^2} + u^* \int_\Omega \frac{\chi_1(w)}{u}\nabla u \cdot \nabla w - d_2 v^* \delta_1 \frac{a_1\mu_1}{a_2\mu_2}\int_\Omega \frac{|\nabla v|^2}{v^2}$$
$$+ v^*\delta_1 \frac{a_1\mu_1}{a_2\mu_2}\int_\Omega \frac{\chi_2(w)}{v}\nabla v \cdot \nabla w - d_3\delta_2 \int_\Omega |\nabla w|^2. \tag{2.5}$$

At first, we shall show from Lemma 2.2 that there exists $\varepsilon_1 > 0$ such that

$$I_3(t) \leq -\varepsilon_1 \left(\int_\Omega (u-u^*)^2 + \int_\Omega (v-v^*)^2 + \int_\Omega (w-w^*)^2 \right). \tag{2.6}$$

To see this, we put

$$\begin{aligned} g_1(\varepsilon) &:= \mu_1 - \varepsilon, \\ g_2(\varepsilon) &:= \left(\frac{a_1}{a_2}\mu_1\delta_1 - \varepsilon \right) - \frac{a_1^2\mu_1^2(1+\delta_1)^2}{4(\mu_1-\varepsilon)}, \\ g_3(\varepsilon) &:= (-\delta_2 h_w - \varepsilon) - \frac{h_u^2}{4(\mu_1-\varepsilon)}\delta_2^2 - \frac{(2h_v(\mu_1-\varepsilon) - h_u a_1\mu_1(1+\delta))^2}{4(\mu_1-\varepsilon)(4\frac{a_1}{a_2}\mu_1\delta_1(\mu_1-\varepsilon) - a_1^2\mu_1^2(1+\delta_1)^2)}\delta_2^2. \end{aligned}$$

Since $\mu_1 > 0$, we have $g_1(0) = \mu_1 > 0$. Due to (1.11), we infer

$$g_2(0) = \frac{a_1\mu_1}{4a_2}(4\delta_1 - a_1a_2(1+\delta_1)^2) > 0.$$

In light of (1.5) and the definitions of $\delta_2 > 0$, $\alpha_i, \beta_i \geq 0$ (defined in (1.10)) we obtain

$$\begin{aligned} g_3(0) &= \delta_2 \left(-h_w - \left(\frac{h_u^2}{4\mu_1} + \frac{a_2(2h_v - h_u a_1(1+\delta_1))^2}{4a_1\mu_1(4\delta_1 - a_1a_2(1+\delta_1)^2)} \right)\delta_2 \right) \\ &\geq \delta_2 \left(\gamma - \left(\frac{\alpha_2^2 a_1\delta_1 + \beta_2^2 a_2 - \alpha_1\beta_1 a_1 a_2(1+\delta_1)}{a_1\mu_1(4\delta_1 - a_1a_2(1+\delta_1)^2)} \right)\delta_2 \right) > 0. \end{aligned}$$

Combination of the above inequalities and the continuity argument yields that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ hold for $i = 1, 2, 3$. Thanks to Lemma 2.2 with

$$\begin{aligned} &a = \mu_1 - \varepsilon_1, && b = a_1\mu_1(1+\delta_1), && c = -\delta_2 h_u, \\ &d = \delta_1\tfrac{a_1\mu_1}{a_2} - \varepsilon_1, && e = -\delta_2 h_v, && f = -\delta_2 h_w - \varepsilon_1, \\ &x = u(t) - u^*, && y = v(t) - v^*, && z = w(t) - w^*, \end{aligned}$$

we obtain (2.6) with $\varepsilon_1 > 0$. Lastly we will find $\varepsilon_2 > 0$ satisfying

$$I_4(t) \leq -\varepsilon_2 \int_\Omega |\nabla w|^2. \tag{2.7}$$

By virtue of the definition of $\delta_2 > 0$, we can find $\delta_3 \in \left(\frac{\chi_i(0)^2 u^*(1+\delta_1)}{4d_1d_3\delta_2}, 1 \right)$. Noting that $\chi_i' < 0$ (from (1.8)) and then using the Young inequality, we have

$$\begin{aligned} u^* \int_\Omega \frac{\chi_1(w)}{u}\nabla u \cdot \nabla w &\leq \chi_1(0)u^* \int_\Omega \frac{|\nabla u \cdot \nabla w|}{u} \\ &\leq \frac{\chi_1(0)^2 u^{*2}(1+\delta_1)}{4d_3\delta_2\delta_3} \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{d_3\delta_2\delta_3}{1+\delta_1}\int_\Omega |\nabla w|^2 \end{aligned}$$

and

$$\begin{aligned} v^*\delta_1\frac{a_1\mu_1}{a_2\mu_2}\int_\Omega \frac{\chi_2(w)}{v}\nabla v \cdot \nabla w &\leq \chi_2(0)v^*\delta_1\frac{a_1\mu_1}{a_2\mu_2}\int_\Omega \frac{|\nabla v \cdot \nabla w|}{v} \\ &\leq \frac{\chi_2(0)^2 v^{*2}\delta_1(1+\delta_1)}{4d_3\delta_2}\left(\frac{a_1\mu_1}{a_2\mu_2}\right)^2 \int_\Omega \frac{|\nabla v|^2}{v^2} + \frac{d_3\delta_1\delta_2}{1+\delta_1}\int_\Omega |\nabla w|^2. \end{aligned}$$

Plugging these into (2.5) we infer

$$I_4(t) \le -u^*\left(d_1 - \frac{\chi_1(0)^2u^*(1+\delta_1)}{4d_3\delta_2\delta_3}\right)\int_\Omega \frac{|\nabla u|^2}{u^2}$$
$$- v^*\delta_1\frac{a_1\mu_1}{a_2\mu_2}\left(d_2 - \frac{a_1\mu_1\chi_2(0)^2v^*(1+\delta_1)}{4d_3a_2\mu_2\delta_2}\right)\int_\Omega \frac{|\nabla v|^2}{v^2}$$
$$- d_3\delta_2\left(1 - \frac{\delta_1+\delta_3}{1+\delta_1}\right)\int_\Omega |\nabla w|^2.$$

We note from the definitions of $\delta_2 > 0$ and $\delta_3 > 0$ that

$$d_1 - \frac{\chi_1(0)^2u^*(1+\delta_1)}{4d_3\delta_2\delta_3} > 0,$$
$$d_2 - \frac{a_1\mu_1\chi_2(0)^2v^*(1+\delta_1)}{4d_3a_2\mu_2\delta_2} > 0$$

and

$$1 - \frac{\delta_1+\delta_3}{1+\delta_1} = \frac{1-\delta_3}{1+\delta_1} > 0.$$

Therefore we obtain that there exists $\varepsilon_2 > 0$ such that (2.7) holds. Combination of (2.4), (2.6) and (2.7) implies the end of the proof. □

Proof of Theorem 1.2. We let $f_1(t) := \int_\Omega (u-u^*)^2 + \int_\Omega (v-v^*)^2 + \int_\Omega (w-w^*)^2 \ge 0$. We have $f_1(t)$ is a nonnegative function, and thanks to the regularity of u, v, w (see Theorem 1.1) we can see that $f_1(t)$ is uniformly continuous. Moreover, integrating (2.3) over $(1,\infty)$, we infer from the positivity of $E_1(t)$ that

$$\int_1^\infty f_1(t)\,dt \le \frac{1}{\varepsilon}E_1(1) < \infty.$$

Therefore we obtain from Lemma 2.1 that $f_1(t) \to 0$. □

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Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, JAPAN
E-mail address: masaaki.mizukami.math@gmail.com

東京理科大学・理学研究科数学専攻 M2　水上　雅昭