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ASYMPTOTIC BEHAVIOR OF THE TRANSMISSION EIGENVALUES

GEORGI VODEV

1. Definition of the transmission eigenvalues

Let $\Omega \subset \mathbf{R}^d$, $d \geq 2$, be a bounded, connected domain with a C^∞ smooth boundary $\Gamma = \partial\Omega$. A complex number $\lambda \neq 0$ with $\operatorname{Re}\lambda \geq 0$ will be said to be a transmission eigenvalue if the following problem has a non-trivial solution:

$$\begin{cases} \left(\nabla c_1(x)\nabla + \lambda^2 n_1(x)\right) u_1 = 0 & \text{in} \quad \Omega, \\ \left(\nabla c_2(x)\nabla + \lambda^2 n_2(x)\right) u_2 = 0 & \text{in} \quad \Omega, \\ u_1 = u_2, \ \ c_1\partial_\nu u_1 = c_2\partial_\nu u_2 & \text{on} \quad \Gamma, \end{cases} \tag{1}$$

where ν denotes the Euclidean inner unit normal to Γ, $c_j, n_j \in C^\infty(\overline{\Omega})$, $j = 1,2$ are strictly positive real-valued functions. The transmission eigenvalues can be viewed as the eigenvalues of the non-symmetric operator $\mathcal{A}$ defined by

$$\mathcal{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{n_1(x)}\nabla c_1(x)\nabla u_1 \\ -\frac{1}{n_2(x)}\nabla c_2(x)\nabla u_2 \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \big\{(u_1, u_2) \in \mathcal{H} : \nabla c_1(x)\nabla u_1 \in L^2(\Omega),\ \nabla c_2(x)\nabla u_2 \in L^2(\Omega),$$
$$u_1 = u_2,\ c_1\partial_\nu u_1 = c_2\partial_\nu u_2 \quad \text{on} \quad \Gamma\big\}$$

where $\mathcal{H} = H_1 \oplus H_2$, $H_j = L^2(\Omega, n_j(x)dx)$. Then the transmission eigenvalues are the poles of the resolvent $(\mathcal{A} - \lambda^2)^{-1}$ (if it forms a meromorphic family) and the multiplicity of a pole λ_k is defined by

$$\operatorname{mult}(\lambda_k) = \operatorname{rank}(2\pi i)^{-1}\int_{|\lambda-\lambda_k|=\varepsilon}(\lambda^2 - \mathcal{A})^{-1}2\lambda d\lambda$$
$$= \operatorname{tr}(2\pi i)^{-1}\int_{|\lambda-\lambda_k|=\varepsilon}(\lambda^2 - \mathcal{A})^{-1}2\lambda d\lambda.$$

Our goal is to study the asymptotic behavior of the counting function $N(r) = \#\{\lambda - \text{trans. eig.} : |\lambda| \leq r\}$, $r > 1$. We will see that it is closely related to the localization of the transmission eigenvalues on the complex plane.

2. The Dirichlet-to-Neumann map

The Dirichlet-to-Neumann map, $N_j(\lambda) : H^1(\Gamma) \to L^2(\Gamma)$, associated to the pair (c_j, n_j) is defined by

$$N_j(\lambda)f = \partial_\nu u_j|_\Gamma,$$

where u_j solves the equation

$$\begin{cases} \left(\nabla c_j(x)\nabla + \lambda^2 n_j(x)\right) u_j = 0 & \text{in} \quad \Omega, \\ u_j = f & \text{on} \quad \Gamma. \end{cases} \tag{2}$$

Denote by G_j, $j = 1, 2$, the Dirichlet self-adjoint realization of the operator $-n_j^{-1}\nabla c_j \nabla$ on the Hilbert space H_j. It is well-known that $N_j(\lambda)$ is meromorphic with poles the eigenvalues of $\sqrt{G_j}$. Introduce the operator

$$T(\lambda) = c_1 N_1(\lambda) - c_2 N_2(\lambda).$$

We have the following trace formula.

Lemma 1. *Suppose that the inverse $T(\lambda)^{-1}$ exists as a meromorphic function. Then the resolvent of the operator $\mathcal{A}$ is meromorphic, too, and we have the formula*

$$M(\gamma) = M_1(\gamma) + M_2(\gamma) + \mathrm{tr}(2\pi i)^{-1} \int_\gamma \frac{dT(\lambda)}{d\lambda} T(\lambda)^{-1} d\lambda \tag{3}$$

where γ is a simple, positively orientied, piecewise smooth, closed curve in the complex plane, which avoids the poles of $T(\lambda)^{-1}$ and the eigenvalues of $\sqrt{G_1}$ and $\sqrt{G_2}$, $M(\gamma)$ is the number of the transmission eigenvalues inside γ, and $M_j(\gamma)$ is the number of the eigenvalues of the operator $\sqrt{G_j}$ inside γ.

3. Weyl asymptotics for the counting function

The following result is proved in [9].

Theorem 1. *Suppose either the condition*

$$c_1(x) \equiv c_2(x) \equiv 1 \quad in \quad \Omega, \quad n_1(x) \neq n_2(x), \quad \forall x \in \Gamma, \quad (isotropic\ case) \tag{4}$$

or the condition

$$c_1(x) \neq c_2(x), \quad \forall x \in \Gamma. \quad (anisotropic\ case) \tag{5}$$

Suppose also that the operator $T(\lambda)$ is invertible in a region of the form

$$\left\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 1, |\mathrm{Im}\,\lambda| \geq C\,(\mathrm{Re}\,\lambda)^{1-\kappa}\right\}, \quad C > 0, 0 < \kappa \leq 1, \tag{6}$$

and satisfies there the bound

$$\left\|T(\lambda)^{-1}\right\| \leq C_0 |\lambda|^{M_0}, \quad C_0, M_0 > 0.$$

Then we have the asymptotics

$$N(r) = (\tau_1 + \tau_2) r^d + O_\varepsilon(r^{d-\kappa+\varepsilon}), \quad \forall\, 0 < \varepsilon \ll 1, \tag{7}$$

where

$$\tau_j = \frac{\omega_d}{(2\pi)^d} \int_\Omega \left(\frac{n_j(x)}{c_j(x)}\right)^{d/2} dx,$$

ω_d being the volume of the unit ball in $\mathbf{R}^d$.

Known results. In the isotropic case when $n_2 \equiv 1$, $n_1(x) > 1$ on Ω, the asymptotic for $N(r)$ with a remainder term $o(r^d)$ is proved by M. Faierman [3] and by L. Robbiano [12].

Idea of the proof. It is inspired by the paper [1] where Weyl type asymptotics have been proved for the counting function of the resonances associated to an exterior transmission problem. We can get an asymptotic for $N(r) - N(r/2)$ by using the trace formula (3), the Weyl asymptotics for the counting functions of the eigenvalues of G_1 and G_2, and the Theorems of Caratheodory and Jensen. We use in an essential way that $\dim \Gamma = d - 1$.

4. Parabolic eigenvalue-free regions

Thus, the problem of proving Weyl asymptotics for the counting function $N(r)$ is reduced to that one of proving parabolic eigenvalue-free regions. The following result is proved in [14] and concerns the isotropic case.

Theorem 2. *Assume the condition*

$$c_1(x) \equiv c_2(x) \equiv 1 \quad \text{in} \quad \Omega, \quad n_1(x) \neq n_2(x), \quad \forall x \in \Gamma. \tag{8}$$

Then there are no transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq 0,\ |\operatorname{Im}\lambda| \geq C_\epsilon \left(\operatorname{Re}\lambda + 1\right)^{\frac{1}{2}+\epsilon}\right\}, \quad \forall 0 < \epsilon \ll 1.$$

In this case the asymptotic (7) holds with $\kappa = 1/2$.

In the anisotropic case the situation is more interesting and one has to distinguish two subcases. The following result is proved in [14].

Theorem 3. *Assume the condition*

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) < 0, \quad \forall x \in \Gamma. \tag{9}$$

Then there are no transmission eigenvalues in the union of the sets

$$\left\{\lambda \in \mathbf{C} : 0 \leq \operatorname{Re}\lambda \leq 1,\ \operatorname{Re}\lambda \geq C_N \left(|\operatorname{Im}\lambda| + 1\right)^{-N}\right\}, \quad \forall N \gg 1,$$

and

$$\left\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq 1,\ |\operatorname{Im}\lambda| \geq C_\epsilon \left(\operatorname{Re}\lambda\right)^{\frac{1}{2}+\epsilon}\right\}, \quad \forall 0 < \epsilon \ll 1.$$

In this case the asymptotic (7) holds with $\kappa = 1/2$.

Assume the condition

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) > 0, \quad \forall x \in \Gamma. \tag{10}$$

Then there are no transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq 0,\ |\operatorname{Im}\lambda| \geq C \left(\operatorname{Re}\lambda + 1\right)^{\frac{3}{5}}\right\}.$$

In this case the asymptotic (7) holds with $\kappa = 2/5$. Moreover, if in addition to (10) we assume either the condition

$$\frac{n_1(x)}{c_1(x)} \neq \frac{n_2(x)}{c_2(x)}, \quad \forall x \in \Gamma, \tag{11}$$

or the condition

$$\frac{n_1(x)}{c_1(x)} = \frac{n_2(x)}{c_2(x)}, \quad \forall x \in \Gamma, \tag{12}$$

then there are no transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq 0,\ |\operatorname{Im}\lambda| \geq C_\epsilon \left(\operatorname{Re}\lambda + 1\right)^{\frac{1}{2}+\epsilon}\right\}, \quad \forall 0 < \epsilon \ll 1.$$

Remark. One can show that under the condition (9) there are infinitely many transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : 0 \leq \operatorname{Re}\lambda \leq C_N \left(|\operatorname{Im}\lambda| + 1\right)^{-N}\right\}$$

and that their counting function, $N^-(r)$, satisfies an asymptotic of the form

$$N^-(r) = \tau_0 r^{d-1} + O(r^{d-2}).$$

Known results. In the isotropic case when $c_1 \equiv c_2 \equiv 1$, $n_2 \equiv 1$, $n_1(x) > 1$ on Ω, it was proved by M. Hitrik, K. Krupchyk, P. Ola and L. Päivärinta [4] that there are no transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq 0,\ |\operatorname{Im}\lambda| \geq C\,(\operatorname{Re}\lambda + 1)^{\frac{23}{25}}\right\}.$$

To prove the above theorems we make our problem semi-classical by putting $h = |\operatorname{Re}\lambda^2|^{-1/2}$, $z = h^2\lambda^2 = \pm 1 + i\operatorname{Im} z$, if $|\operatorname{Re}\lambda^2| \geq |\operatorname{Im}\lambda^2|$, and $h = |\operatorname{Im}\lambda^2|^{-1/2}$, $z = h^2\lambda^2 = \operatorname{Re} z + i$, if $|\operatorname{Re}\lambda^2| \leq |\operatorname{Im}\lambda^2|$. Clearly, $h \sim |\lambda|^{-1}$. The proof of Theorems 2 and 3 is based on the following semi-classical properties of the Dirichlet-to-Neumann map $N_j(z,h) = -ihN_j(\lambda)$ (see [14]).

Theorem 4. *For every $0 < \epsilon \ll 1$, $0 < h \ll 1$, $|\operatorname{Im} z| \geq h^{1/2-\epsilon}$, the Dirichlet-to-Neumann map $N_j(z,h)$ is an $h-\Psi$ DO of class $OPS^1_{1/2-\epsilon}(\Gamma)$ with a principal symbol*

$$\rho_j(x,\xi) = \sqrt{-r_0(x,\xi) + m_j(x)z} \qquad \textit{with} \qquad \operatorname{Im}\rho_j > 0,$$

where m_j denotes the restriction on Γ of the function n_j/c_j, and r_0 is the principal symbol of the Laplace-Beltrami operator $-\Delta_\Gamma$, Γ being considered as a Riemannian manifold equipped with the Riemannian metric induced by the Euclidean one.

Recall that $a \in S^k_\delta(\Gamma)$, $0 \leq \delta < 1/2$, if $a \in C^\infty(T^*\Gamma)$ satisfies the bounds

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta} h^{-\delta(|\alpha|+|\beta|)} \langle\xi\rangle^{k-|\beta|}.$$

It is well-known that for $h-\Psi$ DOs with such symbols there is a very nice calculus (e.g. see [2]).

Thus getting eigenvalue-free regions is reduced to inverting the operator

$$T(z,h) = c_1 N_1(z,h) - c_2 N_2(z,h)$$

with a principal symbol

$$c_1\rho_1 - c_2\rho_2 = \frac{\widetilde{c}(x)(c_0(x) r_0(x,\xi) - z)}{c_1\rho_1 + c_2\rho_2} \tag{13}$$

where $\widetilde{c}$ and c_0 are the restrictions on Γ of the functions

$$c_1 n_1 - c_2 n_2 \quad \text{and} \quad \frac{c_1^2 - c_2^2}{c_1 n_1 - c_2 n_2}$$

respectively. In the isotropic case we have $c_0 \equiv 0$ on Γ, while in the anisotropic case we have $c_0(x) \neq 0$, $\forall x \in \Gamma$. Under the condition (9) we have $c_0(x) < 0$, $\forall x \in \Gamma$, while under the condition (10) we have $c_0(x) > 0$, $\forall x \in \Gamma$.

The parametrix of $N_j(z,h)$ is bad when $\operatorname{Re} z = 1$ near the glancing region

$$\Sigma_j = \{(x,\xi) \in T^*\Gamma : r_0(x,\xi) - m_j(x) = 0\}.$$

Therefore, to improve the above results one has to improve the parametrix construction in the glancing region. Indeed, a better parametrix has been constructed in [15] for strictly concave domains valid for $|\operatorname{Im} z| \geq h^{1-\epsilon}$, which led to some improvements in this case.

5. Optimal eigenvalue-free regions

We can improve the above eigenvalue-free regions if $\Sigma_1 \cap \Sigma_2 = \emptyset$. More precisely, we have the following (see [16]).

Theorem 5. *Assume either the condition (8) or the condition (9). Then there are no transmission eigenvalues in*

$$\{\lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 1,\ |\operatorname{Im} \lambda| \geq C > 0\}. \tag{14}$$

In this case the asymptotic (7) holds with $\kappa = 1$.

Tthe eigenvalue-free region (14) has been previously proved in [10] in the case of a ball and constant coefficients. It is shown by Leung and Colton [6] that in the isotropic case when Ω is a ball and the refraction indices n_1 and n_2 constants, the eigenvalue-free region (14) is optimal. In the anisotropic case we also have the following (see [16]).

Theorem 6. *Assume the conditions (10) and (11). Then there are no transmission eigenvalues in*

$$\{\lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0,\ |\operatorname{Im} \lambda| \geq C \log(\operatorname{Re} \lambda + 2)\}, \quad C > 0. \tag{15}$$

In this case the asymptotic (7) holds with $\kappa = 1$.

Define the cut-off function $\chi_j^0 \in C_0^\infty(T^*\Gamma)$ by

$$\chi_j^0(x,\xi) = \phi\left((r_0(x,\xi) - m_j(x))\delta^{-2}\right)$$

where $0 < \delta \ll 1$ is a small parameter independent of h and z, and $\phi \in C_0^\infty(\mathbf{R})$, $0 \leq \phi \leq 1$, $\phi(t) = 1$ for $|t| \leq 1$, $\phi(t) = 0$ for $|t| \geq 2$, is also independent of h and z. Theorems 5 and 6 follow from the following (see [16]).

Theorem 7. *Let $\operatorname{Re} z = 1$ and let $0 < \epsilon < 1$ be arbitrary. Then, for every $0 < \delta \ll 1$ there are constants $C_\delta > 1$ and $0 < h_0(\epsilon, \delta) \ll 1$ such that we have*

$$\left\| N_j(z,h) - \operatorname{Op}_h(\rho_j(1-\chi_j^0) + hb_j) \right\|_{L^2(\Gamma) \to H_h^1(\Gamma)} \leq C\delta \tag{16}$$

for $C_\delta h \leq |\operatorname{Im} z| \leq h^\epsilon$, $0 < h \leq h_0(\epsilon, \delta)$, where $C > 0$ is a constant independent of h, z and δ, and $b_j \in S_0^0(\Gamma)$ is independent of h, z and the function n_j.

Here $H_h^1(\Gamma)$ denotes the Sobolev space equipped with the semi-classical norm.

6. The degenerate isotropic case

We will consider the case when

$$c_1(x) \equiv c_2(x) \equiv 1 \quad \text{in} \quad \Omega, \quad n_1(x) = n_2(x), \quad \forall x \in \Gamma.$$

We have the following (see [17]).

Theorem 8. *Assume that there is an integer $j \geq 1$ such that*

$$\partial_\nu^s(n_1(x) - n_2(x)) = 0, \quad \forall x \in \Gamma, \quad 0 \leq s \leq j-1, \tag{17}$$

and

$$\partial_\nu^j(n_1(x) - n_2(x)) \neq 0, \quad \forall x \in \Gamma. \tag{18}$$

Then there are no transmission eigenvalues in

$$\left\{\lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0,\ |\operatorname{Im} \lambda| \geq C\,(\operatorname{Re} \lambda + 1)^{1-\kappa_j}\right\},$$

where $\kappa_j = 2(3j+2)^{-1}$. In this case the asymptotic (7) holds with $\kappa = \kappa_j$.

It has been previously proved by Lakshtanov and Vainberg [5] that under the conditions (17) and (18) there are no transmission eigenvalues in $|\arg \lambda| \geq \varepsilon$, $|\lambda| \geq C_\varepsilon \gg 1$, $\forall\, 0 < \varepsilon \ll 1$.

7. Open problems

Conjecture 1. *For an arbitrary domain Ω, the counting function of the transmission eigenvalues satisfies the Weyl asymptotics*

$$N(r) = (\tau_1 + \tau_2)r^d + O(r^{d-1}). \tag{19}$$

References

[1] F. Cardoso, G. Popov and G. Vodev, *Asymptotics of the number of resonances in the transmission problem*, Commun. Partial Diff. Equations **26** (2001), 1811-1859.

[2] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in semi-classical limit*, London Mathematical Society, Lecture Notes Series, **268**, Cambridge University Press, 1999.

[3] M. Faierman, *The interior transmission problem: spectral theory*, SIAM J. Math. Anal. **46** (1) (2014), 803-819.

[4] M. Hitrik, K. Krupchyk, P. Ola and L. Päivärinta, *The interior transmission problem and bounds of transmission eigenvalues*, Math. Res. Lett. **18** (2011), 279-293.

[5] E. Lakshtanov and B. Vainberg, *Application of elliptic theory to the isotropic interior transmission eigenvalue problem*, Inverse Problems **29** (2013), 104003.

[6] Y.-J. Leung and D. Colton, *Complex transmission eigenvalues for spherically stratified media*, Inverse Problems **28** (2012), 075005.

[7] H. Pham and P. Stefanov, *Weyl asymptotics of the transmission eigenvalues for a constant index of refraction*, Inverse problems and imaging **8**(3) (2014), 795-810.

[8] V. Petkov, *Location of eigenvalues for the wave equation with dissipative boundary conditions*, Inverse Problems and imaging **10**(4) (2016), 1111-1139.

[9] V. Petkov and G. Vodev, *Asymptotics of the number of the interior transmission eigenvalues*, J. Spectral Theory **7**(1) (2017), to appear.

[10] V. Petkov and G. Vodev, *Localization of the interior transmission eigenvalues for a ball*, Inverse Problems and imaging, **11**(2) (2017), to appear.

[11] L. Robbiano, *Spectral analysis of interior transmission eigenvalues*, Inverse Problems **29** (2013), 104001.

[12] L. Robbiano, *Counting function for interior transmission eigenvalues*, Mathematical Control and Related Fields **6**(1) (2016), 167-183.

[13] J. Sylvester, *Transmission eigenvalues in one dimension*, Inverse Problems **29** (2013), 104009.

[14] G. Vodev, *Transmission eigenvalue-free regions*, Comm. Math. Phys. **336** (2015), 1141-1166.

[15] G. Vodev, *Transmission eigenvalues for strictly concave domains*, Math. Ann. **366** (2016), 301-336.

[16] G. Vodev, *High-frequency approximation of the interior Dirichlet-to-Neumann map and applications to the transmission eigenvalues*, preprint 2017.

[17] G. Vodev, *Parabolic transmission eigenvalue-free regions in the degenerate isotropic case*, preprint 2017.

Université de Nantes, Laboratoire de Mathématiques Jean Leray, 2 rue de la Houssinière, BP 92208, 44322 Nantes Cedex 03, France

E-mail address: Georgi.Vodev@univ-nantes.fr