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SINGULARITIES OF THE SCATTERING KERNEL AND INVERSE SCATTERING PROBLEMS

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ABSTRACT. We present a survey of results concerning the singularities of the scattering kernel $s(t, \theta, \omega)$ which is the Fourier transform of the scattering amplitude $a(\lambda, \theta, \omega)$. These singularities are easy to be measured and they are related to the sojourn times of the (ω, θ)-rays incoming with direction $\omega \in \mathbb{S}^{n-1}$ and outgoing with direction $\theta \in \mathbb{S}^{n-1}$. The rays with back-scattering directions $\theta = -\omega$ are used in all radar applications and this lead to a reconstruction of the convex hull of the obstacle. The problem is to show that if we know the sojourn times of (ω, θ)-rays for almost all directions $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ we can determine uniquely the obstacle. We present several results leading to a solution of this problem for a class of obstacles.

1. SCATTERING KERNEL

Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^∞ boundary ∂K and connected complement $\Omega = \overline{\mathbb{R}^n \setminus K}$. Such K is called an *obstacle* in $\mathbb{R}^n$. We consider the Dirichlet problem for the Laplacian but similar considerations can be applied to other boundary value problems as Neumann and Robin ones. Let $(\theta, \omega) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Consider the *outgoing solution* $v_s = v_s(x, \omega, \lambda)$ of the problem

$$\begin{cases} (\Delta + \lambda^2) v_s = 0 \text{ in } \mathring{\Omega}, \\ v_s + e^{-\mathbf{i}\lambda\langle x, \omega\rangle} = 0 \,\text{on}\, \partial K \end{cases}$$

satisfying the so called $(\mathbf{i}\lambda)$ - outgoing Sommerfeld radiation condition. This means that as $|x| = r \longrightarrow \infty$ we have

$$v_s(r\theta, \omega, \lambda) = \frac{e^{-\mathbf{i}\lambda r}}{r^{(n-1)/2}}\Big(a(\lambda, \theta, \omega) + \mathcal{O}\Big(\frac{1}{r}\Big)\Big),\ x = r\theta\,.$$

The scattering amplitude has the representation (see [5],[10])

$$a(\lambda, \theta, \omega) = \frac{(\mathbf{i}\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}} \int_{\partial K} \Big(\mathbf{i}\lambda < \nu(x), \theta > e^{\mathbf{i}\lambda<x, \theta-\omega>} + e^{\mathbf{i}\lambda<x,\theta>} \frac{\partial v_s}{\partial \nu}(x, \lambda)\Big) dS_x, \tag{1.1}$$

where $< ., . >$ denotes the scalar product in $\mathbb{R}^n$.

We assume for simplicity n odd but similar results are true for n even. Let $\theta \neq \omega$ and consider the *scattering kernel* $s(t, \theta, \omega)$ defined as the Fourier transform of the scattering amplitude:

$$s(t, \theta, \omega) = \mathcal{F}_{\lambda \to t}\Big(\Big(\frac{\lambda}{2\pi \mathrm{i}}\Big)^{(n-1)/2} \overline{a(\lambda, \theta, \omega)} \Big),$$

where $\Big(\mathcal{F}_{\lambda \to t}\varphi\Big)(t) = (2\pi)^{-1} \int e^{\mathrm{i}t\lambda}\varphi(\lambda) d\lambda$ for functions $\varphi \in \mathcal{S}(\mathbb{R})$. Let $V(t, x; \omega)$ be the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)V = 0 \text{ in } \mathbb{R} \times \mathring{\Omega}, \\ V + \delta(t - \langle x, \omega \rangle) = 0 \text{ on } \mathbb{R} \times \partial K, \\ V|_{t < -t_0} = 0. \end{cases}$$

Then we have

$$s(\sigma, \theta, \omega) = (-1)^{(n+1)/2} 2^{-n} \pi^{1-n} \int_{\partial K} \partial_t^{n-2} \partial_\nu V(\langle x, \theta \rangle - \sigma, x; \omega) dS_x,$$

where the integral is interpreted in the sense of distributions. Our aim will be to examine the singularities of $s(t, \theta, \omega)$ with respect to t.

First we define the so called reflecting (ω, θ)-rays. Given two directions $(\theta, \omega) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, consider a curve $\gamma \in \Omega$ having the form

$$\gamma = \cup_{i=0}^m l_i, \; m \geq 1,$$

where $l_i = [x_i, x_{i+1}]$ are finite segments for $i = 1, ..., m-1$, $x_i \in \partial K$, and l_0 (resp. l_m) is the infinite segment starting at x_1 (resp. at x_m) and having direction $-\omega$ (resp. θ). The curve γ is called a reflecting (ω, θ)-ray in Ω if for $i = 0, 1, ..., m-1$ the segments l_i and l_{i+1} satisfy the law of reflection at x_{i+1} with respect to ∂K. The points $x_1, ..., x_m$ are called *reflection points* of γ and this ray is called *ordinary reflecting* (or *simply reflecting*) if γ has no segments tangent to ∂K.

Next, we define two notions related to (ω, θ)-rays (also called scattering rays). Fix an arbitrary open ball U_0 with radius $a > 0$ containing K. For $\xi \in \mathbb{S}^{n-1}$ introduce the hyperplane Z_ξ orthogonal to ξ and such that ξ is pointing into the interior of the open half space H_ξ with boundary Z_ξ containing U_0 (see Figure). Let $\pi_\xi : \mathbb{R}^n \longrightarrow Z_\xi$ be the orthogonal projection. For a reflecting (ω, θ)-ray γ in Ω with successive reflecting points $x_1, ..., x_m$ the *sojourn time* T_γ of γ is defined by

$$T_\gamma = \|\pi_\omega(x_1) - x_1\| + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| + \|x_m - \pi_{-\theta}(x_m)\| - 2a\,.$$

Obviously, $T_\gamma + 2a$ coincides with the length of this part of γ which lies in $H_\omega \cap H_{-\theta}$. In fact, the sojourn time T_γ does not depend on the choice of the ball U_0 since it follows easily that

$$\|\pi_\omega(x_1) - x_1\| = a + \langle x_1, \omega\rangle, \ \ \|x_m - \pi_{-\theta}(x_m)\| = a - \langle x_m, \theta\rangle \ ,$$

therefore

$$T_\gamma = \langle x_1, \omega\rangle + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| - \langle x_m, \theta\rangle \,.$$

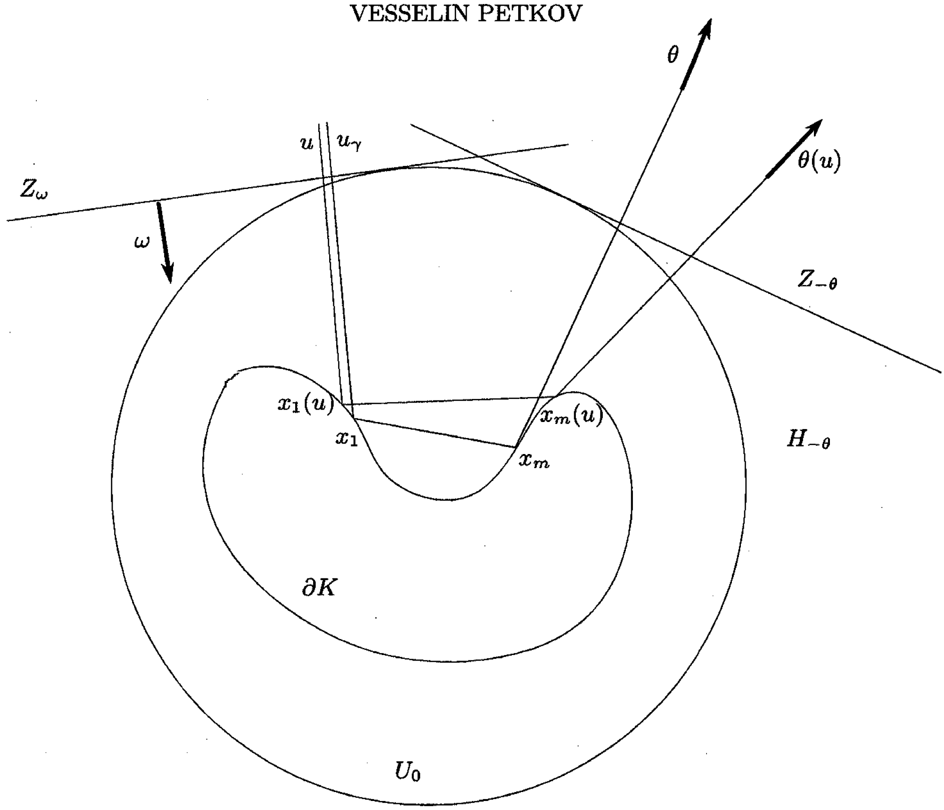
Given an ordinary reflecting (ω, θ)-ray γ set $u_\gamma = \pi_\omega(x_1)$. There exists a small neighborhood W_γ of u_γ in Z_ω such that for every $u \in W_\gamma$ there is an unique direction $\theta(u) \in \mathbb{S}^{n-1}$ and points $x_1(u), ..., x_m(u)$ which are the successive reflection points of a reflecting $(u, \theta(u))$-ray in Ω with $\pi_\omega(x_1(u)) = u$. This defines a smooth map

$$J_\gamma : W_\gamma \ni u \longrightarrow \theta(u) \in \mathbb{S}^{n-1}$$

and $dJ_\gamma(u_\gamma)$ is called a differential cross section related to γ. We say that γ is non-degenerate if

$$\det dJ_\gamma(u_\gamma) \neq 0 \,.$$

The notion of sojourn time as well as that of differential cross section are well known in the physical literature. The definitions given above are due to Guillemin [2].



Sojourn time

For strictly convex obstacles all (non-trivial) reflecting rays have only one reflection point x^+ in the illuminated region and the corresponding sojourn time is equal to $T_+ =< x^+, \omega - \theta >$. For $\theta \neq \omega$, A. Majda [4] proved that $\overline{a(\lambda, \omega, \theta)}$ has a complete asymptotic expansion

$$\overline{a(\lambda, \omega, \theta)} = e^{\mathrm{i}\lambda\langle x^+, \omega - \theta\rangle} \sum_{j=0}^{N} c_j \lambda^{-j} + \mathcal{O}(|\lambda|^{-N-1}),\ \forall N \in \mathbb{N}\,,$$

and for t close to $-T_+$ we have

$$s(t, \theta, \omega) = \left(\frac{-1}{2\pi}\right)^{(n-1)/2} \Big| dJ_{\gamma_+}(u_{\gamma_+}) \Big|^{-1/2} \delta^{(n-1)/2}(t + T_+)$$
$$+\text{lower order singularities.} \qquad (1.2)$$

which gives

$$\text{sing supp}_t\ s(t, \theta, \omega) = \{-T_+\}\,.$$

A simple geometric argument implies that

$$|\det dJ_{\gamma_+}(u_{\gamma_+})| = 4|\theta - \omega|^{(n-3)} \mathcal{K}(x_+)\,,$$

where $\mathcal{K}(y)$ is the Gauss curvature at $y \in \partial K$.

Here x^+ denotes the point in the illuminated region (see Figure 1)

$$\partial K_+(\omega) = \{y \in \partial K :< \nu(y), \omega >< 0\}$$

related to ω, and we have used the convention that the obstacle lies in the half-space

$$\{x \in \mathbb{R}^n :< x, \theta - \omega >< 0\}.$$

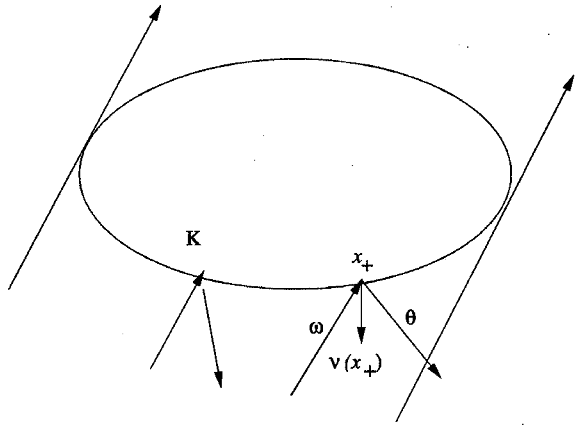


FIGURE 1. Strictly convex obstacle

2. GENERALIZED (ω, θ)-RAYS

It is much more complicated to get similar results in the case of non-convex obstacles. Now the information obtained by means of rays having only one reflection is no longer sufficient. One needs to consider multiple reflecting (ω, θ)-rays leading to isolated singularities of $s(t, \theta, \omega)$. Roughly speaking, the singularities of the scattering kernel are amongst the sojourn times of (ω, θ)-rays, however now one has to consider not only simply reflecting (ω, θ)-rays but all generalized geodesics *incoming* with direction ω and *outgoing* with direction θ.

These rays are simply called (ω, θ)-rays. In general, there exist (ω, θ)-rays with grazing or gliding segments (see Figure 2).

The precise definition of an (ω, θ)-ray is based on the notion of a generalized bicharacteristic of the operator $\Box = \partial_t^2 - \Delta_x$ given as trajectories of the generalized Hamilton flow $\mathcal{F}_t$ in Ω generated by the symbol $\sum_{i=1}^n \xi_i^2 - \tau^2$ of $\Box$ (see [7], Chapter 1, [10]). In general, $\mathcal{F}_t$ is not smooth and in some cases there may exist two different integral curves issued from the same point in the phase space as it was shown by an example of M. Taylor [14]. To avoid this situation we assume

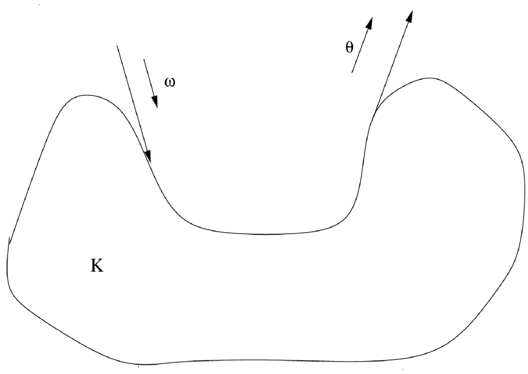


FIGURE 2. Generalized (ω, θ)-ray with gliding segments

that the following generic condition is satisfied.

$(\mathcal{G})$ If for $(x, \xi) \in T^*(\partial K)$ the normal curvature of ∂K vanishes of infinite order in direction ξ, then ∂K is convex at x in direction ξ.

More generally, working with the restriction of the principal symbol of $\Box$ to a level surface $\tau = \tau_0 \neq 0$, one defines generalized bicharacetristics on the set $\dot{T}^*(\Omega)$ of all $(x, \xi) \in T^*(\Omega)$ such that $\xi \neq 0$. Given $\sigma = (x, \xi) \in \dot{T}^*(\Omega)$, there exists a unique generalized bicharacteristic $(x(t), \xi(t)) \in \dot{T}^*(\Omega)$ such that $x(0) = x$ and $\xi(0) = \xi$. Set $\mathcal{F}_t(x, \xi) = (x(t), \xi(t))$ for all $t \in \mathbb{R}$. This defines a flow $\mathcal{F}_t : \dot{T}^*(\Omega) \longrightarrow \dot{T}^*(\Omega)$ which is called the *generalized geodesic flow* on $\dot{T}^*(\Omega)$ (see [7], [10]). Obviously, it leaves the *cosphere bundle* $S^*(\Omega)$ invariant.

At points of transversal reflection at $\dot{T}^*_{\partial K}(\Omega)$ the flow $\mathcal{F}_t$ is discontinuous. To make it continuous, consider the *quotient space* $T^*_b(\Omega) = \dot{T}^*(\Omega)/\sim$ of $\dot{T}^*(\Omega)$ with respect to the following equivalence relation: $\rho \sim \sigma$ if and only if $\rho = \sigma$ or $\rho, \sigma \in T^*_{\partial K}(\Omega)$ and either $\lim_{t \nearrow 0} \mathcal{F}_t(\rho) = \sigma$ or $\lim_{t \searrow 0} \mathcal{F}_t(\rho) = \sigma$. Let $S^*_b(\Omega)$ be the image of $S^*(\Omega)$ in $\dot{T}^*(\Omega)/\sim$. Melrose and Sjöstrand [7] proved that the natural projection of $\mathcal{F}_t$ on $T^*_b(\Omega)$ is continuous.

After these definitions, a curve $\gamma = \{x(t) \in \Omega : t \in \mathbb{R}\}$ is called an (ω, θ)-*ray* if there exist real numbers $t_1 < t_2$ so that

$$\tilde{\gamma}(t) = (x(t), \xi(t)) \in S^*_b(\Omega)$$

is a *generalized bicharacteristic* of $\Box$ and

$$\xi(t) = \omega \text{ for } t \leq t_1, \; \xi(t) = \theta \text{ for } t \geq t_2,$$

provided that the time t increases when we move along $\tilde{\gamma}$. Denote by $\mathcal{L}_{\omega,\theta}(\Omega)$ the set of all (ω,θ)-rays in Ω. The *sojourn time* T_δ of $\delta \in \mathcal{L}_{\omega,\theta}(\Omega)$ is defined as the length of the part of δ lying in $H_\omega \cap H_{-\theta}$.

3. Singularity of an isolated (ω,θ)-reflecting ray

Turning to the problem of the behavior of $s(t,\theta,\omega)$ near singularities, assume that γ is a fixed non-degenerate ordinary reflecting (ω,θ)-ray such that

$$T_\gamma \neq T_\delta \text{ for every } \delta \in \mathcal{L}_{\omega,\theta}(\Omega) \setminus \{\gamma\}. \tag{3.1}$$

By using the continuity of the generalized Hamilton flow, it is easy to show that

$$(-T_\gamma - \epsilon, -T_\gamma + \epsilon) \cap \operatorname{sing\,supp}_t\, s(t,\theta,\omega) = \{-T_\gamma\}$$

for $\epsilon > 0$ sufficiently small. The singularity of $s(t,\theta,\omega)$ at $t = -T_\gamma$ can be investigated by using a global construction of a parametrix of the problem

$$\begin{cases} (\partial_t^2 - \Delta)V = F(t,x) \text{ in } \mathbb{R} \times \mathring{\Omega}, \\ V + \delta(t - \langle x, \omega\rangle) = g(t,x) \text{ on } \mathbb{R} \times \partial K, \\ V|_{t<-t_0} = 0 \end{cases}$$

with smooth $F(t,x)$, $g(t,x)$, where V is given by a sum of global Fourier integral operator, related to the composition of Fourier integral operators determined by the successful reflexions of γ (see Chapter 4 in [10]).

Theorem 3.1 ([9], [10]). *Under the assumption* (3.1) *we have*

$$-T_\gamma \in \operatorname{sing\,supp}_t\, s(t,\theta,\omega) \tag{3.2}$$

and for t close to $-T_\gamma$ the scattering kernel has the form

$$s(t,\theta,\omega) = \Big(\frac{1}{2\pi i}\Big)^{(n-1)/2} (-1)^{m_\gamma - 1} \exp\Big(\mathbf{i}\frac{\pi}{2}\beta_\gamma\Big) \tag{3.3}$$

$$\times \Big|\det dJ_\gamma(u_\gamma)\Big|^{-1/2} \delta^{(n-1)/2}(t + T_\gamma) + \text{lower order singularities}.$$

Here m_γ is the number of reflections of γ, and $\beta_\gamma \in \mathbb{Z}$ is related to Maslov index and to a signature of a matrix.

Remark 3.2. *For strictly convex obstacles the Maslov index is zero, we have $m_\gamma = 1$, $\beta_\gamma = -\frac{n-1}{2}$ and we obtain the result (1.2) of Majda.*

To apply the above result, we need the condition (3.1) and it is desirable to prove that there exists a subset $\mathcal{S} \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ with zero Lebesgue measure such that for all directions $(\omega,\theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \setminus \mathcal{S}$ the corresponding (ω,θ) - rays satisfy (3.1). Here one has to deal with all (generalized) (ω,θ)-rays and this makes the problem rather difficult. We start with a result concerning the ordinary reflecting (ω,θ)-rays only.

Theorem 3.3 ([10]). *For every $\omega \in \mathbb{S}^{n-1}$ there exists a set $S(\omega) \subset \mathbb{S}^{n-1}$ the complement of which is a countable union of compact subsets of $\mathbb{S}^{n-1}$ of measure zero such that if $\theta \in S(\omega)$, then any two different ordinary reflecting (ω,θ)-rays in Ω have distinct sojourn times.*

To deal with reflecting rays with tangent segments, we introduce a more general type of trajectories. A curve γ in $\mathbb{R}^n$ is called an (ω,θ)-*trajectory* for Ω if it has the form $\gamma = \bigcup_{i=0}^{s} l_i$, where $l_i = [x_i, x_{i+1}]$, $i = 1,\dots,s-1$, $x_i \in \partial K$ for all $i = 1,\dots,s$, while l_0 (resp. l_s) is the infinite ray starting at x_1 (resp. x_s) with direction $-\omega$ (resp. θ) and for every $i = 0,1,\dots,s-1$, l_i and l_{i+1} satisfy the law of reflection at x_i with respect to ∂K. It is clear that every reflecting (ω,θ)-ray is an (ω,θ)-trajectory, but the converse is not true in general since some (ω,θ)-trajectory may intersect transversally ∂K. On the other hand, every (ω,θ)-reflecting ray with tangent segment is an (ω,θ)-trajectory. We have the following.

Theorem 3.4 ([10]). *There exists $\mathcal{T} \subset \mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ the complement of which is a countable union of compact subsets of measure zero in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ such that for $(\omega,\theta) \in \mathcal{T}$ all (ω,θ)-trajectories for Ω are ordinary.*

The analysis of the generalized (ω,θ)-rays leads to many difficulties. However it is quite natural to expect that for almost all (ω,θ) in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ there are no generalized (ω,θ)-rays different from reflecting ones. This will be discussed in the next section.

4. Poisson relation for the scattering kernel

To study the singularities of the scattering kernel introduce following generic condition

$(\mathcal{U}_{\omega,\theta})$ Each (ω,θ)-ray in Ω is the projection of a uniquely extendible generalized bicharacteristics γ of $\Box$.

This condition is satisfied for all directions (ω,θ) lying outside a set with zero measure in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ (see Theorem 4.2 below).

Let $\pi : T^*(\mathbb{R}\times\Omega) \longrightarrow \Omega$ be the natural projection. The following result shows that for $\omega \neq \theta$ all singularities in t of $s(t,\theta,\omega)$ are given by the (negative) sojourn times.

Theorem 4.1 ([1]). *Let $\omega \neq \theta$ and assume $(\mathcal{U}_{\omega,\theta})$ satisfied. Then we have*

$$\text{sing supp}_t\ s(t,\theta,\omega) \subset \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}. \tag{4.1}$$

In analogy with the well-known Poisson relation for the Laplacian on Riemannian manifolds, (4.1) is called the Poisson relation for the scattering kernel, while the set of all T_γ, where $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega)$, $(\omega,\theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, is called the *scattering length spectrum* (SLS) of K. The proof of the above result is based on the results of propagation of singularities of Melrose-Sjöstrand [7] and the properties of the generalized flow $\mathcal{F}_t$ (see Chapter 5, [10] for a detailed proof).

While in general the relation (4.1) is not an equality, it turns out that there exists a set $\mathcal{R}$ of full measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that for $(\omega,\theta) \in \mathcal{R}$ the Poisson relation becomes an equality. This is important for some inverse scattering problems.

It was proved by Stoyanov [12] that for each $T > 0$, $S^*(\Omega)$ can be represented as a countable union of Borel subsets S_i such that on each S_i, $\{\mathcal{F}_t\}_{0\leq t\leq T}$ coincides with the restriction of an one-parameter family $\mathcal{G}_t^{(i)}$ of Lipschitz maps defined in a neighborhood of S_i in $\dot{T}^*(\Omega)$, taking values in $T^*(\mathbb{R}^n)$ and such that for all but finitely many t, $\mathcal{G}_t^{(i)}$ is smooth and its restriction to smooth local cross-sections is a contact transformation. As a consequence of this regularity property, one gets the following.

Theorem 4.2 ([12]). *The generalized geodesic flow $\mathcal{F}_t$ preserves the Hausdorff dimension of Borel subsets of $S^*(\Omega)$. There exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that for each $(\omega,\theta) \in \mathcal{R}$ the only (ω,θ)-rays in Ω are reflecting (ω,θ)-rays and*

$$\text{sing supp}_t\ s(t,\theta,\omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}\ .$$

The proof is based on Theorems 3.1, 3.3, 3.4, 4.1 above combined with geometric and dynamical systems arguments.

The fact concerning the dimension of Borel sets would be a trivial one if the maps $\mathcal{F}_t$ were Lipschitz. However, it is well-known and easy to see that this not the case. Locally near a point $\rho \in S^*(\Omega)$, the map $\mathcal{F}_t$ is Lipschitz on a neighborhood of ρ for small $|t|$ when $\rho \notin S^*_{\partial K}(\Omega)$ or ρ is a transversal reflection point. Whenever $\rho \in G$, (G is the glancing set) the map $\mathcal{F}_t$ is not Lipschitz. For example, in the simplest case of a diffractif tangent point $\rho \in G_d$, the map $\mathcal{F}_t$ has a singularity of "square root type" at ρ, so it is clearly not Lipschitz.

5. Inverse scattering problems related to SLS

The *scattering length spectrum* (SLS) of K is by definition the family of sets of real numbers $\mathcal{S}L_K = \{\mathcal{S}L_K(\omega,\theta)\}_{(\omega,\theta)}$ where (ω,θ) runs over $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ and $\mathcal{S}L_K(\omega,\theta)$ is the set of sojourn times T_γ of all (ω,θ)-rays γ in Ω_K. Thus, $\mathcal{S}L_K$ is a map which assigns to each pair of directions (ω,θ) a set $\mathcal{S}L_K(\omega,\theta)$ of real numbers.

In this section we discuss the problem of recovering information about the geometry of the obstacle K from its SLS. Two obstacles K and L in $\mathbb{R}^n$ are said to have almost the same SLS if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ such that $\mathcal{S}L_K(\omega,\theta) = \mathcal{S}L_L(\omega,\theta)$ for all $(\omega,\theta)\in\mathcal{R}$.

It follows from results of A. Majda [5] and P. Lax and R. Phillips [3] that the *convex hull* of K can be recovered from the sojourn times of back-scattering $(\omega,-\omega)$-rays . Consequently, in the class of convex obstacles and also in the class of connected obstacles with real analytic boundaries, K is completely determined by its SLS.

The following example of M. Livshits (see Figure 3 taken from [6]) shows that in general $\mathcal{S}L_K$ does not determine K uniquely. Here the part E is half of an ellipse with foci F_1 and F_2. The ellipse has the property that any ray intersecting the segment connecting the foci, after reflection at the boundary, intersects the same segment again. It is now clear that no scattering ray in the exterior of the obstacle K has a common point with the parts between A and F_1 and between F_2 and B, so these two "pockets" cannot be recovered from the SLS of the obstacle.

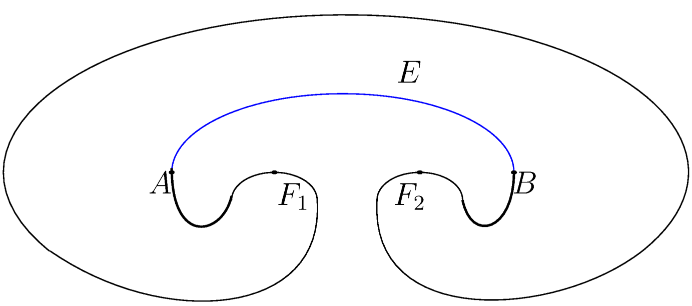


Figure 3. Example of Livshits

It should be mentioned that this example is in $\mathbb{R}^2$ but recently Noakes and Stoyanov [8] constructed examples for arbitrary dimensions. Stoyanov proved that if two obstacles K and L have almost the same SLS, then their generalized geodesic flows are conjugate with a time preserving conjugacy on the non-trapping parts of their phase spaces.

Let $\mathrm{Trap}(\Omega_K)$ be the set of points which are not accessible by scattering rays. Using the existence of the conjugacy Φ and the fact that it is measure-preserving with respect to the canonical measures on $S_b^*(\Omega_K)$ and $S_b^*(\Omega_L)$, one derives the following.

Proposition 5.1 ([13]). *Let the obstacles K and L have almost the same SLS. If the sets of trapped points of both K and L have Lebesgue measure zero, then* $\mathrm{Vol}\,(K) = \mathrm{Vol}\,(L)$.

It seems natural to conjecture that in the case of non-trapping obstacles the SLS uniquely determines the obstacle. While this is still an open problem, one can prove this conjecture at least for star-shaped obstacles. Notice that star-shaped obstacles are necessarily non-trapping.

Theorem 5.2 ([13]). *Let K and L have almost the same SLS and let K be star-shaped. Then $\partial K \subset \partial L$. If L is star-sharped, we have $K = L$.*

The reader may consult Chapter 13 in [10] for other inverse scattering results and for detailed proofs.

6. Trapping obstacles and singularities of $s(t, \omega, \theta)$

Given a generalized bicharacteristic γ in $S^*(\Omega)$, its projection $\tilde{\gamma} =\sim (\gamma)$ in $S_b^*(\Omega)$ is called a compressed generalized bicharacteristic. Let U_0 be an open ball containing K and let C be its boundary sphere. For an arbitrary point $z = (x, \xi) \in S_b^*(\Omega)$, consider the compressed generalized bicharacteristic

$$\gamma_z(t) = (x(t), \xi(t)) \in S_b^*(\Omega)$$

parameterized by the time t and passing through z for $t = 0$. Denote by $T(z) \in \mathbb{R}^+ \cup \infty$ the maximal $T > 0$ such that $x(t) \in \mathcal{U}_0$ for $0 \leq t \leq T(z)$. The so called *trapping set* is defined by

$$\Sigma_\infty = \{(x, \xi) \in S_b^*(\Omega) : x \in C,\, T(z) = \infty\}\,.$$

The trapping set Σ_∞ is closed in $S_b^*(\Omega)$. For simplicity, in the following the compressed generalized bicharacteristics will be called simply generalized ones. The obstacle K is called trapping if $\Sigma_\infty \neq \emptyset$, i.e. when there exists at least one point $(\hat{x}, \hat{\xi}) \in C \times \mathbb{S}^{n-1}$ such that the generalized trajectory $\delta_\mu(t)$ issued from $\mu = (\hat{x}, \hat{\xi})$ stays in U_0 for all $t \geq 0$. This provides some information about the behavior of the rays issued from the points (y, η) sufficiently close to $(\hat{x}, \hat{\xi})$, however in general it does not yield any information about the geometry of (ω, θ)-rays.

Now for every trapping obstacle we have the following

Theorem 6.1 ([11]). *Let the obstacle K be trapping and satisfy the condition $(\mathcal{G})$. Then there exists a sequence of ordinary reflecting (ω_m, θ_m)-rays γ_m with sojourn times $T_{\gamma_m} \longrightarrow \infty$.*

To prove this we use the following

Proposition 6.2. *The set of points $(x, \xi) \in S^*_C(\Omega) = \{(x, \xi) \in T^*(\Omega) : x \in C, |\xi| = 1\}$ such that the trajectory $\{\mathcal{F}_t(x, \xi) : t \geq 0\}$ issued from (x, ξ) is bounded has Lebesgue measure zero in $S^*_C(\Omega)$.*

Let $\mathcal{O}(W)$ be the set of all pairs of directions $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that there exists an ordinary reflecting (ω, θ)-ray issued from $(x, \omega) \in W$ with *outgoing* direction $\theta \in \mathbb{S}^{n-1}$. To obtain convenient approximations with (ω, θ)-rays issued from W, it is desirable to know that $\mathcal{O}(W)$ has a positive measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ for all sufficiently small neighborhoods $W \subset C \times \mathbb{S}^{n-1}$ of (z_0, ω_0). Roughly speaking this means that a trapping generalized bicharacteristic $\delta_\mu(t)$ is non-degenerate in some sense. More precisely, we introduce the following

Definition 1. *The generalized bicharacteristic γ issued from $(y, \eta) \in C \times \mathbb{S}^{n-1}$ is called weakly non-degenerate if for every neighborhood $W \subset C \times \mathbb{S}^{n-1}$ of (y, η) the set $\mathcal{O}(W)$ has a positive measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.*

The above definition generalizes that of a non-degenerate ordinary reflecting ray γ given in section 1.

Remark 6.3. *In general a weakly non-degenerate ordinary reflecting ray does not need to be non-degenerate. To see this, first notice that the set of those $(y, \eta) \in C \times \mathbb{S}^{n-1}$ that generate weakly non-degenerate bicharacteristics is closed in $C \times \mathbb{S}^{n-1}$. For example we can consider the special case when K is convex with vanishing Gauss curvature at some point $x_0 \in \partial K$ and strictly positive Gauss curvature at any other point of ∂K.*

Now we have a stronger version of Theorem 6.1.

Theorem 6.4 ([10]). *Let the obstacle K have at least one trapping weakly non-degenerate bicharacteristic δ issued from $(y, \eta) \in C \times \mathbb{S}^{n-1}$ and let K satisfy $(\mathcal{G})$. Then there exists a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays γ_m with sojourn times $T_{\gamma_m} \longrightarrow \infty$ such that $-T_{\gamma_m}$ is a singularity of $s(t, \theta_m, \omega_m)$.*

For simplicity below we assume that n is odd. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\chi(x) = 1$ on a neighborhood of K. It is well known (see [3]) that the modified cut-off resolvent

$$R_\chi(\lambda) = \chi(-\Delta_D - \lambda^2)^{-1}\chi$$

has a meromorphic continuation in $\mathbb{C}$ with poles in $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and the poles of $R_\chi(\lambda)$ are independent of the choice of χ. These poles are called resonances. On the other hand, the scattering amplitude $a(\lambda, \theta, \omega)$ has a representation involving $R_\chi(\lambda)$, hence $a(\lambda, \theta, \omega)$ also admits a meromorphic continuation in $\mathbb{C}$ and the poles of this continuation are included in the set of resonances. An obstacle K is called non-trapping if the set Σ_∞ is empty. From the general results on propagation of singularities given by Melrose and Sjöstrand [7], it follows that if K is non-trapping, there exist $\epsilon > 0$ and $d > 0$ so that $R_\chi(\lambda)$ has no poles in the domain

$$U_{\epsilon,d} = \{\lambda \in \mathbb{C} : \ 0 \leq \operatorname{Im} \lambda \leq \epsilon \log(1 + |\lambda|) - d\}.$$

Moreover, for non-trapping obstacles we have the following estimate (see [15])

$$\|R_\chi(\lambda)\|_{L^2(\Omega) \longrightarrow L^2(\Omega)} \leq \frac{C}{|\lambda|} e^{\alpha |\operatorname{Im} \lambda|}, \ \forall \lambda \in U_{\epsilon,d}, \ \alpha \geq 0. \tag{6.1}$$

We conjecture that the existence of singularities $t_m \longrightarrow -\infty$ of the scattering kernel $s(t, \theta_m, \omega_m)$ implies that for every $\epsilon > 0$ and every $d > 0$ we have resonances in $U_{\epsilon,d}$.

By using Theorem 6.4, we prove a weaker result assuming an estimate of the scattering amplitude weaker than (6.1)

Theorem 6.5 ([10]). *Suppose that there exist $m \in \mathbb{N}$, $\alpha \geq 0$, $\epsilon > 0$, $d > 0$ and $C > 0$ so that $a(\lambda, \theta, \omega)$ is analytic in $U_{\epsilon,d}$ for all $\lambda \in U_{\epsilon,d}$ we have*

$$|a(\lambda, \theta, \omega)| \leq C(1 + |\lambda|)^m e^{\alpha |\operatorname{Im} \lambda|}, \ \forall (\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}. \tag{6.2}$$

Then if K satisfies $(\mathcal{G})$, there are no trapping weakly non-degenerate (ω, θ) rays in Ω.

It is an open problem to examine the optimal estimate of the scattering amplitude, provided that $a(\lambda, \theta, \omega)$ is analytic in $U_{\epsilon,d}$ for all (ω, θ).

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