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Stability of Delaunay surfaces as steady states for a geometric evolution equation

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1 Introduction

Let $\Gamma_t \subset \mathbb{R}^3$ be a evolving surface with respect to time t. The surface diffusion equation

$$V = -\Delta_{\Gamma_t} H \quad \text{on} \ \Gamma_t \tag{1}$$

is one of the geometric evolution laws, where V is the normal velocity of Γ_t, H is the mean curvature of Γ_t, and Δ_{Γ_t} is the Laplace-Beltrami operator on Γ_t. In our sign convention, the mean curvature H for spheres with outer unit normal is negative.

The mean curvature flow

$$V = H \quad \text{on} \ \Gamma_t \tag{2}$$

is a well-known geometric law and represented as the L^2-gradient flow for the area functional of Γ_t. This implies a variational structure that the area of the surface Γ_t decreases with respect to time t. On the other hand, the surface diffusion equation (1) is the H^{-1}-gradient flow for the area functional of Γ_t, so that this geometric evolution equation has a variational structure that the area of the surface Γ_t decreases with respect to time t whereas the volume of the region enclosed by the surface Γ_t is preserved.

In this paper, we consider the following problem. For $\phi_\pm : \mathbb{R}_+ \to \mathbb{R}$, set

$$\begin{aligned}
&\Pi_\pm = \{(\phi_\pm(|\boldsymbol{\eta}|), \boldsymbol{\eta})^T \mid \boldsymbol{\eta} \in \mathbb{R}^2\},\\
&\Omega = \{(x, \boldsymbol{\eta})^T \mid \phi_-(|\boldsymbol{\eta}|) \le x \le \phi_+(|\boldsymbol{\eta}|),\ \boldsymbol{\eta} \in \mathbb{R}^2\},\\
&\partial\Omega = \Pi_- \cup \Pi_+.
\end{aligned}$$

Note that $\Pi_\pm$ are the axisymmetric surfaces. Let us assume that $\Gamma_t \subset \Omega$ and the motion of Γ_t is governed by

$$\begin{cases}
V = -\Delta_{\Gamma_t} H \quad \text{on} \ \Gamma_t,\\
(N_{\Gamma_t}, N_{\Pi_\pm})_{\mathbb{R}^3} = \cos\theta_\pm \quad \text{on} \ \Gamma_t \cap \Pi_\pm,\\
(\nabla_{\Gamma_t} H, \nu_\pm)_{\mathbb{R}^3} = 0 \quad \text{on} \ \Gamma_t \cap \Pi_\pm,\\
\Gamma_t|_{t=0} = \Gamma_0.
\end{cases} \tag{3}$$

Here, N_{Γ_t} and $N_{\Pi_\pm}$ are the outer unit normals to Γ_t and $\Pi_\pm$, respectively, and $\nu_\pm$ are the outer unit co-normals to $\partial\Gamma_t$ on $\Gamma_t \cap \Pi_\pm$.

Let Γ_* be the steady states for (3) and H_* be the mean curvature of Γ_*. Then Γ_* satisfies

$$\begin{cases} \Delta_{\Gamma_*} H_* = 0 \text{ on } \Gamma_*, \\ (\nabla_{\Gamma_*} H_*, \nu_\pm)_{\mathbb{R}^3} = 0 \text{ on } \Gamma_* \cap \Pi_\pm. \end{cases}$$

Multiplying H_* by the both side of the equation $\Delta_{\Gamma_*} H_* = 0$ and applying the Green's formula, we obtain

$$\|\nabla_{\Gamma_*} H_*\|^2_{L^2(\Gamma_*)} = 0.$$

Thus we see that the steady states of (3) are the constant mean curvature surfaces (CMC surfaces). In this paper, we only consider the axisymmetric CMC surfaces, which is so called the Delaunay surfaces, as the steady states Γ_*. For an axisymmetric perturbation from Γ_*, we derive the eigenvalue problem corresponding to the linearized problem for (3) and obtain the criteria of the stability of Γ_*.

As regards the results on the stability of the Delaunay surfaces as the variational problem for the capillary energy, we refer to Athanassenas [2], Fel and Rubinstein [6, 14], and Vogel [15, 16, 17, 18, 19]. Concerning the results on the stability as steady states for the surface diffusion equation, we refer to Abels, Garcke, and Müller [1], Depner [5], and LeCrone and Simonett [12].

2 The eigenvalue problem

Let Γ_* be a axisymmetric steady states of (3) and set

$$\Gamma_* = \{(x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T \mid s \in [0, d], \zeta \in [0, 2\pi]\},$$

where s is the arc-length parameter of a generating curve $(x_*(s), y_*(s))^T$. In the following theorem, we introduce the representation formula of the Delaunay surfaces with a non-zero constant mean curvature.

Theorem 2.1 ([9, 13]) Let H_* be a constant satisfying $H_* \neq 0$ (assuming $H_* < 0$). Then a generating curve $(x_*(s), y_*(s))^T$ of the Delaunay surface with a constant mean curvature H_* is given by

$$\begin{cases} x_*(s) = \displaystyle\int_0^s \frac{1 - B\sin(2H_*(\sigma - \tau))}{\sqrt{1 + B^2 - 2B\sin(2H_*(\sigma - \tau))}}\, d\sigma, \\ y_*(s) = -\dfrac{1}{2H_*}\sqrt{1 + B^2 - 2B\sin(2H_*(s - \tau))}, \end{cases} \tag{4}$$

where $B \geq 0$ and $\tau \in \mathbb{R}$ are constants.

The Delaunay surface is a cylinder for $B = 0$ (Fig. 1), an unduloid for $0 < B < 1$ (Fig. 2), a series of spheres for $B = 1$ (Fig. 3), and a nodoid for $B > 1$ (Fig. 4).

Figure 1: Cylinder ($B = 0$)

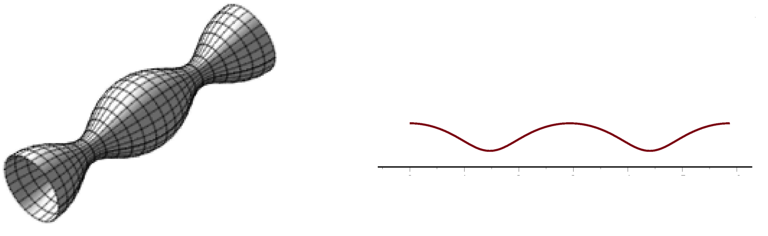


Figure 2: Unduloid ($0 < B < 1$)

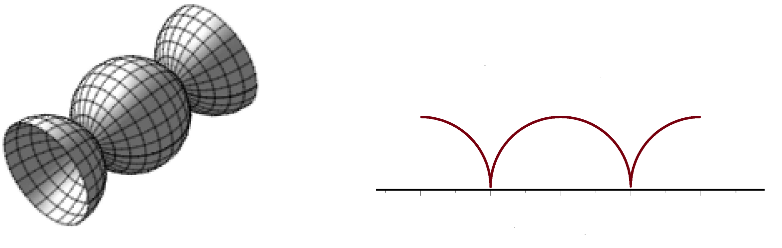


Figure 3: Series of spheres ($B = 1$)

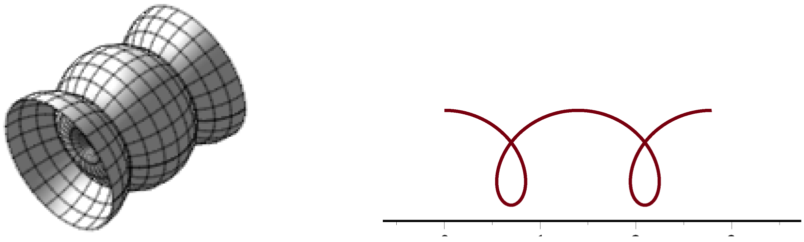


Figure 4: Nodoid ($B > 1$)

Applying an axisymmetric perturbation $v(s,t)$ for the Delaunay surfaces Γ_* and linearizing the nonlinear problem for $v(s,t)$, we have

$$\begin{cases} v_t = -\dfrac{1}{2}\Delta_{\Gamma_*} L[v] \ \text{ for } \ (s,t) \in [0,d] \times [0,T], \\ \partial_s v \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) v = 0 \ \text{ for } \ s = 0, d, \ t \in [0,T], \\ \partial_s L[v] = 0 \ \text{ for } \ s = 0, d, \ t \in [0,T], \end{cases} \tag{5}$$

where $L[v] = \Delta_{\Gamma_*} v + |A_*|^2 v$ with

$$\Delta_{\Gamma_*} = \frac{1}{y_*}\left\{\partial_s(y_*\partial_s) + \frac{1}{y_*}\partial_\zeta^2\right\}, \quad |A_*|^2 = (-x_*'' y_*' + x_*' y_*'')^2 + \left(\frac{x_*'}{y_*}\right)^2,$$

and

$$\kappa_{\Pi_\pm} = \pm\frac{\ddot{\phi}_\pm(y_*)}{\{1 + (\dot{\phi}_\pm(y_*))^2\}^{3/2}}, \quad \kappa_{\Gamma_*} = -x_*'' y_*' + x_*' y_*''.$$

Note that κ_{Π_-} and κ_{Π_-} are the curvature of $x = -\phi_-(y)$ at $y = y_*(0)$ and $x = \phi_+(y)$ at $y = y_*(d)$, respectively, and κ_{Γ_*} is the curvature of the generating curve $(x_*(s), y_*(s))^T$. Taking account of the fact that v is independent of ζ, we have

$$\Delta_{\Gamma_*} v = \frac{1}{y_*}\left\{\partial_s(y_*\partial_s v)\right\}.$$

For this linearized problem the corresponding eigenvalue problem is given by

$$\begin{cases} -\Delta_{\Gamma_*} L[w] = \lambda w \ \text{ for } \ s \in [0,d], \\ \partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \ \text{ at } \ s = 0, d, \\ \partial_s L[w] = 0 \ \text{ at } \ s = 0, d. \end{cases} \tag{6}$$

We say that the steady states Γ_* is linearly stable under an axisymmetric perturbation if and only if all of eigenvalues of (6) are negative.

Set

$$\mathcal{E} = \left\{ w \in H^1(\Gamma_*) \,\middle|\, \int_0^d w\, y_*\, ds = 0 \right\},$$
$$\mathcal{X} = \left\{ w \in (H^1(\Gamma_*))^* \,\middle|\, \langle w, 1 \rangle = 0 \rangle \right\},$$

where $(H^1(\Gamma_*))^*$ is the duality space of $H^1(\Gamma_*)$ and $\langle \cdot, \cdot \rangle$ is the duality pairing $(H^1(\Gamma_*))^*$ and $H^1(\Gamma_*)$. Also, set

$$\mathcal{D}(\mathcal{A}) = \Big\{ w \in H^3(\Gamma_*) \,\Big|\, w \text{ satisfies}$$
$$\partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \ \text{ at } \ s = 0, d,$$
$$\text{and} \int_0^d w\, y_*\, ds = 0 \Big\}$$

and define the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ by

$$\langle \mathcal{A}w, \psi \rangle = \int_0^d \partial_s L[w]\, \partial_s \psi\, y_* ds \quad (w \in \mathcal{D}(\mathcal{A}),\ \psi \in \mathcal{E}).$$

Taking the symmetric bilinear form

$$\begin{aligned} I[w_1, w_2] = & \int_0^d \left\{ \partial_s w_1 \partial_s w_2 - |A_*|^2 w_1 w_2 \right\} y_*\, ds \\ & + y_*(\kappa_{\Pi_+} \csc\theta_+ - \kappa_{\Gamma_*} \cot\theta_+) w_1 w_2 \big|_{s=d} \\ & + y_*(\kappa_{\Pi_-} \csc\theta_- - \kappa_{\Gamma_*} \cot\theta_-) w_1 w_2 \big|_{s=0}, \end{aligned}$$

and the H^{-1}-inner product

$$(w_1, w_2)_{-1} = \int_0^d \partial_s u_{w_1} \partial_s u_{w_2}\, y_* ds,$$

where u_{w_i} is a weak solution of

$$\begin{cases} -\Delta_{\Gamma_*} u_{w_i} = w_i & \text{for } \ s \in (0, d), \\ \partial_s u_{w_i} = 0 & \text{at } \ s = 0, d \end{cases}$$

for $w_i \in \mathcal{X}$, we obtain

$$(\mathcal{A}w, \psi)_{-1} = -I[w, \psi] \quad (\psi \in \mathcal{E}).$$

For the linear operator $\mathcal{A}$ and its eigenvalues, we have the following properties.

(P1) The operator $\mathcal{A}$ is self-adjoint with respect to the H^{-1}-inner product.

(P2) The spectrum of $\mathcal{A}$ contains a countable system of real eigenvalues.

(P3) Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be eigenvalues of $\mathcal{A}$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Then $\{\lambda_n\}_{n\in\mathbb{N}}$ are characterized by

$$\lambda_1 = -\inf_{w\in\mathcal{E}\setminus\{0\}} \frac{I[w,w]}{(w,w)_{-1}}, \qquad \lambda_n = -\sup_{\mathcal{W}\in\Sigma_{n-1}} \inf_{w\in\mathcal{W}^\perp\setminus\{0\}} \frac{I[w,w]}{(w,w)_{-1}}.$$

Here, Σ_n is the class of subspaces of $\mathcal{E}$ with n-dimension and $\mathcal{W}^\perp$ is the orthogonal subspace of $\mathcal{W}$ with respect to the H^{-1}-inner product.

(P4) The eigenvalues of $\mathcal{A}$ depend continuously on $\kappa_{\Pi_\pm}$, $\kappa_{\Gamma_{\rho_*}}$, d, and $\theta_\pm$, and are monotone decreasing with respect to $\kappa_{\Pi_\pm}$.

Concerning proofs, see [5, 8] for (P1) and (P2), and [4, Chapter VI] for (P3) and (P4).

3 Criteria of Stability

If the maximal eigenvalue λ_1 for (6) is negative, the steady states Γ_* are linearly stable under an axisymmetric perturbation. First, we show the following lemma.

Lemma 3.1 Set

$$\Lambda_\pm := \kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm.$$

Then there exists $m > 0$ and $\delta > 0$ such that

$$I[w,w] > 0 \quad (w \in \mathcal{E}\setminus\{0\}),$$

provided that $\Lambda_-, \Lambda_+ > m$ and $d < \delta$.

Regarding a proof, see [10].

Lemma 3.1 implies that there exist $m > 0$ and $\delta > 0$ such that the maximal eigenvalue λ_1 is non-positive, provided that $\kappa_{\Pi_-}, \kappa_{\Pi_+} > m$ and $d < \delta$. That is, all of eigenvalues are non-positive. According to (P4), the eigenvalues depend continuously on the parameters and are monotone decreasing with respect to $\kappa_{\Pi_\pm}$. Thus we want to know the condition for the parameters that the zero is an eigenvalue for the eigenvalue problem (6). Now we consider the zero-eigenvalue problem

$$\Delta_{\Gamma_*} L[w] = 0 \ \text{ for } \ s \in [0,d], \tag{7}$$

$$\partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \ \text{ at } \ s = 0, d, \tag{8}$$

$$\partial_s L[w] = 0 \ \text{ at } \ s = 0, d. \tag{9}$$

Multiplying $L[w]$ by the both side of (7) and integrating it by parts with (9), we have

$$\|\partial_s L[w]\|^2_{L^2(\Gamma_*)} = 0.$$

Hence $L[w]$ must be constants, so that we can obtain the solutions of (7) satisfying the boundary condition (9) if we solve

$$L[w]=0, \quad L[w]=\gamma\,(\neq 0). \tag{10}$$

Let w_1, w_2 be fundamental solutions of $L[v]=0$ and w_3 be a solution of $L[v]=\gamma$. Then a solution of (7) satisfying the boundary condition (9) is represented by

$$w(s)=c_1w_1(s)+c_2w_2(s)+c_3w_3(s), \tag{11}$$

where $c_i\,(i=1,2,3)$ are arbitrary constants. Deriving the condition of parameters that w given by (11) is a non-trivial solution satisfying the boundary condition (8) and

$$\int_0^d v\,y_*ds=0,$$

it gives the condition of parameters that the zero is an eigenvalue for (6). That is, the zero is an eigenvalue if and only if the parameters satisfy

$$\begin{vmatrix} w_1'(0)-\Lambda_-w_1(0) & w_2'(0)-\Lambda_-w_2(0) & w_3'(0)-\Lambda_-w_3(0) \\ w_1'(d)+\Lambda_+w_1(d) & w_2'(d)+\Lambda_+w_2(d) & w_3'(d)+\Lambda_+w_3(d) \\ \int_0^d w_1\,y_*ds & \int_0^d w_2\,y_*ds & \int_0^d w_3\,y_*ds \end{vmatrix}=0, \tag{12}$$

where $\Lambda_\pm=\kappa_{\Pi_\pm}\csc\theta_\pm-\kappa_{\Gamma_*}\cot\theta_\pm$. Setting

$$\boldsymbol{w}(s)=(w_1(s),w_2(s),w_3(s))^T, \quad \boldsymbol{I}(d)=\left(\int_0^d w_1\,y_*ds,\int_0^d w_2\,y_*ds,\int_0^d w_3\,y_*ds\right)^T,$$

(12) is equivalent to

$$A^w\kappa_{\Pi_-}\kappa_{\Pi_+}+B_-^w\kappa_{\Pi_-}+B_+^w\kappa_{\Pi_+}+C^w=0, \tag{13}$$

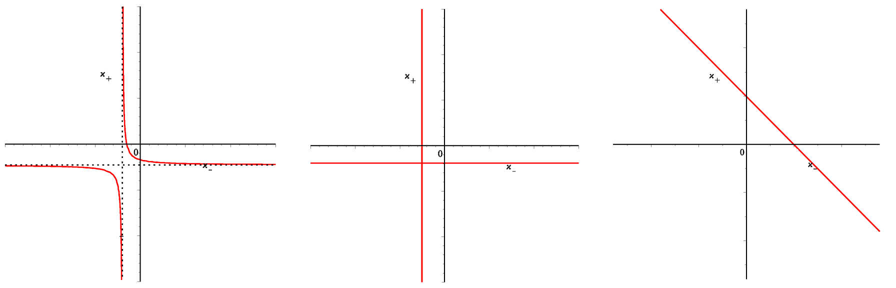
where

$$\begin{aligned}
A^w &= -\,(\boldsymbol{w}(0)\times\boldsymbol{w}(d),\boldsymbol{I}(d))_{\mathbb{R}^3},\\
B_-^w &= \left\{-(\boldsymbol{w}(0)\times\boldsymbol{w}'(d),\boldsymbol{I}(d))_{\mathbb{R}^3}+(\boldsymbol{w}(0)\times\boldsymbol{w}(d),\boldsymbol{I}(d))_{\mathbb{R}^3}\kappa_{\Gamma_*}(d)\cot\theta_+\right\}\sin\theta_+,\\
B_+^w &= \left\{(\boldsymbol{w}'(0)\times\boldsymbol{w}(d),\boldsymbol{I}(d))_{\mathbb{R}^3}+(\boldsymbol{w}(0)\times\boldsymbol{w}(d),\boldsymbol{I}(d))_{\mathbb{R}^3}\kappa_{\Gamma_*}(0)\cot\theta_-\right\}\sin\theta_-,\\
C^w &= \left\{(\boldsymbol{w}'(0)\times\boldsymbol{w}'(d),\boldsymbol{I}(d))_{\mathbb{R}^3}\right.\\
&\qquad \left.-\,(\boldsymbol{w}(0)\times\boldsymbol{w}(d),\boldsymbol{I}(d))_{\mathbb{R}^3}\kappa_{\Gamma_*}(d)\kappa_{\Gamma_*}(0)\cot\theta_+\cot\theta_-\right\}\sin\theta_+\sin\theta_-.
\end{aligned}$$

Then we obtain the following three representaitons of (13).

(I) If $A^w\neq 0$ and $B_-^wB_+^w-A^wC^w\neq 0$,

$$(13) \quad\Leftrightarrow\quad \kappa_{\Pi_+}=-\frac{B_-^w}{A^w}+\frac{\dfrac{B_-^wB_+^w-A^wC^w}{(A^w)^2}}{\kappa_{\Pi_-}-\left(-\dfrac{B_+^w}{A^w}\right)}.$$



(I) $(B_-^w B_+^w - A^w C^w > 0)$ (II) (III)

Figure 5: The configurations of (I), (II), and (III).

(II) If $A^w \neq 0$ and $B_-^w B_+^w - A^w C^w = 0$,

$$(13) \quad \Leftrightarrow \quad \left\{\kappa_{\Pi_-} - \left(-\frac{B_+^w}{A^w}\right)\right\}\left\{\kappa_{\Pi_+} - \left(-\frac{B_-^w}{A^w}\right)\right\} = 0.$$

(III) If $A^w = 0$,

$$(13) \quad \Leftrightarrow \quad B_-^w \kappa_{\Pi_-} + B_+^w \kappa_{\Pi_+} + C^w = 0.$$

The coefficients A^w, $B_\pm^w$, and C^w depend on the configurations of the steady states Γ_*. Thus, let us derive w_i when Γ_* are the Delaunay surfaces with non-zero constant mean curvature. Since the generating curves $(x_*(s), y_*(s))^T$ are given by (4), the coefficient $|A_*|^2$ in the operator $L[w]$ and κ_{Γ_*} in the boundary condition (8) are

$$|A_*|^2 = \frac{4H_*^2\{B^2(B - \sin(2H_*(s-\tau))^2 + (1 - B\sin(2H_*(s-\tau))^2\}}{(1 + B^2 - 2B\sin(2H_*(s-\tau)))^2},$$
$$\kappa_{\Gamma_*} = \frac{2BH_*(B - \sin(2H_*(s-\tau))}{1 + B^2 - 2B\sin(2H_*(s-\tau))}.$$

Solving $L[w] = 0$ and $L[w] = 1$ (we choose 1 as γ in (10)), we obtain

$$\begin{cases} w_1(s) = \dfrac{\cos(2H_*(s-\tau))}{\sqrt{1 + B^2 - 2B\sin(2H_*(s-\tau))}}, \\ w_2(s) = \sin(2H_*(s-\tau)) + 2H_*\left\{\dfrac{1+B^2}{2} I_1(s) - \dfrac{1}{2} I_2(s)\right\}, \\ w_3(s) = \dfrac{1}{4H_*^2} + \dfrac{B}{2H_*} I_1(s)\, w_1(s), \end{cases} \qquad (14)$$

where

$$I_1(s) = I_1(s; H_*, B, \tau) := \int_0^s \frac{1}{\sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}}\,d\sigma,$$
$$I_2(s) = I_2(s; H_*, B, \tau) := \int_0^s \sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}\,d\sigma.$$

Set

$$H_*^+ = -H_*\,(>0), \quad \alpha = H_*^+\tau + \frac{\pi}{4}, \quad \beta = H_*^+\tau - \frac{\pi}{4}$$

and let $\alpha \in [-\pi/2, \pi/2)$, $\beta < 0$, and

$$\begin{cases} -\dfrac{\pi}{2} + m\pi < H_*^+ s - \alpha < -\dfrac{\pi}{2} + (m+1)\pi \quad (m \in \mathbb{N}\cup\{0\}) \ \text{ for } B \neq 1, \\ 0 < H_*^+ s - \beta < \pi \ \text{ for } B = 1. \end{cases}$$

Then $I_1(s; -H_*^+, B, \tau)$ and $I_2(s; -H_*^+, B, \tau)$ are represented by

$$I_1(s; -H_*^+, B, \tau)$$
$$= \begin{cases} \dfrac{1}{H_*^+(1+B)}\{2mK(k) + (-1)^m F(\sin(H_*^+ s - \alpha); k) - F(\sin(-\alpha); k)\} \quad (B \neq 1), \\ \dfrac{1}{2H_*^+}\left\{\log\left(\tan\left(\dfrac{H_*^+ s - \beta}{2}\right)\right) - \log\left(\tan\left(-\dfrac{\beta}{2}\right)\right)\right\} \quad (B = 1), \end{cases}$$
$$I_2(s; -H_*^+, B, \tau)$$
$$= \begin{cases} \dfrac{1+B}{H_*^+}\{2mE(k) + (-1)^m E(\sin(H_*^+ s - \alpha); k) - E(\sin(-\alpha); k)\} \quad (B \neq 1), \\ \dfrac{2}{H_*^+}\{\cos\beta - \cos(H_*^+ s - \beta)\} \quad (B = 1), \end{cases}$$

where $k = 2\sqrt{B}/(1+B)$, $K(k)$ and $E(k)$ are complete elliptic integrals of the 1st and 2nd kind, and $F(\eta; k)$ and $E(\eta; k)$ are incomplete elliptic integrals of the 1st and 2nd kind. In this paper, the elliptic integrals are given by

$$K(k) = \int_0^1 \frac{1}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}}\,d\xi, \quad E(k) = \int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\,d\xi,$$
$$F(\eta; k) = \int_0^\eta \frac{1}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}}\,d\xi, \quad E(\eta; k) = \int_0^\eta \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\,d\xi.$$

Substituting (14) for (13), we are led to

$$A^D(H_*^+, B, d, \tau)\kappa_{\Pi_-}\kappa_{\Pi_+} + B_-^D(H_*^+, B, d, \tau, \theta_+)\kappa_{\Pi_-} + B_+^D(H_*^+, B, d, \tau, \theta_-)\kappa_{\Pi_+}$$
$$+ C^D(H_*^+, B, d, \tau, \theta_+, \theta_-) = 0. \tag{15}$$

The precise forms of A^D, $B^D_\pm$, and C^D are obtained by using Maple 17. Here, we show only the form of A^D:

$$
\begin{aligned}
&A^D(H_*^+, B, d, \tau)\\
&= \frac{1}{8(H_*^+)^3 PQ}\Big[(H_*^+)^2(1-B^2)^2 I_1^2 \cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\\
&\quad - 4(H_*^+)^2(1+B^2) I_1 I_2 \cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\\
&\quad + 3(H_*^+)^2 I_2^2 \cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\\
&\quad + 2H_*^+(1+B^2) I_1\{P\sin(2H_*^+\tau)\cos(2H_*^+(d-\tau)) + Q\cos(2H_*^+\tau)\sin(2H_*^+(d-\tau))\}\\
&\quad - 4H_*^+ B I_1\{P\cos(2H_*^+(d-\tau)) - Q\cos(2H_*^+\tau)\}\\
&\quad - 4H_*^+ I_2\{P\sin(2H_*^+\tau)\cos(2H_*^+(d-\tau)) + Q\cos(2H_*^+\tau)\sin(2H_*^+(d-\tau))\}\\
&\quad + 2PQ\{1 + \sin(2H_*^+\tau)\sin(2H_*^+(d-\tau))\}\\
&\quad - (P^2+Q^2)\cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
P(H_*^+, B, \tau) &= \sqrt{1 + B^2 - 2B\sin(2H_*^+\tau)},\\
Q(H_*^+, B, d, \tau) &= \sqrt{1 + B^2 + 2B\sin(2H_*^+(d-\tau))}.
\end{aligned}
$$

Moreover, by the help with Maple 17, we have

$$
\begin{aligned}
&B_-^D(H_*^+, B, d, \tau, \theta_+) B_+^D(H_*^+, B, d, \tau, \theta_-) - A^D(H_*^+, B, d, \tau) C^D(H_*^+, B, d, \tau, \theta_+, \theta_-)\\
&= \frac{1}{16(H_*^+)^4 PQ}\Big[H_*^+\{(1+B^2)(1+\sin(2H_*^+\tau)\sin(2H_*^+(d-\tau))) - (P^2+Q^2)\} I_1\\
&\quad + H_*^+(3 - \sin(2H_*^+(d-\tau))\sin(2H_*^+\tau)) I_2\\
&\quad - P\cos(2H_*^+\tau)\sin(2H_*^+(d-\tau)) - Q\sin(2H_*^+\tau)\cos(2H_*^+(d-\tau))\Big]^2 \geq 0.
\end{aligned}
$$

Theorem 3.1 Set

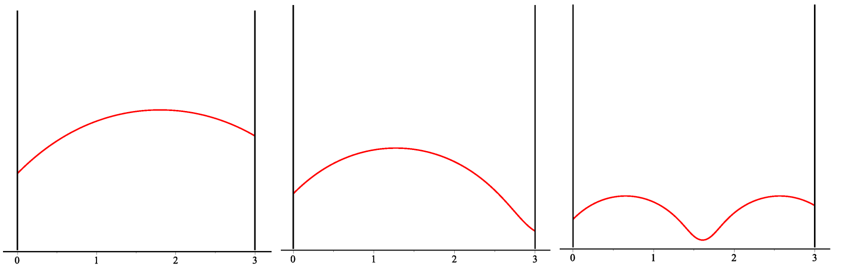
$$
\begin{aligned}
&D(\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm)\\
&:= A^D(H_*^+, B, d, \tau)\kappa_{\Pi_-}\kappa_{\Pi_+} + B_-^D(H_*^+, B, d, \tau, \theta_+)\kappa_{\Pi_-} + B_+^D(H_*^+, B, d, \tau, \theta_-)\kappa_{\Pi_+}\\
&\quad + C^D(H_*^+, B, d, \tau, \theta_+, \theta_-),
\end{aligned}
$$

and let q_1 be the value of $H_*^+ d$ which is the 1st zero-point of A^D. If the parameters $\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm$ satisfy

$$
\hat{D}(\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm) > 0, \quad \kappa_{\Pi_-} > -\frac{B_+^D(H_*^+, B, d, \tau, \theta_-)}{A^D(H_*^+, B, d, \tau)}, \quad \text{and} \quad H_*^+ d < q_1, \tag{16}
$$

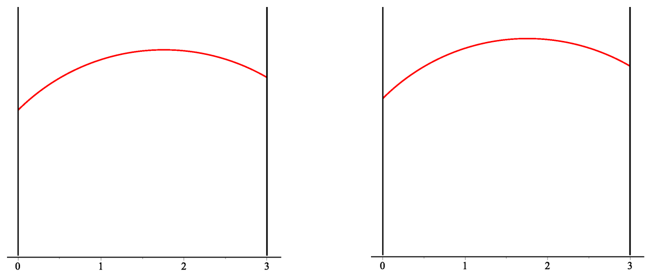
then the Delaunay surfaces are linearly stable under an axisymmetric perturbation.

Theorem 3.2 If $H_*^+ d \geq q_1$, then there are no pairs of $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$ such that the Delaunay surfaces are stable.



$H_*^+ d \approx 1.6348$ $H_*^+ d \approx 2.4759$ $H_*^+ d \approx 4.7764$

A part of unduloids ($B = 0.75$).



$H_*^+ d \approx 1.3089$ A part of sphere.

$H_*^+ d \approx 1.2720$ A part of nodoid ($B = 1.05$).

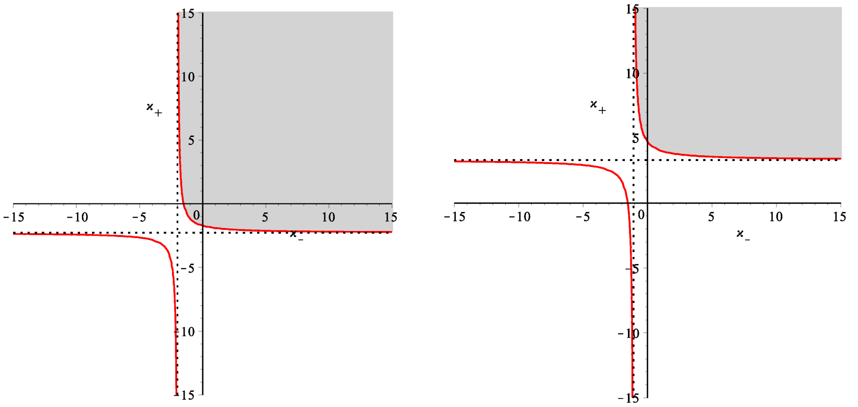
Figure 6: The Delaunay surfaces with $\theta_- = \dfrac{\pi}{4}$ and $\theta_+ = \dfrac{\pi}{3}$.

4 Examples

Concerning criteria of stability for cylinders and unduloids with $\tau = \pi/(4H_*^+)$ under $\theta_\pm = \pi/2$, see [10, 11]. In this paper, we consider the stability of unduloids, sphere, and nodoid given by Fig. 6.

For unduloids in this setting, we can obtain $q_1 \approx 2.6310$ by the help with Maple 17. Thus, by Theorem 3.2, the unduloid with $H_*^+ d \approx 4.7764$ is unstable. In the cases $H_*^+ d \approx 1.6348$ and $H_*^+ d \approx 2.4759$, the criteria of the unduloids are given by Fig. 7. By Theorem 3.1, unduloids are stable under an axisymmetric perturbation, provided that $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$ is included in the gray parts in Fig. 7. For $H_*^+ d \approx 1.6348$, $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0, 0)$ is included in the gray part, so that the unduloid with $H_*^+ d \approx 1.6348$ is stable under an axisymmetric perturbation. On the other hand, for $H_*^+ d \approx 2.4759$, $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0, 0)$ is not included in the gray part. Thus the unduloid with $H_*^+ d \approx 2.4759$ is unstable.

For sphere in this setting, we consider the problem in the interval $[0, 2.3561]$. In this



$H_*^+ d \approx 1.6348$ $\quad$ $H_*^+ d \approx 2.4759$

Figure 7: The criteria of unduloids with $H_*^+ d \approx 1.6348$ and $H_*^+ d \approx 2.4759$.

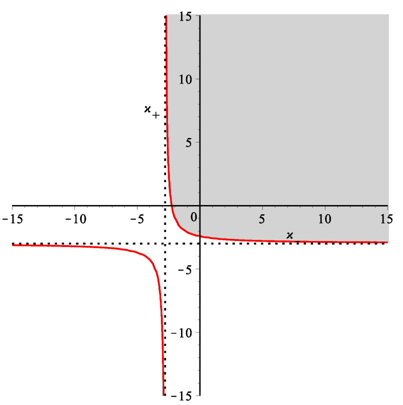


Figure 8: The criterion of the sphere with $H_*^+ d \approx 1.3089$.

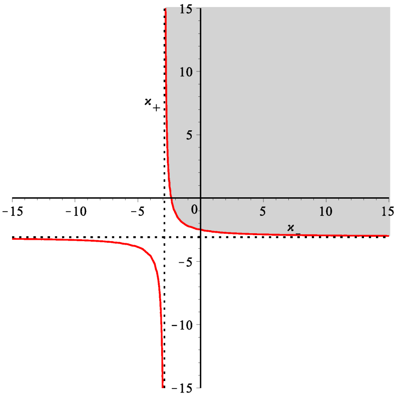


Figure 9: The criterion of the nodoid with $H_*^+ d \approx 1.2720$.

interval, we have no value of $H_*^+ d$ which is the zero-point of A^D. Thus we can jude the stability by using Fig. 8. $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0, 0)$ is included in the gray part, so that the sphere with $H_*^+ d \approx 1.3089$ is stable under an axisymmetric perturbation.

For nodoid in this setting, we can obtain $q_1 \approx 2.3389$ by the help with Maple 17. The criterion of the nodoid with $H_*^+ d \approx 1.2720$ is given by Fig. 9. Then we see that $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0, 0)$ is included in the gray part. Thus the nodoid with $H_*^+ d \approx 1.2720$ is stable under an axisymmetric perturbation.

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