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Splitting formulas for the rational lift of the Kontsevich integral: a survey.

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1 Introduction

In [Kri00], Kricker constructed a rational lift of the Kontsevich integral of knots in integral homology 3-spheres ($\mathbb{Z}$-spheres). In [GK04], he proved with Garoufalidis that his construction provides an invariant of knots in $\mathbb{Z}$-spheres. They also proved that the Kricker invariant satisfies some splitting formulas with respect to the so-called null-move. Kricker's construction easily generalizes to null-homologous knots in rational homology 3-spheres ($\mathbb{Q}$-spheres). The question arises to know wether one can get splitting formulas for the Kricker invariant of these knots with respect to null Lagrangian-preserving surgery, a move which generalizes the null-move.

In [CHM08], Cheptea, Habiro and Massuyeau extended the LMO invariant of $\mathbb{Q}$-spheres to a functor defined on a category of Lagrangian cobordisms. Massuyeau [Mas15] used this functor to obtain splitting formulas for the LMO invariant of $\mathbb{Q}$-spheres with respect to Lagrangian-preserving surgeries.

In [Mou17], we extend the LMO functorial invariant of Cheptea-Habiro-Massuyeau to a category of Lagrangian cobordisms with paths, inserting the Kricker's idea in the construction. We obtain a functorial invariant from which the Kricker invariant of null-homologous knots in $\mathbb{Q}$-spheres is recovered. Following Massuyeau, we use the functoriality to obtain splitting formulas for our invariant and, as a consequence, for the Kricker invariant. This article is a survey of this construction.

Notations and conventions. For $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$, a $\mathbb{K}$-*sphere*, (resp. a $\mathbb{K}$-*cube*, a *genus* g $\mathbb{K}$-*handlebody*) is a 3-manifold, compact and oriented, which has the same homology with coefficients in $\mathbb{K}$ as the standard 3-sphere (resp. 3-cube, genus g handlebody).

2 Statement of the splitting formulas

We first give the definitions we need to state the splitting formulas for the Kricker lift of the Kontsevich integral.

Null LP-surgeries. The *Lagrangian* $\mathcal{L}_C$ of a $\mathbb{Q}$-handlebody C is the kernel of the map $i_*: H_1(\partial C;\mathbb{Q}) \to H_1(C;\mathbb{Q})$ induced by the inclusion. The Lagrangian of a $\mathbb{Q}$-handlebody C is indeed a Lagrangian subspace of $H_1(\partial C;\mathbb{Q})$ with respect to the intersection form. A *Lagrangian-preserving pair*, or *LP-pair*, is a pair $\mathbf{C} = \left(\frac{C'}{C}\right)$ of $\mathbb{Q}$-handlebodies equipped with a homeomorphism $h: \partial C \xrightarrow{\cong} \partial C'$ such that $h_*(\mathcal{L}_C) = \mathcal{L}_{C'}$.

Given a 3-manifold M, a *Lagrangian-preserving surgery*, or *LP-surgery*, on M is a family $\mathbf{C} = (\mathbf{C}_1, \dots, \mathbf{C}_n)$ of LP-pairs such that the C_i are embedded in M and disjoint. The manifold obtained from M by LP-surgery on $\mathbf{C}$ is defined as

$$M(\mathbf{C}) = (M \setminus (\sqcup_{1\le i\le n} C_i)) \cup_\partial (\sqcup_{1\le i\le n} C_i').$$

Let M be a 3-manifold and let K be a disjoint union of knots or paths properly embedded in M. A *$\mathbb{Q}$-handlebody null in $M \setminus K$* is a $\mathbb{Q}$-handlebody $C \subset M \setminus K$ such that the map $i_*: H_1(C;\mathbb{Q}) \to H_1(M\setminus K;\mathbb{Q})$ induced by the inclusion has a trivial image. A *null LP-surgery* on (M,K) is an LP-surgery $\mathbf{C} = (\mathbf{C}_1, \dots, \mathbf{C}_n)$ on $M \setminus K$ such that each C_i is null in $M \setminus K$. The pair obtained by surgery is denoted $(M,K)(\mathbf{C})$.

The tensor $\mu(\mathbf{C})$. Given an LP-pair $\mathbf{C} = \left(\frac{C'}{C}\right)$, define the associated *total manifold* $\mathcal{C} = (-C) \cup C'$ and define

$$\mu(\mathbf{C}) \in \mathrm{hom}(\Lambda^3 H^1(\mathcal{C};\mathbb{Q}),\mathbb{Q}) \cong \Lambda^3 H_1(\mathcal{C};\mathbb{Q})$$

by associating with a triple of cohomology classes the evaluation of their triple cup products on the fondamental form of $\mathcal{C}$. For a family $\mathbf{C} = (\mathbf{C}_1, \dots, \mathbf{C}_n)$ of LP-pairs, let $\mathcal{C} = \mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n$ and set:

$$\mu(\mathbf{C}) = \mu(\mathcal{C}_1) \otimes \dots \otimes \mu(\mathcal{C}_n) \ \in \otimes_{i=1}^n \Lambda^3 H_1(\mathcal{C}_i;\mathbb{Q}) \subset S^n\Lambda^3 H_1(\mathcal{C};\mathbb{Q}),$$

where we use the natural identification $H_1(\mathcal{C};\mathbb{Q}) \cong \oplus_{i=1}^n H_1(\mathcal{C}_i;\mathbb{Q})$.

The bilinear form $\ell_{(S,\kappa)}(\mathbf{C})$. Let (S,κ) be a $\mathbb{Q}$*SK-pair*, *i.e.* a pair made of a $\mathbb{Q}$-sphere S and a null-homologous knot $\kappa \subset S$. Let E be the *exterior of κ in S*. Let $\tilde{E}$ be the maximal free abelian covering of E. Given two knots ζ and ξ in $\tilde{E}$ whose projections in E are disjoint, denote $\mathrm{lk}_e(\zeta,\xi) \in \mathbb{Q}(t)$ their equivariant linking number.

Let $\mathbf{C} = (\mathbf{C}_1, \dots, \mathbf{C}_n)$ be a null LP-surgery on (S,κ). Let $\mathcal{C} = \mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n$ be the disjoint union of the associated total manifolds. Fix a lift $\tilde{C}_i$ of each C_i in $\tilde{E}$. We will define a *hermitian* form:

$$\ell_{(S,\kappa)}(\mathbf{C}) : H_1(\mathcal{C};\mathbb{Q}) \times H_1(\mathcal{C};\mathbb{Q}) \to \mathbb{Q}(t),$$

i.e. a $\mathbb{Q}$-bilinear form such that reversing the order of the arguments changes t to t^{-1}. Let $a \in H_1(\mathcal{C}_i;\mathbb{Q})$ and $b \in H_1(\mathcal{C}_j;\mathbb{Q})$ be homology classes that can be represented by simple closed curves $\alpha \subset \partial C_i$ and $\beta \subset \partial C_j$, disjoint if $i = j$. Note that such homology classes generate $H_1(\mathcal{C};\mathbb{Q})$ over $\mathbb{Q}$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the copies of α and β in $\tilde{C}_i$ and $\tilde{C}_j$. Set:

$$\ell_{(S,\kappa)}(\mathbf{C})(a,b) = \mathrm{lk}_e(\tilde{\alpha},\tilde{\beta}).$$

We get a well-defined hermitian form $\ell_{(S,\kappa)}(\mathbf{C})$ associated with a choice of lifts of the C_i's. We will keep this choice implicit; the statement of Theorem 2.1 is valid for any such choice.

Diagrammatic representations. Let V be a rational vector space. A *V-colored Jacobi diagram* is a unitrivalent graph whose trivalent vertices are oriented and whose univalent vertices are labelled by V, where an *orientation* of a trivalent vertex is a cyclic order of the three edges that meet at this vertex – fixed as  in the pictures. Set:

$$\mathcal{A}_{\mathbb{Q}}(V)=\frac{\mathbb{Q}\langle V\text{-colored Jacobi diagrams}\rangle}{\mathbb{Q}\langle \text{AS, IHX, LV}\rangle},$$

where the relations are depicted in Figure 1. A symmetric tensor in $S^n\Lambda^3V$ can be

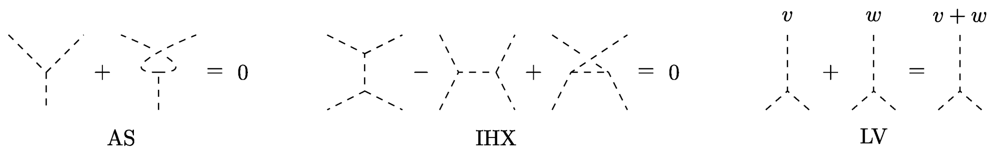
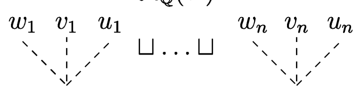


Figure 1: Relations AS, IHX and LV on Jacobi diagrams.

represented by a Jacobi diagram *via* the following embedding.

$$\begin{array}{ccc} S^n\Lambda^3V & \to & \mathcal{A}_{\mathbb{Q}}(V) \\ (u_1\wedge v_1\wedge w_1)\dots(u_n\wedge v_n\wedge w_n) & \mapsto & \end{array}$$



Now define a *$\mathbb{Q}(t)$-beaded Jacobi diagram* as a trivalent graph whose vertices are oriented and whose edges are oriented and labelled by $\mathbb{Q}(t)$. Set:

$$\widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(\varnothing)=\frac{\mathbb{Q}\langle \mathbb{Q}(t)\text{-beaded Jacobi diagrams}\rangle}{\mathbb{Q}\langle \text{AS, IHX, LE, Hol, OR}\rangle},$$

where the relations are depicted in Figures 1 and 2, with the IHX relation defined with the

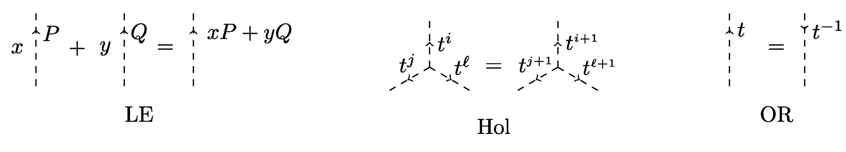


Figure 2: Relations LE, Hol and OR on Jacobi diagrams.

central edge labelled by 1. Define the *i-degree*, or *internal degree*, of any Jacobi diagram as its number of trivalent vertices. Given a hermitian form $\ell : V\times V\to\mathbb{Q}(t)$, one can *glue with ℓ* some legs of a V-colored Jacobi diagram as depicted in Figure 3. If n is even, one can pairwise glue all legs of an i-degree n V-colored Jacobi diagram in order to get an element of $\widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(\varnothing)$. This latter space is the target space of the Kricker invariant $\tilde{Z}$ of $\mathbb{Q}$SK-pairs.

We can now state the splitting formulas for the invariant $\tilde{Z}$ with respect to null LP-surgeries.

Figure 3: Gluing some legs of a Jacobi diagram with ℓ.

Theorem 2.1 *Let (S,κ) be a $\mathbb{Q}$SK-pair. Let $\mathbf{C}=(\mathbf{C}_1,\dots,\mathbf{C}_n)$ be a null LP-surgery on (S,κ). Then:*

$$\sum_{I\subset\{1,\dots,n\}}(-1)^{|I|}\tilde{Z}\left((S,\kappa)(\mathbf{C}_I)\right)\equiv_n\left(\begin{array}{c}\textit{sum of all ways of gluing all legs}\\ \textit{of } \mu(\mathbf{C}) \textit{ with } \ell_{(S,\kappa)}(\mathbf{C})/2\end{array}\right),$$

where $\equiv_n$ means "equal up to i-degree at least $n+1$ terms".

3 Strategy

In this Section, we give an overview of the strategy developed in [Mou17] to prove Theorem 2.1.

The idea is to construct a functorial LMO invariant defined on a category of Lagrangian cobordisms with paths. The morphisms of this category are cobordisms between compact surfaces with one boundary component, satisfying a Lagrangian-preserving condition, with finitely many disjoint paths with fixed extremities which we think of as knots with a fixed part on the boundary. This category is equivalent to a category of bottom-top tangles in $\mathbb{Q}$-cubes, whose top part has a trivial linking matrix, with paths with fixed extremities. These bottom-top tangles can be viewed as morphisms in a category of (general) tangles with paths in $\mathbb{Q}$-cubes, with an important difference in the composition law. Now a tangle with paths in a $\mathbb{Q}$-cube can be expressed as the result of a surgery on a link in a tangle with trivial paths – segment lines – in $[-1,1]^3$. To sum up, with a Lagrangian cobordism with paths, we associate a tangle with disks – whose boundaries define the paths – in $[-1,1]^3$ with a surgery link. This is represented in the first line of the scheme in Figure 4. We initiate the construction of the invariant at the "tangle with disks" level.

On the above mentioned categories, we define functorial invariants valued in categories of Jacobi diagrams with beads, *i.e.* unitrivalent graphs whose univalent vertices are labelled by some finite set or embedded in some 1-manifold – the squeleton –, and whose edges are labelled (beaded) by powers of t, polynomials in $\mathbb{Q}[t^{\pm1}]$ or rational functions in $\mathbb{Q}(t)$. At the first step, we define a functor $Z^\bullet$ on the category of tangles with disks by applying the Kontsevich integral and adding a bead $t^{\pm1}$ on the squeleton when the corresponding component meets a disk of the tangle. At a second step, we apply the invariant $Z^\bullet$ to surgery presentations of tangles with paths in $\mathbb{Q}$-cubes. We use the formal Gaussian integration methods introduced by Bar-Natan, Garoufalidis, Rozansky and Thurston in [ÅI02, ÅII02] and adapted to the beaded setting in [Kri00, GK04]. We get a functor Z on the category of tangles with paths in $\mathbb{Q}$-cubes. At the last step, given

$$\begin{tikzcd}[column sep=large, row sep=huge]
\boxed{\begin{array}{c}\text{cobordisms with paths}\\ \wr \\ \text{bottom-top tangles}\\ \text{with paths}\end{array}} \arrow[r, dashed] \arrow[d, "\tilde{Z}"] & \boxed{\begin{array}{c}\text{tangles with paths}\\ \text{in } \mathbb{Q}\text{-cubes}\end{array}} \arrow[r, dashed] \arrow[d, "Z"] & \boxed{\begin{array}{c}\text{tangles with disks}\\ \text{in } [-1,1]^3\end{array}} \arrow[d, "Z^\bullet"] \\
\boxed{\begin{array}{c}\mathbb{Q}(t)\text{-beaded}\\ \text{top-substantial}\\ \text{Jacobi diagrams}\end{array}} & \boxed{\begin{array}{c}\mathbb{Q}(t)\text{-winding}\\ \text{Jacobi diagrams}\end{array}} \arrow[l, dashed, "\textit{normalization}"'] & \boxed{\begin{array}{c}\mathbb{Q}[t^{\pm 1}]\text{-winding}\\ \text{Jacobi diagrams}\end{array}} \arrow[l, dashed, "\textit{Formal Gaussian Integration}"']
\end{tikzcd}$$

Figure 4: Scheme of construction for the invariant $\tilde{Z}$.

a Lagrangian cobordism with paths, we apply Z to the associated bottom-top tangles with paths and normalize it following [CHM08] to obtain a functor $\tilde{Z}$ on the category of Lagrangian cobordisms with paths. Functoriality allows to prove splitting formulas for this invariant with respect to null Lagrangian-preserving surgeries.

Given a Lagrangian cobordism with one path between genus 0 surfaces, *i.e.* a $\mathbb{Q}$-cube with one path, one can glue a 3-ball to the boundary to get a $\mathbb{Q}$-sphere with a knot. In this way, the functor $\tilde{Z}$ provides an invariant of $\mathbb{Q}$SK-pairs which coincides with the Kricker invariant for knots in $\mathbb{Z}$-spheres. Splitting formulas for this invariant are deduced from the splitting formulas for the functor $\tilde{Z}$.

3.1 Preliminaries: Jacobi diagrams

For a compact oriented 1-manifold X and a finite set C, a *Jacobi diagram on* (X, C) is a unitrivalent graph whose trivalent vertices are oriented and whose univalent vertices are embedded in X or labelled by C, where an orientation of a trivalent vertex is a cyclic order of the three edges that meet at this vertex – fixed as in the pictures. The manifold X is the *squeleton* of the diagram. Next, let R be the ring $\mathbb{Q}[t^{\pm 1}]$ or $\mathbb{Q}(t)$. An *R-beaded Jacobi diagram on* (X, C) is a Jacobi diagram on (X, C) whose graph edges are oriented and labelled by R. Last, an *R-winding Jacobi diagram on* (X, C) is an R-beaded Jacobi diagram on (X, C) whose squeleton is viewed as a union of edges – defined by the embedded vertices – that are labelled by powers of t, with the condition that the product of the labels on each component of X is 1. As defined in the introduction, the i-degree of a trivalent diagram is its number of trivalent vertices. Set:

$$\mathcal{A}(X, *_C) = \frac{\mathbb{Q}\langle \text{Jacobi diagrams on } (X, C)\rangle}{\mathbb{Q}\langle \text{AS, IHX, STU}\rangle},$$

$$\widetilde{\mathcal{A}}_R(X, *_C) = \frac{\mathbb{Q}\langle R\text{-beaded Jacobi diagrams on } (X,C)\rangle}{\mathbb{Q}\langle \text{AS, IHX, STU, LE, OR, Hol}\rangle},$$

$$\widetilde{\mathcal{A}}^{\mathrm{w}}_R(X, *_C) = \frac{\mathbb{Q}\langle R\text{-winding Jacobi diagrams on } (X,C)\rangle}{\mathbb{Q}\langle \text{AS, IHX, STU, LE, OR, Hol, Hol}_{\mathrm{W}}\rangle},$$

with the relations in Figures 1, 2 and 5, where the IHX relation for beaded and winding diagrams is defined with the central edge labelled by 1. In the pictures, the squeleton

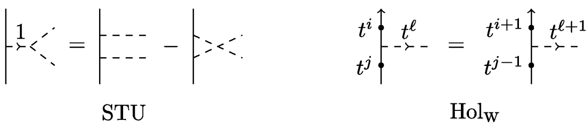


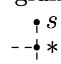
Figure 5: Relations STU and Hol$_{\mathrm{W}}$ on Jacobi diagrams.

is represented with full lines and the graph with dashed lines. We indeed consider the i-degree completion of these vector spaces, keeping the same notation.

For a finite set S, denote by $\uparrow_S$ (resp. $\circlearrowleft_S$) the manifold made of $|S|$ intervals (resp. circles) indexed by the elements of S. We have a formal PBW isomorphism (see [BN95, Theorem 8]):

$$\chi_S : \widetilde{\mathcal{A}}^{\mathrm{w}}_R(X, *_{C\cup S}) \xrightarrow{\cong} \widetilde{\mathcal{A}}^{\mathrm{w}}_R(X \cup \uparrow_S, *_C).$$

For a Jacobi diagram D, the image $\chi_S(D)$ is the average of all possible ways to attach the s-colored vertices of D on the corresponding s-indexed interval in $\uparrow_S$ for each $s \in S$. When $|S| = 1$, closing the S-labelled component gives an isomorphism from $\widetilde{\mathcal{A}}^{\mathrm{w}}_R(X \cup \uparrow_S, *_C)$ to $\widetilde{\mathcal{A}}^{\mathrm{w}}_R(X \cup \circlearrowleft_S, *_C)$ [BN95, Lemma 3.1]. However, this isomorphism does not hold for $|S| > 1$. To recover an isomorphism onto $\widetilde{\mathcal{A}}^{\mathrm{w}}_R(X \cup \circlearrowleft_S, *_C)$, we need additional relations.

Given a winding Jacobi diagram D on $(X, C \cup S)$, and a univalent vertex $*$ of D labelled by $s \in S$, define the associated *link relation* as the vanishing of the sum of all diagrams obtained from D by gluing the vertex $*$ on the edges adjacent to a univalent s-labelled vertex, as follows:  , see Figure 6. Given a winding Jacobi diagram D

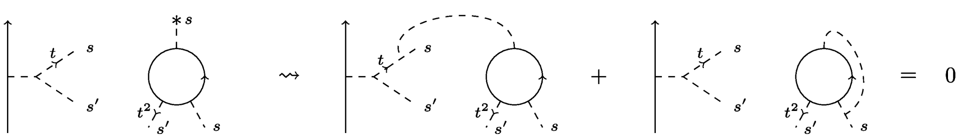


Figure 6: A link relation.

on $(X, C \cup S)$, a label $s \in S$ and an integer k, the associated *winding relation* identifies D with the diagram obtained from D by *pushing t^k at each s-labelled vertex*, *i.e.* by multiplying the label of each edge adjacent to a univalent s-labelled vertex by t^k if the orientation of the edge goes backward the vertex and by t^{-k} otherwise, see Figure 7.

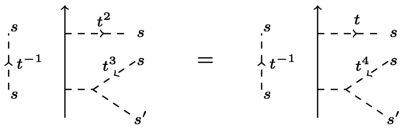


Figure 7: A winding relation.

Denote $\widetilde{\mathcal{A}}^{\mathrm{w}}_R(X, *_C, \circledast_S)$ the quotient of $\widetilde{\mathcal{A}}^{\mathrm{w}}_R(X, *_{C\cup S})$ by all link and winding relations on S-labelled vertices. When some of the sets X, C, S are empty, we simply drop the corresponding notation, mentionning $\varnothing$ only when they are all empty.

Proposition 3.1 *The isomorphism* $\chi_S : \widetilde{\mathcal{A}}^{w}_R(X, *_{C\cup S}) \xrightarrow{\cong} \widetilde{\mathcal{A}}^{w}_R(X \cup \uparrow_S, *_C)$ *descends to an isomorphism:*

$$\chi_S : \widetilde{\mathcal{A}}^{w}_R(X, *_C, \circledast_S) \xrightarrow{\cong} \widetilde{\mathcal{A}}^{w}_R(X \cup \circlearrowleft_S, *_C).$$

We now define a formal Gaussian integration along S on $\widetilde{\mathcal{A}}^{\mathrm{w}}_R(X, *_{C\cup S})$.

Definition 3.2 A (beaded, winding) Jacobi diagram on $(X, C \cup S)$ is *substantial* if it has no *strut*, *i.e.* no isolated dashed edge. It is S*-substantial* if it has no S*-strut*, *i.e.* no strut with both vertices labelled in S.

Given two (beaded, winding) Jacobi diagrams D and E on $(X, C \cup S)$, one of whose is S-substantial, define $\langle D, E\rangle_S$ as the sum of all diagrams obtained by gluing all s-colored vertices of D with all s-colored vertices of E for all $s \in S$ – if the numbers of s-colored vertices in D and E do not match for some $s \in S$, then $\langle D, E\rangle_S = 0$. In the beaded and winding cases, we must precise the orientation and label of the created edges. Such an edge is the gluing of two or three edges in the initial diagrams. Fix arbitrarily the orientation of the new edge. Let $P(t)$ (resp. $Q(t)$) be the product of the labels of the initial edges whose orientation coincides (resp. does not coincide). Define the label of the new edge as $P(t)Q(t^{-1})$, see Figure 8.

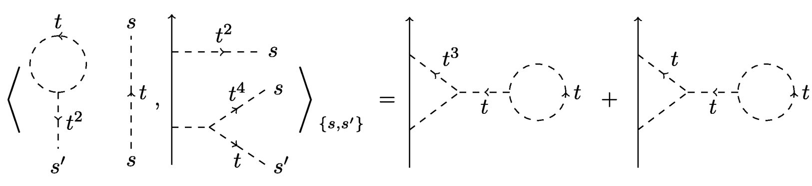
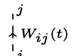


Figure 8: Bracketting diagrams.

Notation 3.3 *If* $W = (W_{ij}(t))_{i,j\in S}$ *is an* (S, S)*-matrix with coefficients in* $\mathbb{Q}(t)$*, we also denote* $W = \sum_{i,j\in S}$  .

Definition 3.4 An element $G \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(X, *_{C\cup S})$ is *Gaussian* if $G = \exp_{\sqcup}(\frac{1}{2}W(t)) \sqcup H$ where $W(t)$ is an (S,S)-matrix with coefficients in $\mathbb{Q}[t^{\pm 1}]$ and H is S-substantial. If $\det(W(t)) \neq 0$, G is *non degenerate* and we set:

$$\int_S G = \langle \exp_{\sqcup}(-\frac{1}{2}W^{-1}(t)), H \rangle_S \ \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}(t)}(X, *_C).$$

Lemma 3.5 *If $G \in \widetilde{\mathcal{A}}^{w}_{\mathbb{Q}[t^{\pm 1}]}(X, *_C, \circledast_S)$ is the image of a non degenerate Gaussian in $\widetilde{\mathcal{A}}^{w}_{\mathbb{Q}[t^{\pm 1}]}(X, *_{C\cup S})$, then $\int_S G \in \widetilde{\mathcal{A}}^{w}_{\mathbb{Q}(t)}(X, *_C)$ is well-defined.*

We now define categories of Jacobi diagrams. For $\bar{\mathcal{A}} = \mathcal{A}$, $\widetilde{\mathcal{A}}_R$, or $\widetilde{\mathcal{A}}^{\mathrm{w}}_R$, define a category $\bar{\mathcal{A}}$ whose objects are associative words in the letters $(+,-)$ and whose set of morphisms are $\bar{\mathcal{A}}(v,u) = \oplus_X \bar{\mathcal{A}}(X)$, where X runs over all compact oriented 1-manifolds with boundary identified with the set of letters of u and v, with the following sign convention: for u, a $+$ when the orientation of X goes towards the boundary point and a $-$ when it goes backward, and the converse for v. Composition is given by vertical juxtaposition, where the label of the created edges in the case of beaded or winding diagrams is defined with the same rule as in the definition of $\langle D, E\rangle$. The tensor product given by disjoint union defines a strict monoidal structure on $\bar{\mathcal{A}}$.

We finally define the target category of our functor.

Notation 3.6 *Given a positive integer g and a symbol $\natural$, set $\lfloor g \rceil^{\natural} = \{1^{\natural}, \dots, g^{\natural}\}$. Set $\lfloor 0 \rceil^{\natural} = \varnothing$.*

Definition 3.7 Fix non-negative integers f and g. An R-beaded Jacobi diagram on $(*_{\lfloor g\rceil^+ \cup \lfloor f \rceil^-})$ is *top-substantial* if it is $\lfloor g \rceil^+$-substantial.

Given two such diagrams D and E, define their composition $D \circ E$ as the sum of all ways of gluing all i^+-labelled vertices of D with all i^--labelled vertices of E, fixing the orientations and labels of the created edges as in the definition of $\langle D, E\rangle_S$. We get a category ${}^{ts}\!\widetilde{\mathcal{A}}$ whose objects are non-negative integers, with set of morphisms $\widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g\rceil^+ \cup \lfloor f \rceil^-})$ from g to f. The identity of g is $\exp_{\sqcup}(\sum_{i=1}^{g} \overset{i^+}{\underset{i^-}{\vdots}})$. The tensor product defined by disjoint union of diagrams provides ${}^{ts}\!\widetilde{\mathcal{A}}$ a strict monoidal structure.

3.2 At the level of tangles with disks

A *q-tangle with disks* is an equivalence class of pairs (γ, k), where γ is a q-tangle in $[-1,1]^3$, k is a non-negative integer understood as the datum of k disks $d_i = [0,1] \times [-1,1] \times \{\frac{i}{k+1}\}$, and each component of γ has a trivial algebraic intersection number with each disk d_i. Equivalence of such pairs is defined as isotopy relative to $(\partial[-1,1]^3) \cup (\cup_{i=1}^{k} \partial d_i)$. Define two categories $\mathcal{T}_q$ and $\widetilde{\mathcal{T}}_q$ with objects the non-associative words in the letters $(+,-)$ and morphisms the q-tangles for $\mathcal{T}_q$ and the q-tangles with disks for $\widetilde{\mathcal{T}}_q$. Composition is given by vertical juxtaposition. Given a q-tangle γ and a q-tangle with disks (v,k), define the tensor product $\gamma \otimes (v,k)$ in $\widetilde{\mathcal{T}}_q((w_t(\gamma))(w_t(v)), (w_b(\gamma))(w_b(v)))$ by horizontal juxtaposition in the x direction.

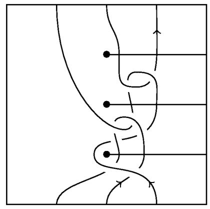


Figure 9: Diagram of a tangle with disks.

The definition of the functor $Z^\bullet : \widetilde{\mathcal{T}}_q \to \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}$ is based on the functor $Z : \mathcal{T}_q \to \mathcal{A}$ of [CHM08], which is a renormalization of the Le-Murakami functor [LM95, LM96].

Let (γ, k) be a q-tangle with disks. Assume γ is transverse to $[-1,1]^2 \times \{\frac{i}{k+1}\}$ for all $i \in \{1, \dots, k\}$, and write γ as a composition of q-tangles γ_i by cutting along these levels, see Figure 10. Write the bottom word of γ_i as $w_b(\gamma_i) = (v_i)(w_i)$, where w_i corresponds to

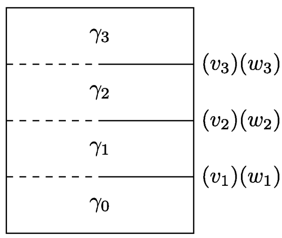


Figure 10: Cutting a q-tangle with disks $(\gamma, 3)$.

the part of the tangle which meets the disk d_i. Set:

$$Z^\bullet(\gamma, k) = Z(\gamma_0) \circ (I_{v_1} \otimes G_{w_1}) \circ Z(\gamma_1) \circ \ldots \circ (I_{v_k} \otimes G_{w_k}) \circ Z(\gamma_k) \ \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(\gamma),$$

where I_v is the identity on the word v and G_v is obtained from I_v by adding a label t (resp. t^{-1}) on squeleton components associated with a $-$ sign (resp. a $+$ sign), see Figure 11. At the level of objects, $Z^\bullet$ forgets the parentheses.

Figure 11: The diagrams I_v and G_v.

Proposition 3.8 *The functor $Z^\bullet : \widetilde{\mathcal{T}}_q \to \widetilde{\mathcal{A}}^{w}_{\mathbb{Q}[t^{\pm 1}]}$ is well-defined and preserves the tensor product on $\mathcal{T}_q \otimes \widetilde{\mathcal{T}}_q$.*

3.3 At the level of tangles with paths in $\mathbb{Q}$-cubes

Let B be a cobordism between two disks with a fixed parametrization $b : \partial[-1,1]^3 \to \partial B$ of its boundary. A *tangle* γ *in* B is an isotopy (rel. ∂B) class of framed oriented tangles whose boundary points lie on the top and bottom surfaces and are uniformly distributed along the line segments $[-1,1] \times \{0\} \times \{1\}$ and $[-1,1] \times \{0\} \times \{-1\}$ in $\partial B = b(\partial[-1,1]^3)$. Associate with each boundary point of γ the sign $+$ if γ is oriented downwards at that point and the sign $-$ otherwise. This provides two words in the letters $+$ and $-$, one for the top surface and the other for the bottom surface. Lifting these two words into non-associative words $w_t(\gamma)$ and $w_b(\gamma)$ in the letters $(+,-)$, one gets a *q-tangle*.

A *q-tangle with paths* (B, K, γ) is a q-tangle γ in a cobordism (B, b) with a disjoint union $K = \sqcup_{i=1}^k K_i \subset (B \setminus \gamma)$ of oriented paths K_i from $b(0,-1,\frac{i}{k+1})$ to $b(0,1,\frac{i}{k+1})$ with $k \geq 0$, such that $\hat{K} = \sqcup_{i=1}^k \hat{K}_i$ is an oriented boundary link, where $\hat{K}_i$ is the knot defined as the union of K_i with the line segments $[(0,-1,\frac{i}{k+1}),(1,-1,\frac{i}{k+1})]$, $[(1,-1,\frac{i}{k+1}),(1,1,\frac{i}{k+1})]$ and $[(1,1,\frac{i}{k+1}),(0,1,\frac{i}{k+1})]$.

Define two categories $\mathcal{T}_q\mathcal{C}ub$ and $\widetilde{\mathcal{T}_q}\mathcal{C}ub$ with objects the non-associative words in the letters $(+,-)$ and morphisms the q-tangles in $\mathbb{Q}$-cubes for $\mathcal{T}_q\mathcal{C}ub$ and the q-tangles with paths in $\mathbb{Q}$-cubes for $\widetilde{\mathcal{T}_q}\mathcal{C}ub$, up to orientation-preserving homeomorphism respecting the boundary parametrization. Composition is given by vertical juxtaposition. Given a morphism (C, v) in $\mathcal{T}_q\mathcal{C}ub$ and a morphism (B, K, γ) in $\widetilde{\mathcal{T}_q}\mathcal{C}ub$, define the tensor product $(C, v) \otimes (B, K, \gamma)$ by horizontal juxtaposition in the x direction.

In order to get a functor $Z : \widetilde{\mathcal{T}_q}\mathcal{C}ub \to \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}(t)}$, we wish to evaluate $Z^\bullet$ on the surgery presentation of a q-tangle with paths in a $\mathbb{Q}$-cube. Let $(B, K, \gamma) \in \widetilde{\mathcal{T}_q}\mathcal{C}ub(w, v)$. Let $([-1,1]^3, \Xi, \eta)$ be a q-tangle with paths where Ξ is a union of line segments. Let $L \subset [-1,1]^3 \setminus (\Xi \cup \eta)$ be a framed link null-homotopic in $[-1,1]^3 \setminus \Xi$ such that (B, K, γ) is obtained from $([-1,1]^3, \Xi, \eta)$ by surgery on L. We have a q-tangle with disks $(\eta \cup L, k)$ naturally associated with $([-1,1]^3, \Xi, \eta)$, where k is the number of components of Ξ, and $Z^\bullet(\eta \cup L, k) \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(\eta \cup L)$. Set:

$$Z^\circ((\Xi, \eta), L) = \chi^{-1}_{\pi_0(L)}(\nu^{\otimes \pi_0(L)} \sharp_{\pi_0(L)} Z^\bullet(\eta \cup L, k)) \ \in \widetilde{\mathcal{A}}^{\mathrm{w}}_{\mathbb{Q}[t^{\pm 1}]}(\eta, \circledast_{\pi_0(L)})$$

where the connected sum means that a copy of ν is summed to each component of L.

Proposition 3.9 *Let (B, K, γ) be a q-tangle with paths in a $\mathbb{Q}$-cube. Fix a surgery presentation $(([-1,1]^3, \Xi, \eta), L)$ of (B, K, γ). Then:*

$$Z(B, K, \gamma) = U_+^{-\sigma_+(L)} \sqcup U_-^{-\sigma_-(L)} \sqcup \int_{\pi_0(L)} Z^\circ((\Xi, \eta), L) \quad \in \widetilde{\mathcal{A}}^{w}_{\mathbb{Q}(t)}(\gamma),$$

where $U_\pm = Z^\circ((\varnothing, \varnothing), \bigcirc^{\pm 1})$, defines a functor $Z : \widetilde{\mathcal{T}_q}\mathcal{C}ub \to \widetilde{\mathcal{A}}^{w}_{\mathbb{Q}(t)}$ which preserves the tensor product on $\mathcal{T}_q\mathcal{C}ub \otimes \widetilde{\mathcal{T}_q}\mathcal{C}ub$.

3.4 At the level of Lagrangian cobordisms with paths

Given $g \in \mathbb{N}$, we fix a model surface F_g, compact, connected, oriented, of genus g, with one boundary component represented in Figure 12. It is equipped with a fixed base point

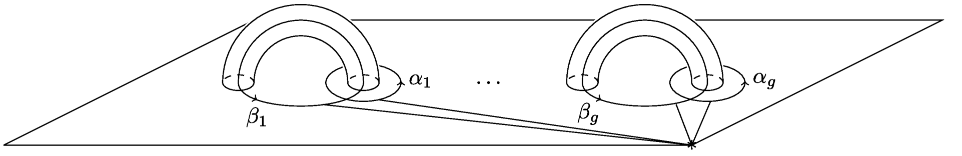


Figure 12: The model surface F_g.

$*$ and a fixed basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ of $\pi_1(F_g, *)$. Denote by $C_{g^-}^{g^+}$ the cube $[-1,1]^3$ with g^+ handles on the top boundary and g^- tunnels in the bottom boundary. We have canonical embeddings $F_{g^+} \hookrightarrow \partial C_{g^-}^{g^+}$ and $F_{g^-} \hookrightarrow \partial C_{g^-}^{g^+}$. A *cobordism with paths* from F_{g^+} to F_{g^-} is an equivalence class of triples (M, K, m) where:

- M is a compact, connected, oriented 3-manifold,
- $m : \partial C_{g^-}^{g^+} \xrightarrow{\cong} \partial M$ is an orientation-preserving homeomorphism,
- $K = \sqcup_{i=1}^k K_i \subset M$ is a union of oriented paths K_i from $m(0, -1, \frac{i}{k+1})$ to $m(0, 1, \frac{i}{k+1})$, with $k \geq 0$,
- $\hat{K} = \sqcup_{i=1}^k \hat{K}_i$ is an oriented boundary link, where the $\hat{K}_i$ are defined as in the previous subsection.

Two such triples are *equivalent* if they are related by an orientation-preserving homeomorphism which respects the boundary parametrizations and identifies the paths. We get embeddings $m_+ : F_{g^+} \hookrightarrow \partial M$ and $m_- : F_{g^-} \hookrightarrow \partial M$.

Set $A_g = \ker(\mathrm{incl}_* : H_1(F_g; \mathbb{Q}) \to H_1(C_0^g; \mathbb{Q}))$ and $B_g = \ker(\mathrm{incl}_* : H_1(F_g; \mathbb{Q}) \to H_1(C_g^0; \mathbb{Q}))$. These are Lagrangian subspaces of $H_1(F_g; \mathbb{Q})$ with respect to the intersection form, and A_g (resp. B_g) is generated by the homology classes of the curves α_i (resp. β_i). A cobordism with paths (M, K, m) from F_{g^+} to F_{g^-} is *Lagrangian(-preserving)* if the following conditions are satisfied:

- $H_1(M; \mathbb{Q}) = (m_-)_*(A_{g^-}) \oplus (m_+)_*(B_{g^+})$,
- $(m_+)_*(A_{g^+}) \subset (m_-)_*(A_{g^-})$ as subspaces of $H_1(M; \mathbb{Q})$.

Define a category $\widetilde{\mathcal{LC}ob}$ of Lagrangian cobordisms with paths whose objects are non-negative integers and whose set of morphisms $\widetilde{\mathcal{LC}ob}(g^+, g^-)$ is the set of Lagrangian cobordisms with paths from F_{g^+} to F_{g^-}. The composition of a cobordism (M, K, m) from F_g to F_f with a cobordism (N, J, n) from F_h to F_g is given by gluing N on the top of M. Let $lcob$ be the subcategory of $\widetilde{\mathcal{LC}ob}$ of Lagrangian cobordisms with no path. For a cobordism (M, m) and a cobordism with paths (N, J, n), define the tensor product $(M, m) \otimes (N, J, n)$ by horizontal juxtaposition in the x direction.

Define categories $\mathcal{LC}ob_q$ and $\widetilde{\mathcal{LC}ob}_q$ of q-cobordisms with objects the non-commutative words in the single letter $\bullet$, and with set of morphisms from a word on g^+ letters to a word on g^- letters the set of morphisms from $g+$ to $g-$ in $\mathcal{LC}ob$ and $\widetilde{\mathcal{LC}ob}$ respectively.

In order to define the functor $\tilde{Z} : \widetilde{\mathcal{LC}ob}_q \to {}^{ts}\widetilde{\mathcal{A}}$, we represent Lagrangian cobordisms with paths by Lagrangian bottom-top tangles with paths.

For a positive integer $g \geq 0$, let $(p_1, q_1), \ldots, (p_g, q_g)$ be g pairs of points uniformly distributed on $[-1,1] \times \{0\} \subset [-1,1]^2 \cong F_0$ as represented Figure 13. A *bottom-top*

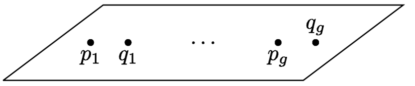


Figure 13: The pairs of points (p_i, q_i) on $[-1,1]^2$.

tangle with paths of type (g^+, g^-) is an equivalence class of triples (B, K, γ) where

- $(B, K) = (B, K, b)$ is a cobordism with paths form F_0 to F_0,
- $\gamma = (\gamma^+, \gamma^-)$ is a framed oriented tangle in B with g^+ components γ_i^+ from $b(\{p_i\} \times \{1\})$ to $b(\{q_i\} \times \{1\})$ and g^- components γ_i^- from $b(\{q_i\} \times \{-1\})$ to $b(\{p_i\} \times \{-1\})$,
- $\hat{K}$ is a boundary link in $B \setminus \gamma$.

Two such triples (B, K, γ) and (B', K', γ') are *equivalent* if (B, K) and (B', K') are related by an equivalence which identifies γ and γ'. A bottom-top tangle with paths (B, K, γ) is Lagrangian if B is a $\mathbb{Q}$-cube and the linking matrix $\mathrm{Lk}(\gamma^+)$ is trivial.

In order to define the composition, we need the bottom-top tangle $([-1,1]^3, \varnothing, T_g)$ represented in Figure 14. The composition of a bottom-top tangle (B, K, γ) of type (g, f)

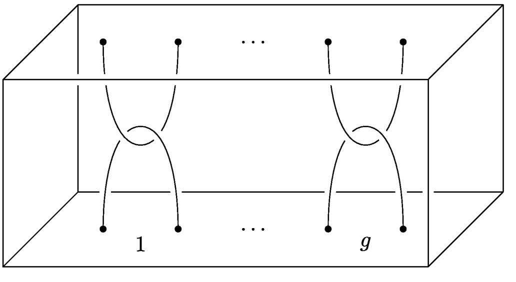


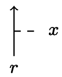
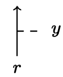
Figure 14: The bottom-top tangle T_g in $[-1,1]^3$.

with a bottom-top tangle (C, J, v) of type (h, g) is given by first making the composition $(B, K) \circ ([-1,1]^3, \varnothing) \circ (C, J)$ in the category $\widetilde{\mathcal{LC}ob}$ and then perfoming the surgery on the $2g$ components link $\gamma^+ \cup T_g \cup v^-$. We get a category ${}_b^t\widetilde{\mathcal{LT}}$ whose objects are non-negative integers and whose set of morphisms ${}_b^t\widetilde{\mathcal{LT}}(g^+, g^-)$ is the set of Lagrangian bottom-top tangles with paths of type (g^+, g^-). Denote ${}_b^t\mathcal{LT}$ the subcategory of Lagrangian bottom-top tangles with no path. For a Lagrangian bottom-top tangle (B, γ) and a Lagrangian bottom-top tangle with paths (C, J, v), define the tensor product $(B, \gamma) \otimes (C, J, v)$ by horizontal juxtaposition in the x direction. Define categories ${}_b^t\mathcal{LT}_q$ and ${}_b^t\widetilde{\mathcal{LT}}_q$ of bottom-top q-tangles with objects the non-commutative words in the single letter $\bullet$.

In the following result, the map D is defined by digging tunnels around the components of the tangle.

Proposition 3.10 *There is an isomorphism $D : {}^t_b\widetilde{\mathcal{LT}} \to \widetilde{\mathcal{LC}ob}$ which identifies ${}^t_b\mathcal{LT}$ with $\mathcal{LC}ob$ and preserves the tensor product on ${}^t_b\mathcal{LT} \otimes {}^t_b\widetilde{\mathcal{LT}}$.*

Now we can define a functor on Lagrangian q-cobordisms with paths by applying the invariant Z on bottom-top q-tangles with paths in $\mathbb{Q}$-cubes. The invariant Z is functorial on q-tangles but not on bottom-top q-tangles, due to the different composition laws. To deal with this, we introduce some specific elements $\top_g \in {}^{ts}\widetilde{\mathcal{A}}(*_{\lfloor g\rceil^+ \cup \lfloor g\rceil^-})$ following [CHM08, Sec. 4]. Set:

$$\lambda(x, y; r) = \chi^{-1}_{\{r\}}(\exp(\ $$  $$\ ) \circ \exp(\ $$  $$\ )) \in \widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\{x,y,r\}}),$$

$$\top(x^+, x^-) = U_+^{-1} \sqcup U_-^{-1} \sqcup \int_{\{r^+,r^-\}} \langle \lambda(x^+, 1^+; r^+) \sqcup \lambda(x^-, 1^-; r^-), \chi^{-1}(T_1) \rangle_{\{1^+,1^-\}},$$

$$\top_g = \top(1^+, 1^-) \sqcup \ldots \sqcup \top(g^+, g^-) \in \widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g\rceil^+ \cup \lfloor g\rceil^-}),$$

where the bottom-top tangle T_1 is drawn in Figure 14. Set:

$$\tilde{Z}(M, K) = \chi^{-1}(Z(B, K, \gamma)) \circ \top_g.$$

At the level of objects, $\tilde{Z}$ sends a word on its number of letters.

Proposition 3.11 *The functor $\tilde{Z} : \widetilde{\mathcal{LC}ob_q} \to {}^{ts}\widetilde{\mathcal{A}}$ is weel-defined and preserves the tensor product on $\mathcal{LC}ob_q \otimes \widetilde{\mathcal{LC}ob_q}$.*

3.5 Splitting formulas

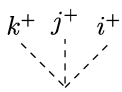
We first mention useful lemmas from [Mas15, Lemmas 4.3 & 4.4]. Recall the tensor $\mu(\mathbf{C})$ was defined in the introduction.

Lemma 3.12 *For a $\mathbb{Q}$-handlebody C of genus g, there exists a boundary parametrization $c : \partial C_0^g \to C$ such that $(C, c) \in \mathcal{LC}ob(g, 0)$.*

Lemma 3.13 *Let $\mathbf{C} = \binom{C'}{C}$ be an LP-pair of genus g. Take boundary parametrizations $c : \partial C_0^g \to C$ and $c' : \partial C_0^g \to C'$ compatible with the fixed identification $\partial C \cong \partial C'$ such that $(C, c) \in \mathcal{LC}ob(g, 0)$ and $(C', c') \in \mathcal{LC}ob(g, 0)$. Then:*

$$\mu(\mathbf{C}) = \tilde{Z}_1(C, c) - \tilde{Z}_1(C', c'),$$

where $\tilde{Z}_1$ is the i-degree 1 part of $\tilde{Z}$ and $\mu(\mathbf{C})$ is considered as an element of $\widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g\rceil^+})$ via *the inclusion $\Lambda^3 H_1(\mathcal{C}; \mathbb{Q}) \hookrightarrow \widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g\rceil^+})$ defined by:*

$$[c_+(\beta_i)] \wedge [c_+(\beta_j)] \wedge [c_+(\beta_k)] \mapsto$$  .

Let $(M,K) \in \widetilde{\mathcal{LC}ob_q}(w,v)$. Let $\mathbf{C} = (\mathbf{C}_1, \ldots, \mathbf{C}_n)$ be a null LP-surgery on (M,K). Let e_i be the genus of C_i. For $1 \leq i \leq n$, take boundary parametrizations $c_i : \partial C_0^{e_i} \to C_i$ and $c_i' : \partial C_0^{e_i} \to C_i'$ compatible with the fixed identification $\partial C_i \cong \partial C_i'$ such that $(C_i, c_i) \in \mathcal{LC}ob(e_i, 0)$ and $(C_i', c_i') \in \mathcal{LC}ob(e_i, 0)$. Set $e = \sum_{i=1}^n e_i$. Take a collar neighborhood $m_-(F_f) \times [-1, \varepsilon - 1]$ of the bottom surface $m_-(F_f)$. Take pairwise disjoint solid tubes T_i, $i = 1, \ldots, n$, such that T_i connects $(c_i)_-(F_0)$ to a disk in $m_-(F_f) \times \{\varepsilon - 1\}$ in the complement of the C_j, the collar neighborhood and K. This provides a decomposition of the cobordism (M,K) as:

$$(M,K) = ((C_1, \varnothing) \otimes \ldots \otimes (C_n, \varnothing) \otimes Id_f) \circ (N, J),$$

where f is the number of letters of v. It is proved in [Mas15, Section 4.4] that N is a Lagrangian cobordism. The nullity condition on the surgery ensures that $\hat{J}$ is a boundary link. Thus (N,J) is a Lagrangian cobordism with paths.

From such a decomposition of a cobordism associated with a null LP-surgery, one can obtain splitting formulas for the functor $\tilde{Z}$. We only state here a specific version of these formulas for a cobordism with one path.

Given a null LP-surgery $\mathbf{C} = (\mathbf{C}_1, \ldots, \mathbf{C}_n)$ on a Lagrangian cobordism (M,K), define a hermitian form $\ell_{(M,K)}(\mathbf{C}) : H_1(\mathcal{C};\mathbb{Q}) \times H_1(\mathcal{C};\mathbb{Q}) \to \mathbb{Q}(t)$ in the same way as $\ell_{(S,\kappa)}(\mathbf{C})$ was defined in the introduction. Also define a map $\rho_{\mathbf{c}} : \mathcal{A}_{\mathbb{Q}}(H_1(\mathcal{C};\mathbb{Q})) \to \widetilde{\mathcal{A}}_{\mathbb{Q}(t)}(*_{\lfloor g \rceil^+ \cup \lfloor f \rceil^-})$ which changes the labels of the univalent vertices by first sending them in $H_1(M;\mathbb{Q})$ *via* $H_1(\mathcal{C};\mathbb{Q}) \cong \otimes_{i=1}^n H_1(C_i;\mathbb{Q}) \to H_1(M;\mathbb{Q})$, and then writing them in terms of the $[m_+(\beta_i)]$ and $[m_-(\alpha_i)]$.

Proposition 3.14 *Let $(M,K) \in \widetilde{\mathcal{LC}ob_q}(w,v)$ be a Lagrangian q-cobordism with one path. Let $\mathbf{C} = (\mathbf{C}_1, \ldots, \mathbf{C}_n)$ be a null LP-surgery on (M,K). Let (B,K,γ) be the bottom-top tangle with paths associated with (M,K). Then:*

$$\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} \tilde{Z}\left((M,K)(\mathbf{C}_I)\right) \equiv_n \exp_{\sqcup}\left(\frac{1}{2}\mathrm{Lk}_e(\gamma)\right) \sqcup \rho_{\mathbf{c}} \begin{pmatrix} \text{sum of all ways of gluing} \\ \text{some legs of } \mu(\mathbf{C}) \text{ with} \\ \ell_{(M,K)}(\mathbf{C})/2 \end{pmatrix},$$

where $\mathbf{C}_I = ((\mathbf{C}_i)_{i \in I})$, $\equiv_n$ means "equal up to i-degree at least $n+1$ terms" and $\mathrm{Lk}_e(\gamma)$ is the matrix of equivariant linking numbers of lifts of the components of $\hat{\gamma}$ in the infinite cyclic covering of $B \setminus K$.

To apply this result to the Kricker lift of the Kontsevich integral, we need to recover this invariant from our functor. Let (S,κ) be a $\mathbb{Q}$SK-pair. Let M be the $\mathbb{Q}$-cube obtained from S by removing the interior of a ball B^3 disjoint from κ. Isotoping κ in M and fixing a boundary parametrization m of M, we can view κ as the knot $\hat{K}$ associated with a Lagrangian cobordism with one path (M,K). Since the top and bottom words are empty, we get a Lagrangian q-cobordism with one path.

Proposition 3.15 *Let (S,κ) be a $\mathbb{Q}$SK-pair. Define as above an associated Lagrangian q-cobordism with one path (M,K). Then $\tilde{Z}(S,\kappa) = \tilde{Z}(M,K)$ defines an invariant of $\mathbb{Q}$SK-pairs, which coincides with the Kricker invariant Z^{rat} for knots in $\mathbb{Z}$-spheres.*

Theorem 2.1 can be deduced from Propositions 3.14 and 3.15.

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