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Recent progress of various Volume Conjectures for links as well as 3-manifolds

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Abstract

This is a survey paper. It will report some progress towards various Volume Conjectures including the original one for the colored Jones polynomials of links, the Chen-Yang style ones for the Reshetikhin-Turaev invariants and the Turaev-Viro invariants of hyperbolic 3-manifolds as well as a new vision on the relation between the Habiro type cyclotomic expansion and Volume Conjectures inspired by the large N duality.

1 Introduction

In late 1970's, the area of low dimensional topology especially the geometry and topology of 3-manifolds was revolutionized by Thurston. In 1980's, Jones constructed his famous polynomial for knots/links. Later, Witten explained Jones polynomial as an invariant coming from a quantum $SU(2)$ Chern-Simons theory with fundamental representation, and also predicted new invariants of knots/links as well as 3-manifolds. Then Reshetikhin-Turaev constructed these invariants by using quantum groups. Chern-Simons gauge theory and corresponding quantum invariants of knots/links as well as 3-manifolds forms a hot topic in mathematics since early 1990's. The following **Volume conjecture** formulated by Kashaev [28] in 1997 and Murakami-Murakami [45] in 2001 surprisingly connects two quite different area, namely TQFT and low dimensional topology, which has attracted many talented mathematicians and physicists.

Conjecture 1.1 (Volume Conjecture for colored Jones polynomial, Kashaev-Murakami-Murakami [28, 45]). *For any link $\mathcal{K}$ in S^3, let $J_N(\mathcal{K};q)$ be its $(N+1)$-th colored Jones polynomial. Then*

$$\lim_{N\to+\infty}\frac{2\pi}{N}\log\left|J_N(\mathcal{K};e^{\frac{\pi\sqrt{-1}}{N+1}})\right| = v_3||S^3\backslash\mathcal{K}||,$$

where $||S^3\backslash\mathcal{K}||$ is the Gromov norm of the complement of K in S^3 and $v_3\approx 1.0145$ is the volume of the regular hyperbolic tetrahedron.

Remark 1.1. When link $\mathcal{K}$ is a hyperbolic link, then the right hand side of the above conjecture can be stated as hyperbolic volume $\mathrm{Vol}(S^3\backslash\mathcal{K})$, Since $Vol(S^3\backslash\mathcal{K}) = v_3||S^3\backslash\mathcal{K}||$.

Murakami-Murakami-Okamoto-Takata-Yokota [46] complexified the volume conjecture, including the Chern-Simons invariant in the right-hand side as the imaginary part.

For a very long time, this conjecture has been only proved for very few cases including the figure-eight knot (by Ekholm, cf. [44]), 5_2 by Kashaev-Yokota [30], torus knots [29],

$(2, 2m)$-torus links [26], Borromean rings [21], twisted Whitehead links [60] and Whitehead chains [58]. Actually progress towards hyperbolic knots except the figure-eight knot 4_1 and knot 5_2 were relatively less. Recently T. Ohtsuki [48] obtained the whole asymptotic expansion of Kashaev's invariants (colored Jones polynomial J_n evaluated at $e^{\frac{\pi\sqrt{-1}}{N}}$) of knot 5_2. By a similar fashion procedure, Ohtsuki-Yokota [52], T. Ohtsuki [49] and T. Takata [57] obtained asymptotic expansion of Kashaev's invariants of hyperbolic knots with 6, 7 and 8 crossings (8_6 and 8_{12}) respectively.

Some of my recent work on Volume Conjectures and related topics can be roughly divided to two research directions. In this survey, we will discuss some recent developments in these two directions.

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2 Volume Conjectures for Reshetikhin-Turaev and Turaev-Viro invariants of 3-manifolds

2.1 Volume Conjectures for Reshetikhin-Turaev and Turaev-Viro invariants of 3-manifolds evaluated at roots of unity $q(2) = \exp(\frac{2\pi\sqrt{-1}}{r})$

In [11], Chen-Yang first extended the Turaev-Viro invariant from closed 3-manifolds to 3-manifolds with cusps (orientable case by Benedetti-Petronio [2]) or even with totally geodesic boundary. Recall that Turaev-Viro's original construction gives rise to real valued invariants for closed 3-manifolds and $2+1$ TQFT's for 3-cobordisms containing a link inside. In all of their constructions, they use the usual triangulations, meaning the vertices of the triangulations are inside the manifolds and the cobordisms, and in the case of cobordisms, the edges on the boundary are from the edges of the triangulations. The difference in our construction [11] is that, instead of using the usual triangulations, we use ideal triangulation of a 3-manifold with non-empty boundary.

Let $q(s)$ denotes the roots of unity $e^{\frac{s\pi\sqrt{-1}}{r}}$ for an integer s, where $(r, s) = 1$. Recall that the Reshetikhin-Turaev invariants $\{\tau_r(M; q)\}$ are complex-valued invariants of oriented closed 3-manifolds with $q = q(odd)$. This provided a mathematical construction of the 3-manifold invariants (evaluated at $q(1)$ only) introduced by Witten using Chern-Simons action. Using skein theory, Lickorish [38, 39] redefined the Reshetikhin-Turaev invariants with $q = q(odd)$, and Blanchet-Habegger-Masbaum-Vogel [3] extended them to $q(2) = e^{\frac{2\pi\sqrt{-1}}{r}}$ for odd r. In [11], Chen-Yang proposed the following **Volume Conjecture**.

Conjecture 2.1 (Volume Conjectures for Reshetikhin-Turaev and Turaev-Viro invariants, Chen-Yang [11]). *Let M_1 be a closed oriented hyperbolic 3-manifold and let $\tau_r(M_1; q)$ be its Reshetikhin-Turaev invariants. Let M_2 be a compact 3-manifold (cusped or with totally geodesic boundary). Then for r running over all odd integers and $q = q(2) = e^{\frac{2\pi\sqrt{-1}}{r}}$*

under the convention $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$*, and with a suitable choice of the arguments, we have*

$$\lim_{r\to+\infty} \frac{4\pi}{r-2} \log\left(\tau_r(M_1; q)\right) \equiv Vol(M_1) + \sqrt{-1}CS(M_1) \quad (mod \quad \sqrt{-1}\pi^2),$$

where $CS(M_1)$ *is the usual Chern-Simons invariant of* M_1 *multiplied by* $2\pi^2$*, and*

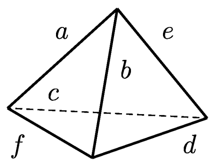
$$\lim_{r\to+\infty} \frac{2\pi}{r-2} \log\left(TV_r(M_2; e^{\frac{2\pi\sqrt{-1}}{r}})\right) = Vol(M_2).$$

In fact, Witten's Asymptotic Expansion Conjecture (WAE) considers Reshetikhin-Turaev (also Turaev-Viro) invariants evaluated at usual root of unity $q(1) = e^{\frac{\pi\sqrt{-1}}{r}}$. This turns out to be of only polynomial growth w.r.t. r. For past 25 years, people have thought Reshetikhin-Turaev and Turaev-Viro invariants evaluated at other roots of unity should have similar asymptotic behavior which is also polynomial growth w.r.t. r.

Thus Chen-Yang's conjecture not only corrects a long existing wrong feeling about Reshetikhin-Turaev and Turaev-Viro invariants but also largely extends the original Volume Conjecture (Kashaev-Murakami-Murakami) from knot complements in S^3 to all kinds of hyperbolic 3-manifolds, no matter they are closed, cusped or even with totally geodesic boundaries.

Some further developments of Chen-Yang's Volume Conjectures have been announced by various mathematicians in several recent international conferences or on arXiv. These results are listed as follows

- T. Ohtsuki[50] first generalize our Volume Conjecture for Reshetikhin-Turaev invariants at $q(2) = e^{\frac{2\pi\sqrt{-1}}{r}}$ to a full asymptotic expansion conjecture (physics flavour conjecture by D. Gang-M. Romo-M. Yamazaki[20]) and then he proved our Volume Conjecture for a series cases, the closed hyperbolic 3-manifolds M_K obtained by integer Dehn surgery in S^3 along the figure-eight knot 4_1.
- T. Ohtsuki-T. Takata[51] recognized the Reidemeister torsion term appearing in the above full asymptotic expansion.
- R. Detcherry-E. Kalfagianni-T. Yang[14] proved our Turaev-Viro Volume Conjecture for cases of hyperbolic cusped 3-manifolds $S^3\backslash 4_1$, $S^3\backslash$Borromean rings via establishing a relation involving Turaev-Viro invariants of link complements in S^3 and certain sum of colored Jones polynomials of that link.
- R. Detcherry-E. Kalfagianni[13] first established a relation between the asymptotics of the Turaev-Viro invariants at $q(2) = e^{\frac{2\pi\sqrt{-1}}{r}}$ that Chen-Yang considered in [11] and the Gromov norm of 3-manifolds. Furthermore, they obtained a lower bound for the Gromov norm of any compact, oriented 3-manifold with empty or toroidal boundary, in terms of the Turaev-Viro invariants. They also proved Turaev-Viro Volume Conjecture for Gromov norm zero links (knots proved by Detcherry-Kalfagianni-Yang[14], several torus knots/links such as Trefoil knot, Hopf link, link $T(2,4)$ proved in Chen-Yang's original paper [11]). Finally they constructed infinitely families of 3-manifolds for Turaev-Viro invariants with exponential growth predicted by Chen-Yang's Volume conjecture



Usual (compact) tetrahedron

Figure 1: Edges of a tetrahedron

- A. Kolpakov-J. Murakami[34] formulated the corresponding Volume Conjecture for Kirillov-Reshetikhin invariants at $q(2) = e^{\frac{2\pi\sqrt{-1}}{r}}$ that Chen-Yang considered in [11]
- Q. Chen-J. Murakami[10] proposed a Conjecture for dominant term (Volume) and secondary term (Gram matrix of the tetrahedra) in asymptotics of the quantum $6j$ symbol at $q(2) = e^{\frac{2\pi\sqrt{-1}}{r}}$ that Chen-Yang considered in [11] and proved majority cases (at least one of the vertex of the tetrahedra is ideal or Ultra-ideal. Furthermore, Q. Chen-J. Murakami [10] also proposed a Conjecture for a symmetric property of asymptotics of quantum $6j$ symbol at $q(2) = e^{\frac{2\pi\sqrt{-1}}{r}}$ observed from big cancellations.

Conjecture 2.2 (Symmetry of asymptotics of quantum $6j$ symbols, Chen-Murakami [10]). *Let T be a hyperbolic tetrahedron and θ_a, θ_b, θ_c, θ_d, θ_e, θ_f be dihedral angles at edges a, $\cdots$, f in above Figure. Let a_k, b_k, $\cdots$, f_k be sequences of non-negative half integers satisfying*

$$\lim_{k\to\infty} \frac{4\pi}{k} a_k = \pi - \theta_a, \quad \lim_{k\to\infty} \frac{4\pi}{k} b_k = \pi - \theta_b, \quad \cdots, \quad \lim_{k\to\infty} \frac{4\pi}{k} f_k = \pi - \theta_f$$

so that the triplets (a_r, b_r, e_r), (a_r, d_r, f_r), (b_r, d_r, f_r), (c_r, d_r, e_r) are all r-admissible for odd $r \geq 3$. Let a'_r be a sequence of non-negative half integers satisfying

$$\lim_{k\to\infty} \frac{4\pi}{k} a'_k = \pi + \theta_a$$

so that the triplets (a'_r, b_r, e_r), (a'_r, d_r, f_r) are r-admissible. Then the asymptotic expansions of $\begin{Bmatrix} a_r & b_r & e_r \\ d_r & c_r & f_r \end{Bmatrix}^{RW}_{q(2)}$ and $\begin{Bmatrix} a'_r & b_r & e_r \\ d_r & c_r & f_r \end{Bmatrix}^{RW}_{q(2)}$ with respect to r are equal.

Remark 2.1. Since the volume function and the gram matrix of a tetrahedron are not changed by switching the sign of a dihedral angle, the above conjecture is true up to the second leading term (by main theorem in [10]). We also checked that the coincidence of the third term for some cases by numerical computation.

2.2 Volume Conjectures for Reshetikhin-Turaev and Turaev-Viro invariants of 3-manifolds evaluated at roots of unity $q(odd)$

Now we have new Volume Conjectures for Reshetikhin-Turaev and Turaev-Viro invariants at $q(2)$. So it is natural to ask whether there also exists Volume Conjectures for

Reshetikhin-Turaev and Turaev-Viro invariants at $q(odd)$.

In fact, the example of Turaev-Viro invariant of non-orientable 3-manifold N_{2_1} (Callahan-Hildebrand-Weeks census) vanishes at roots of unity $q(s)$, where r and s are both odd numbers and $(r,s)=1$ (required by a condition from the definition of the Turaev-Viro invariant). Numerical evidence shows that it is nonzero at $q(s)$ and also goes exponentially large as $r\to\infty$, when s is an odd number other than 1 but r is an even number.

Take into account the above example it is natural to propose the following Volume Conjecture,

Conjecture 2.3. *For any 3-manifold M with boundary (orientable or non-orientable, or even with totally geodesic boundary), for a fixed odd number s other than* 1 *and such integer r that condition $(*)$ $TV_r(M,q(s))\neq 0$ is satisfied, then we have*

$$\lim_{\substack{r\to\infty,\ (r,s)=1\\ r\ satisfy\ (*)}} \frac{s\pi}{r}\log|TV_r(M,q(s))| = Vol_{cpx}(M).$$

Remark 2.2. If $TV_r(M,q(s))\neq 0$ for all any even integer r and any 3-manifold M with boundary, we could change condition "r satisfy $(*)$" to "r is even".

Remark 2.3. When M has cusps, as in [10], we consider ideal tetrahedral decomposition of M, and consider Turaev-Viro invariant of this tetrahedral decomposition. When M has totally geodesic boundary, we consider the singular 3-manifold obtained from M by collapsing each boundary component, and consider Turaev-Viro invariant of a triangulation of this singular 3-manifold.

Similar phenomenon also happens to the Reshetikhin-Turaev invariants. The Reshetikhin-Turaev invariants of closed 3-manifold M obtained from a $4k+2$-surgery along a knot K, $RT_r(M,q(s))$, vanishes at roots of unity $q(s)$, where r and s are both odd numbers and $(r,s)=1$ and (See Kirby-Melvin and see also Chen-Liu-Peng-Zhu [7]). Numerical evidence shows that Reshetikhin-Turaev of certain examples we tested are nonzero at $q(s)$ and also goes exponentially large as $r\to\infty$, when s is an odd number other than 1 but r is an even number.

Take into account the above example it is natural to propose the following Volume Conjecture,

Conjecture 2.4. *For any closed 3-manifold M (orientable or non-orientable, or even with totally geodesic boundary), for a fixed odd number s other than* 1 *and such integer r that condition $(**)$ $RT_r(M,q(s))\neq 0$ is satisfied, then we have*

$$\lim_{\substack{r\to\infty,\ (r,s)=1\\ r\ satisfy\ (**)}} \frac{2s\pi}{r}\log\big(RT_r(M,q(s))\big) = Vol_{cpx}(M) \qquad (\mathrm{mod}\ \sqrt{-1}\pi^2\mathbb{Z}).$$

Remark 2.4. If $RT_r(M,q(s))\neq 0$ for all any even integer r and any 3-manifold closed manifold M, we could change condition "r satisfy $(**)$" to "r is even".

Here is a summary of various results.

	Colored Jones polynomials for links	Reshetikhin-Turaev -Witten invariants for closed oriented 3-manifolds ($q(odd)$ by Reshetikhin-Turaev 90', extended to $q(4t+2)$ by Blanchet-Hagegger-Masbaum-Vogel [3], 92')	Turaev-Viro invariants for 3-manifolds with boundary (orientable cusped by Benedetti-Petronio [2] 96', nonorientable cusped or with totally geodesic boundary by Chen-Yang [11] 15')	Quantum 6j Symbols (Krillov-Reshetikhin [35])
$q(1)$	Original Volume Conjecture (Kashaev [28] 97' & Murakami-Murakami [45] 01')	Polynomial growth, Witten's Asymptotic Expansion Conjecture (WAE)	Polynomial growth, should be something similar to WAE conjecture	Polynomial growth, Woodward's conjecture [56]
$q(2)$	Similar Volume Conjecture	Exponential growth and Volume Conjecture (Chen-Yang [11], 15')	Exponential growth and Volume Conjecture (Chen-Yang [11] 15')	Exponential growth and Volume Conjecture (proposed by and majority cases proved by Chen-Murakami [10] 17')
$q(odd \geq 3)$	Similar Volume Conjecture	Identical to 0 or Exponential growth then Volume Conjecture (Chen 17')	Identical to 0 or Exponential growth then Volume Conjecture (Chen 17')	In Preparation

Volume Conjectures for 3-manifolds with or without boundaries and quantum 6j symbols

2.3 New Volume Conjectures in mathematics may indicate new physics

By many numerical checks and proofs of several non-trivial examples of our Volume Conjecture, it is clear that Reshetikhin-Turaev/Turaev-Viro invariants evaluated at root of unity $q(2)$ display a much deeper hidden relation to the hyperbolic geometry than those evaluated at usual roots of unity $q(1)$.

Connections of the original Volume Conjecture with physics were explored by Gukov [23], Dijkgraaf-Fuji-Manabe [15] and Witten [59] and the physics meaning of Reshetikhin-Turaev invariants evaluated at root of unity $q(1)$ was the original quantum Chern-Simons theory studied by Witten.

So it is very natural to ask **the physical meaning of the Reshetikhin-Turaev / Turaev-Viro invariants evaluated at the root of unity $q(2)$. Due to the negative quantum integer involved, their nature should be a non-unitary physics theory, which looks a bit crazy, but it is confirmed by many top mathematical physicists such as G. Felder, R. Kashaev, N. Reshetikhin and E. Witten etc.**

"Drastic cancellation" appears in the computation of the Turaev-Viro invariants of hyperbolic 3-manifolds evaluated at roots of unity $q(2)$, when T. Yang and the author numerically check those examples in [11]. This means that one still obtain exponentially large identities of Turaev-Viro invariants evaluated at roots of unity $q(2)$ even after those drastic cancellations. This never happens to the original Volume Conjecture, so we believe our Volume Conjecture display a very unexpected/wild nature of quantum invariants, which make its corresponding physics meaning more mysterious.

The above new discovery shed some new lights on postulating a new quantum Chern-Simons theory corresponding to the Reshetikhin-Turaev invariants evaluated at non-conventional roots of unity such as $q(2)$ etc. we expect that such a potential physics explanation will have a very wide application just like Witten's quantum Chern-Simons theory which corresponds to the Reshetikhin-Turaev invariant evaluated at $q(1)$ and will shed some new lights on the current study of High Energy Physics.

3 A new vision on proposing Volume Conjectures

3.1 Some backgrounds of large N duality and LMOV Conjectures

The physics background of the problem the author concerned can be traced back to the seminal work of 't Hooft on large N expansion of $U(N)$ gauge field theories in 1974. It was discovered by Gopakumar and Vafa that the topological string theory on the resolved conifold is dual to the $U(N)$ Chern-Simons theory on S^3. This striking duality means the partition functions of two different theories exactly agree up to all orders. The former one corresponds to the (open) Gromov-Witten theory which is still in its infancy, while the latter one corresponds to quantum invariants of links and 3-manifolds. The open large N duality was established by Ooguri-Vafa [53].

Quantum group invariants $W^{\mathfrak{g}}_{V^1,\cdots,V^L}(\mathcal{L})(q)$ of link $\mathcal{L}$ were determined by representations V^α of $U_q(\mathfrak{g})$, the quantized universal enveloping algebra of $\mathfrak{g}$. A partition A can be labeled by the Young tableau which corresponds to an irreducible representation V_A for a specific $\mathfrak{g} = sl_N$ and there exists a two-variable colored HOMFLY-PT invariant $W^{\mathfrak{g}}_{A^1,\cdots,A^L}(\mathcal{L})(q,t)$, s.t. $W^{\mathfrak{g}}_{A^1,\cdots,A^L}(\mathcal{L})(q,t)|_{t=q^N} = W^{\mathfrak{g}}_{V_{A^1},\cdots,V_{A^L}}(\mathcal{L})(q)$. Let $\mathcal{P}^L$ denote all the partition sets labeled by the Young tableau. For each link $\mathcal{L}$, the type$-A$ Chern-Simons partition function of $\mathcal{L}$ is defined by

$$Z^{SL}_{CS}(\mathcal{L};q,t) = \sum_{\vec{A}\in\mathcal{P}^L} W^{SL}_{\vec{A}}(\mathcal{L};q,t)s_{\vec{A}}(x) = \sum_{\vec{\mu}\neq 0} F^{SL}_{\vec{\mu}} p_{\vec{\mu}}$$

where $s_{\vec{A}}(x)$ are the Schur polynomials of partition $\vec{A} = (A^1,...,A^L)$ and $p_n(\vec{x}) = \sum_{i=1}^{+\infty}(x_i)^n$.

By using the plethystic exponential method (due to Getzler-Kapranov), we can write the free energy as follows

$$\log Z^{SL}_{CS}(\mathcal{K};q,t;x) = \sum_{A\in\mathcal{P}}\sum_{d=1}^{\infty}\frac{f_A(\mathcal{K};q^d,t^d)}{d}s_A(x^d).$$

Based on the large N duality, Labastida-Marino-Ooguri-Vafa [36, 37, 53] conjectured an amazing algebraic structure for the generating series (Chern-Simons partition function) of colored HOMFLY-PT link invariants and the integrality of the infinite family of new topological invariants.

Conjecture 3.1 (LMOV Conjecture, Labastida-Marino-Ooguri-Vafa, 2000-2002). *There exists a knot invariant $P_B(\mathcal{K};q,t) \in \frac{1}{(q-q^{-1})^2}\mathbb{Z}[(q-q^{-1})^2, t^{\pm 1}]$ s.t.*

$$f_A(\mathcal{K};q,t) = \sum_{|B|=|A|} P_B(\mathcal{K};q,t) M_{AB}(q),$$

where $M_{AB}(q) = \sum_{|\mu|=|A|} \frac{\chi_A(C_\mu)\chi_B(C_\mu)}{\mathfrak{z}_\mu} \prod_{j=1}^{\ell(\mu)} (q^{\mu_j} - q^{-\mu_j})$, and χ_A, C_μ and $\mathfrak{z}_\mu$ are character, conjugacy class and multiplicity labelled by Young tableau respectively.

The original LMOV conjecture describes a very subtle structure of $Z_{CS}^{SL}(\mathcal{L};q,t)$, which was proved by Liu-Peng [41]. Mathematically, the LMOV conjecture confirms that colored HOMFLY-PT invariants also have integrality, symmetry of q, pole order structure of $q-q^{-1}$ just like classical HOMFLY-PT polynomials. Physically, these integer coefficients correspond to the BPS states on Calabi-Yau 3-folds.

3.2 Orthogonal LMOV conjecture

L. Chen and the author [6] gave a mathematically rigorous formulation of orthogonal LMOV conjecture dealing with the colored Kauffman invariants by using the theory of the Birman-Murakami-Wenzl algebras. By using the cabling technique, we obtained [6] a uniform formula of the colored Kauffman polynomial for all torus links. Then we were able to prove [6] many interesting cases of this orthogonal LMOV conjecture. The topological string side of the large N duality of the orthogonal LMOV conjecture is the open Gromov-Witten theory of orientifolds developed purely by physicists, while its math is still in its infancy.

3.3 Congruence skein relations for colored HOMFLY-PT invariants

The reformulated colored HOMFLY-PT invariant $\check{\mathcal{Z}}_{\vec{\mu}}(\mathcal{L};q,t)$ is defined as

$$\check{\mathcal{Z}}_{\vec{\mu}}(\mathcal{L};q,t) = [\vec{\mu}] \sum_{A^\alpha} \prod_{\alpha=1}^{L} \chi_{A^\alpha}(\mu^\alpha) \widetilde{W}_{\vec{A}}(\mathcal{L};q,t),$$

where $\widetilde{W}_{\vec{A}}(\mathcal{L};q,t)$ is the framing dependent colored HOMFLY-PT invariant colored by $\vec{A}$ and χ_{A^α} is the character of the irreducible representation indexed by the Young tableau A^α.

In particular, suppose $\vec{\mu} = ((p),...,(p))$ with L row partitions (p), for $p \in \mathbb{Z}_+$. We use the notation $\check{\mathcal{Z}}_p(\mathcal{L};q,t)$ to denote the reformulated colored HOMFLY-PT invariant $\check{\mathcal{Z}}_{((p),...,(p))}(\mathcal{L};q,t)$ for simplicity. Although the definition of $\check{\mathcal{Z}}_{\vec{\mu}}(\mathcal{L};q,t)$ seems complicated in the above definition, it has a simpler form than the colored HOMFLY-PT invariant

$W_{\vec{A}}(\mathcal{L};q,t)$ in the the HOMFLY-PT skein theory. In fact, $\check{\mathcal{Z}}_{\vec{\mu}}(\mathcal{L};q,t)$ has nice properties and it is natural to study the reformulated colored HOMFLY-PT invariant $\check{\mathcal{Z}}_{\vec{\mu}}(\mathcal{L};q,t)$ instead of $\widetilde{W}_{\vec{A}}(\mathcal{L};q,t)$.

All classical knot invariants can be defined from a simple computational rule, the skein relation. Under the above setups, classical skein relation for HOMFLY-PT polynomials can be restated as follows. For any link $\mathcal{L}$, we have

$$\check{\mathcal{Z}}_1(\mathcal{L}_+;q,t) - \check{\mathcal{Z}}_1(\mathcal{L}_-;q,t) = \check{\mathcal{Z}}_1(\mathcal{L}_0;q,t) \text{ type I}$$
$$\check{\mathcal{Z}}_1(\mathcal{L}_+;q,t) - \check{\mathcal{Z}}_1(\mathcal{L}_-;q,t) = \{1\}^2\check{\mathcal{Z}}_1(\mathcal{L}_0;q,t) \text{ type II},$$

where type I means self-crossing in a link component and type II means crossing in different link components.

The question "whether quantum invariants share the same property of certain skein relation" has perplexed people for quite a long time due to the complexity of definition of quantum invariants. The colored HOMLFY-PT invariants are notoriously hard to compute, and even the case of the figure-eight knot 4_1 with arbitrary shape of Young tableau is not established.

Inspired by studying the framed LMOV conjecture (a generalization of the original LMOV Conjecture mentioned in the last subsection), Chen-Liu-Peng-Zhu [7] discovered a very interesting phenomenon called congruence skein relations, which means that skein relations hold for (reformulated) colored HOMFLY-PT at certain roots of unity.

Conjecture 3.2 (Congruence skein relations for the colored HOMFLY-PT invariants, Chen-Liu-Peng-Zhu [7]). *For any link $\mathcal{L}$ and prime number p, we have*

$$\check{\mathcal{Z}}_p(\mathcal{L}_+;q,t) - \check{\mathcal{Z}}_p(\mathcal{L}_-;q,t) \equiv (-1)^{p-1}\check{\mathcal{Z}}_p(\mathcal{L}_0;q,t) \mod \{p\}^2 \text{ type I}$$
$$\check{\mathcal{Z}}_p(\mathcal{L}_+;q,t) - \check{\mathcal{Z}}_p(\mathcal{L}_-;q,t) \equiv (-1)^{p-1}p\{p\}^2\check{\mathcal{Z}}_p(\mathcal{L}_0;q,t) \mod \{p\}^2[p]^2 \text{ type II}$$

where $\{p\} = q^p - q^{-p}$, $[p] = \{p\}/\{1\}$. The notation $A \equiv B \mod C$ means $\frac{A-B}{C} \in \mathbb{Z}[(q-q^{-1})^2, t^{\pm 1}]$.

Different kinds of examples are confirmed in [7].

More excitingly, such congruence skein relations is not an isolated phenomenon. Chen-Zhu [12] first proved the integrality of composite invariants of full colored HOMFLY-PT invariants and discovered a type II congruence skein relation for them. But it looks like skein relations for Kauffman invariants. This could explain the mysterious mathematical nature of an LMOV type conjecture proposed by Marino in [42], connecting the full colored HOMFLY-PT invariants and the colored Kauffman invariants. We will *study congruence skein relations of the colored Kauffman invariants* and substantial evidence has been obtained [9].

3.4 A new vision on proposing Volume Conjectures inspired by Congruence relations and Habiro type cyclotomic expansion

Habiro [25] established the following cyclotomic expansion formula which has many applications in the area of TQFT.

Theorem 3.3 (Cyclotomic expansion for colored Jones polynomial, Habiro [25]). *For any knot $\mathcal{K}$, there exists $H_k(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of N $(N \geq 0)$, such that*

$$J_N(\mathcal{K}; q) = \sum_{k=0}^{N} C_{N+1,k} H_k(\mathcal{K}),$$

where $C_{N+1,k} = \{N-(k-1)\}\{N-(k-2)\} \cdots \{N-1\}\{N\}\{N+2\}\{N+3\} \cdots \{N+2+(k-1)\}$, for $k = 1, .., N$, and $C_{N+1,0} = 1$. In particular, $J_0(\mathcal{K}; q) = H_0(\mathcal{K}) = 1$.

We discover that the idea of "gap equations" in cyclotomic expansions plays an important role in Volume Conjectures. The root of unity used in original Volume Conjecture is $\frac{\pi\sqrt{-1}}{N+1}$, a solution of "gap equations" $\{N+1\} = 0$, where $\{N+1\}$ serves as a "gap" in $C_{N+1,k}$.

Chen-Liu-Zhu [8] discovered the cyclotomic expansion of colored $SU(n)$ invariants were indicated from their congruence relations. By studying the so called "gap equation" in cyclotomic expansions, we proposed a Volume Conjecture for the colored $SU(n)$ invariants. We think this new machinery will open a window **to understand the very mysterious essence of quantum invariants**, especially between Volume Conjectures and Habiro type cyclotomic expansion

In [7], we proposed a congruence relation conjecture for the colored $SU(n)$ invariants $J_N^{SU(n)}(\mathcal{K}; q)$ which is actually a consequence of the following conjecture.

Conjecture 3.4 (Cyclotomic expansions for colored $SU(n)$ invariants, Chen-Liu-Zhu [8]). *For any knot $\mathcal{K}$, there exist $H_k^{(n)}(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of N $(N \geq 0)$, such that*

$$J_N^{SU(n)}(\mathcal{K}; q) = \sum_{k=0}^{N} C_{N+1,k}^{(n)} H_k^{(n)}(\mathcal{K}),$$

where $C_{N+1,k}^{(n)} = \{N-(k-1)\}\{N-(k-2)\} \cdots \{N-1\}\{N\}\{N+n\}\{N+n+1\} \cdots \{N+n+(k-1)\}$, for $k = 1, .., N$, and $C_{N+1,0}^{(n)} = 1$. In particular, $J_0^{SU(n)}(\mathcal{K}; q) = H_0^{(n)}(\mathcal{K}) = 1$.

We choose solutions of "gap equations" as our roots of unity. In the above conjecture of cyclotomic expansions, we could see that the "gap equations" associated to the basis $C_{N+1,k}^{(n)}$ of the cyclotomic expansion are $\{N+a\} = 0$ for $a \in \{1, 2, ..., n-1\}$. We introduce roots of unity $\xi_{N,a}(s) = \exp(\frac{s\pi\sqrt{-1}}{N+a})$, where $a, s \in \mathbb{Z}$, which satisfy these "gap equations". Then for a fixed $n \geq 2$, we have

Conjecture 3.5 (Volume Conjecture for colored $SU(n)$ invariants, Chen-Liu-Zhu [8]). *If $a \in \{1, 2, ..., n-1\}$, then*

$$2\pi s \lim_{N\to\infty} \frac{\log J_N^{SU(n)}(\mathcal{K}; \xi_{N,a}(s))}{N+1} = Vol(S^3 \backslash \mathcal{K}) + \sqrt{-1} CS(S^3 \backslash \mathcal{K})$$

for any hyperbolic knot $\mathcal{K}$.

3.5 Applying this new vision to Superpolynomials

M. Khovanov introduced the idea of categorification that the reduced Poincare polynomial of Khovanov's homology $\mathcal{P}(\mathcal{K};q,t)$ recovers the Jones polynomial $J(\mathcal{K};q)$ i.e. $\mathcal{P}(\mathcal{K};q,-1) = J(\mathcal{K};q)$. Then Khovanov-Rozansky [32] generalized the categorification of the Jones polynomial to the categorification of the $sl(N)$ invariants, whose corresponding Poincare polynomial $\mathcal{P}^{sl(N)}(\mathcal{K};q,t)$ recovers the classical HOMFLY-PT polynomial with specialization $a = q^N$, i.e. $\mathcal{P}^{sl(N)}(\mathcal{K};q,-1) = P(\mathcal{K};q^N,q)$. The idea of the superpolynomial $\mathcal{P}(\mathcal{K};a,q,t)$ was introduced in [16] by Dunfield-Gukov-Rasmussen so that they could recover the classical HOMFLY-PT polynomial and Alexander polynomial respectively. This was further studied by Khovanov-Rozansky in [33]. The theory of the superpolynomial become a very active area which attracts many mathematician and physicists. Dunfield-Gukov-Rasmussen further argued [16] that the superpolynomial $\mathcal{P}(\mathcal{K};a,q,t)$ could recover $\mathcal{P}^{sl(N)}(\mathcal{K};q,t)$ under a certain differential d_N while the specialized superpolynomial $\mathcal{P}(\mathcal{K};t^{-1},q,t)$ could recover the Poincare polynomial $HFK(\mathcal{K};q^2,t)$ of Heegaard-Floer knot homology $\widehat{HFK}_i(\mathcal{K};s)$ under a certain differential d_0.

The author first proposed a congruence relation conjecture under some homological t-grading shifting just like the non-categorified colored $SU(n)$ invariants[7]. Based on these congruence relations, finally the author formulated the following cyclotomic expansion conjecture,

Conjecture 3.6 (Cyclotomic Expansion Conjecture of the Superpolynomial of colored HOMFLY-PT homology, Chen [5]). *For any knot $\mathcal{K}$, there exists an integer-valued invariant $\alpha(\mathcal{K}) \in \mathbb{Z}$, s.t. the reduced Superpolynomial $\mathcal{P}_N(\mathcal{K};a,q,t)$ of the colored HOMFLY-PT homology of a knot $\mathcal{K}$ has the following cyclotomic expansion formula*

$$\begin{aligned}&(-t)^{N\alpha(\mathcal{K})}\mathcal{P}_N(\mathcal{K};a,q,t)\\&= 1+\sum_{k=1}^{N}H_k(\mathcal{K};a,q,t)\left(A_{-1}(a,q,t)\prod_{i=1}^{k}\left(\frac{\{N+1-i\}}{\{i\}}B_{N+i-1}(a,q,t)\right)\right)\end{aligned}$$

with coefficient functions $H_k(\mathcal{K};a,q,t) \in \mathbb{Z}[a^{\pm1},q^{\pm1},t^{\pm1}]$, where $A_i(a,q,t) = aq^i + t^{-1}a^{-1}q^{-i}$, $B_i(a,q,t) = t^2aq^i + t^{-1}a^{-1}q^{-i}$ and $\{p\} = q^p - q^{-p}$.

Remark 3.1. The above Conjecture-Definition for the invariant $\alpha(\mathcal{K})$ should be understood in this way. If the above conjecture of a knot $\mathcal{K}$ is true for $N = 1$, then $\alpha(\mathcal{K})$ is defined.

We tested many homologically thick knots up to 10 crossings to illustrate this conjecture as well as many examples with higher representation. Based on highly nontrivial computations of torus knots/links studied in [17], we are able to prove the following theorem for torus knots.

Theorem 3.7 (Chen [5]). *For any coprime pair $(m,n) = 1$, where $m < n$, the cyclotomic expansion conjecture is true for torus knot $T(m,n)$ and we have $\alpha(T(m,n)) = -(m-1)(n-1)/2$.*

Now we are considering a problem relating to the sliceness of a knot.

Definition 3.1. The smooth 4-ball genus $g_4(\mathcal{K})$ of a knot $\mathcal{K}$ is the minimum genus of a surface smoothly embedded in the 4-ball B^4 with boundary the knot. In particular, a knot $\mathcal{K} \subset S^3$ is called smoothly slice if $g_4(\mathcal{K}) = 0$.

Conjecture 3.8 (Milnor Conjecture, proved by Kronheimer-Mrowka and Rasmussen)**.** *The smooth 4-ball genus for torus knot $T(m,n)$ is $(m-1)(n-1)/2$.*

Based on all the above results, we are able to propose the following conjecture.

Conjecture 3.9 (Chen [5])**.** *The invariant $\alpha(\mathcal{K})$ (determined by the cyclotomic expansion conjecture for $N=1$) is a lower bound for the smooth 4-ball genus $g_4(\mathcal{K})$, i.e. $|\alpha(\mathcal{K})| \leq g_4(\mathcal{K})$.*

Remark 3.2. Rasmussen [55] introduced a knot concordant invariant $s(\mathcal{K})$, which is a lower bound for the smooth 4-ball genus for knots. For all the knots we tested, it is identical to the Ozsváth-Szabó's τ invariant and Rasmussen's s invariant (up to a factor of 2).

Now we present certain motivation to propose our Volume Conjecture for reduced superpolynomials associated to the colored HOMFLY-PT homologies.

We have the following expression for the figure-eight knot 4_1 [27],

$$\mathcal{P}_N(4_1; a, q, t) = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} \left(\frac{\{N+1-i\}}{\{i\}} A_{i-2}(a,q,t) B_{N-1+i}(a,q,t) \right),$$

where $A_i(a,q,t) = aq^i + t^{-1}a^{-1}q^{-i}$, $B_i(a,q,t) = t^2aq^i + t^{-1}a^{-1}q^{-i}$ and $\{p\} = q^p - q^{-p}$.

Now we apply the idea of "gap equations" on the cyclotomic expansion of reduced superpolynomial of the colored HOMFLY-PT homology. By looking at middle terms $A_{N-2}(q^n,q,t) = q^{N+n-2} + t^{-1}q^{-(N+n-2)}$ and $B_N(q^n,q,t) = (-t)^2q^{N+n} + t^{-1}q^{-(N+n)}$, we get to know that the possible "gap equations" is the equation $(-t)q^{N+n-1} + t^{-1}q^{-(N+n-1)} = 0$.

By studying the above 2-variable "gap equations", we propose a Volume Conjecture for superpolynomials of the HOMFLY-PT homology at its solution $t = q^{-(N+n-1)}$ as follows.

Conjecture 3.10 (Volume Conjecture for the Superpolynomial of the HOMFLY-PT homology, Chen [5])**.** *For any hyperbolic knot $\mathcal{K}$, we have*

$$2\pi \lim_{N\to\infty} \frac{\log \mathcal{P}_N(\mathcal{K}; q^n, q, q^{-(N+n-1)})|_{q=e^{\frac{\pi\sqrt{-1}}{N+b}}}}{N+1} = Vol(S^3\backslash\mathcal{K}) + \sqrt{-1}CS(S^3\backslash\mathcal{K}) \ (mod \sqrt{-1}\pi^2\mathbb{Z}),$$

where $b \geq 1$ and $\frac{n-1-b}{2}$ is not a positive integer.

Remark 3.3. The choices of roots of unity in this conjecture are much more relaxed than those of the original Volume conjectures, because here b can be any large positive integers. For example, the original Volume Conjecture is only proposed for $n=2$ and $b=1$, but this Volume Conjecture is proposed for all positive integer b with $n=2$.

It will be interesting to know the relationship of this volume conjecture to the one proposed in [19], where they used categorified A-polynomials of knots.

We think this new vision on Volume Conjecture creates many problems, which opens a new window **to understand the very mysterious essence of quantum invariants,** and form another key goal of this survey.

Cyclotomic expansion for the reduced superpolynomial $\mathcal{F}_N(\mathcal{K}; a, q, t)$ of the colored Kauffman homology (Gukov-Walcher [24]) were also proposed by the author in [5].

The author address the following question in Problem set of ITLDT conference. From the above discussion and many results we have obtained, the author have a feeling that all quantum invariants may have such cyclotomic expansion and will also have Volume Conjecture from studying of such "Gap equations" indicated from the corresponding cyclotomic expansion.

Here is a summary of various results.

Congruence skein relations for the colored Jones ((6.4), (6.5) of Chen-Liu-Peng -Zhu [7]v2 14')	**indicate** $\leftarrow$	Conjecture of Congruence skein relations for the reformulated colored HOMFLY-PT (Chen-Liu-Peng-Zhu [7] 14')	**indicate** $\leftarrow$	framed LMOV conjecture (still open, original conjecture proved by Liu-Peng [41] 09')
indicate $\downharpoonleft\upharpoonright$ **prove**				
Congruence relations for the colored Jones (Thm 7.3 of Chen-Liu -Peng-Zhu [7]v3 14')	**indicate** $\rightleftharpoons$ **prove**	Cyclotomic Expansion for the colored Jones (Habiro [25] 08')	**indicate** **via "gap** **equations"** $\rightarrow$	Volume Conjecture for the colored Jones (Kashaev [28] 97' & Murakami- Murakami [45] 01')
special case $\uparrow$		**special case** $\uparrow$		**special case** $\uparrow$
Congruence relations for the colored $SU(n)$ invariants (Chen-Liu -Peng-Zhu [7] 14')	**indicate** $\rightleftharpoons$ **prove**	Conjecture of Cyclotomic Expansion for the colored $SU(n)$ invariants (Chen-Liu-Zhu [8] 15')	**indicate** **via "gap** **equations"** $\rightarrow$	Volume Conjecture for the colored $SU(n)$ invariants (Chen-Liu-Zhu [8] 15')
special case $\uparrow$		**special case** $\uparrow$		**special case** $\uparrow$
Congruence relations for the Superpolynomials of the colored HOMFLY-PT homology (Chen [5], 15'-16')	**indicate** $\rightleftharpoons$ **prove**	Conjecture of Cyclotomic Expansion for the Superpolynomial of the colored HOMFLY-PT homology (Chen [5], 15'-16')	**indicate** **via "gap** **equations"** $\rightarrow$	Volume Conjecture for the Superpolynomials of the colored HOMFLY-PT homology (Chen [5], 15'-16')

Relations between various HOMFLY-PT theories

Congruence skein relations for reformulated composite colored HOMFLY (looks like Kauffman style) (Chen-Zhu [12], 14')	**indicate** $\leftharpoonup$	Special linear-Orthogonal LMOV conjecture (formulated by Marino [42], 09')
Congruence skein relations for reformulated colored Kauffman (Chen-Liu-Zhu [9], in preparation)	**indicate** $\leftharpoonup$	Orthogonal LMOV conjecture (formulated and several cases proved by L.Chen -Q.Chen [6], 08')
Conjecture of Cyclotomic Expansion for the superpolynomials of the colored Kauffman homology (Chen [5], 15'-16')	**indicate via "gap equations"** $\rightarrow$	Volume Conjecture for the superpolynomials of the colored Kauffman homology (In preparation)

Relations between various Kauffman theories

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