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On the finite space with a finite group action II

福井大学医学部　藤田亮介 (Ryousuke Fujita)
School of Medical Sciences, University of Fukui

1 Introduction

The purpose of our presentation was to apply the finite topology theory to the subgroup complex theory. A *finite* T_0*-space* is a topological space having finitely many points with the T_0-separation axioms, that is, for each pair of distinct two points, there exists an open set containing one but not the other. Many well-known properties about finite T_0-spaces may be found in [1], [2] and [5]. Moreover we consider the finite space with a finite group G-action, called a *finite* T_0-G*-space.*

On the other hand, we are interested in homotopy properties on subgroup complexes of a finite group. Let G be a finite group and p a prime factor of the order of G. Let $O_p(G)$ be the maximal normal p-subgroup of G. The Bouc poset(= partially ordered set) $B_p(G)$ of a finite group G is the subposet of $S_p(G)$ with $O_p(N_G(P))=P$, where $N_G(P)$ is the normalizer of P and $S_p(G)$ is the poset of the non-trivial p-subgroups of G ordered by inclusion. We remark that the Bouc poset $B_p(G)$ contains all Sylow p-subgroups of G. Let $\Delta(B_p(G))$ denote the order complex of $B_p(G)$, that is, the vertices are the elements of $B_p(G)$ and the n-simplices are the chains of p-subgroups of $B_p(G)$ of length n. This simplicial complex is called the *Bouc complex* of G at p.

Quillen examined the simplicial complex $\Delta(S_p(G))$ associated with the poset $S_p(G)$. In particular, let us take a finite solvable group G. The main theorem of his paper [4] is that $\Delta(S_p(G))$ is contractible if and only if there is a non-trivial normal p-subgroup. Our study is motivated by this result.

McCord's result [3, Theorem 2] provides deep insight into understanding relations between finite T_0-spaces and finite simplicial complexes. For a finite T_0-space X, we can define the order complex $\Delta(X)$. Let $|\Delta(X)|$ be the geometric realization of $\Delta(X)$.

Proposition 1.1. *There exists a weak homotopy equivalence* $\mu_X:|\Delta(X)|\to X$. *Moreover, each map* $\varphi:X\to Y$ *between finite* T_0*-spaces defines a simplicial map* $\Delta(\varphi):\Delta(X)\to\Delta(Y)$ *by* $\Delta(\varphi)(x)=\varphi(x)$, *and* $\varphi\circ\mu_X=\mu_Y\circ|\Delta(\varphi)|$ *where* $|\Delta(\varphi)|:|\Delta(X)|\to|\Delta(Y)|$ *is a continuous map induced by* $\Delta(\varphi)$.

Corollary 1.2. *Let* $\varphi:X\to Y$ *be a map between finite* T_0*-spaces. Then* φ *is a weak homotopy equivalence if and only if* $|\Delta(\varphi)|:|\Delta(X)|\to|\Delta(Y)|$ *is a homotopy equivalence.*

Then we show the following:

Theorem A. *Let* G *be a finite nilpotent group and* p *any prime factor of the order of* G. *Then* $\Delta(B_p(G))$ *is contractible.*

We apply McCord's theorem to give a very short, purely topological proof of the above result.

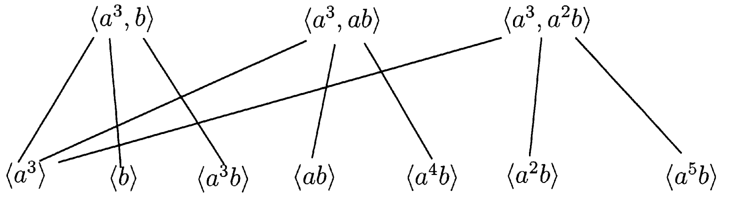
2 Some examples of Bouc posets

For the convenience of the reader, we present some examples of Bouc posets.

Example 2.1. Take $G = D_{12}$, the dihedral group of order 12, and $p = 2$. We can give the abstract presentation of G by the generators and relations:

$$G = \langle a, b \,|\, a^6 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle;$$

where these represent a rotation and a reflection, when G is regarded concretely as the group of a regular hexagon. We find three Sylow 2-subgroups of order 4: $\langle a^3, b\rangle$, $\langle a^3, ab\rangle$, $\langle a^3, a^2b\rangle$, and the minimal members are generated by 7 involutions: $\langle a^3\rangle$, $\langle b\rangle$, $\langle ab\rangle$, $\langle a^2b\rangle$, $\langle a^3b\rangle$, $\langle a^4b\rangle$, $\langle a^5b\rangle$. Thus the poset diagram for $S_2(D_{12})$ is given by:

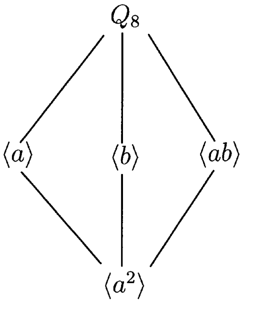


Observe that each of three Sylow 2-subgroups is not the normal subgroup of G and the center $Z(G)$ of G equals $\langle a^3\rangle$. Therefore $B_2(G) = \{\langle a^3, b\rangle,\ \langle a^3, ab\rangle,\ \langle a^3, a^2b\rangle,\ \langle a^3\rangle\}$.

Example 2.2. Take $G = Q_8$, the quaternion group of order 8, and $p = 2$. We can give the abstract presentation of G by the generators and relations:

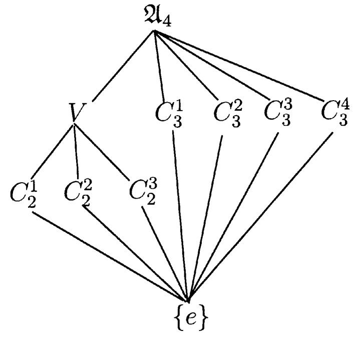
$$G = \langle a, b \,|\, a^4 = 1,\ b^2 = a^2,\ b^{-1}ab = a^{-1} \rangle.$$

We find three Sylow 2-subgroups of order 4: $\langle a\rangle$, $\langle b\rangle$, $\langle ab\rangle$, and each of these three Sylow 2-subgroups contains the unique cyclic subgroup $\langle a^2\rangle$. Thus the poset diagram for $S_2(Q_8)$ is given by:



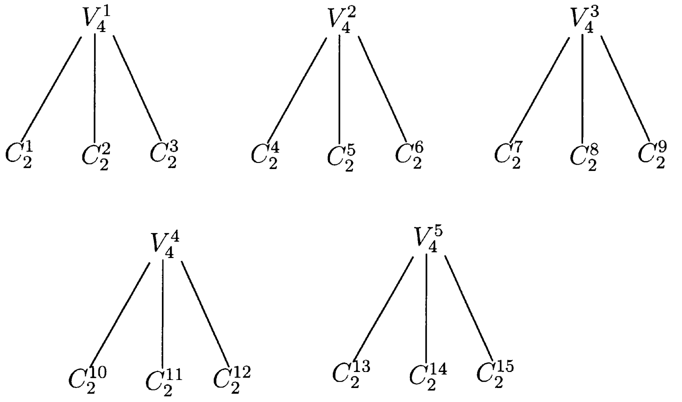
Since any subgroup of G is normal, so that $B_2(G) = \{Q_8\}$.

Example 2.3. Take $G = \mathfrak{A}_4$, the alternative group of letter 4. We find one Sylow 2-subgroup of order 4 and four Sylow 3-subgroups of order 3. The subgroups diagram for $\mathfrak{A}_4$ is given by:



Here V is a Klein group, each C_3^i $(i = 1, 2, 3, 4)$ is a distinct cyclic group of order 3, and each C_2^j $(j = 1, 2, 3)$ is a distinct cyclic group of order 2. Then $B_2(G) = \{V\}$ and $B_3(G) = \{C_3^1, C_3^2, C_3^3, C_3^4\}$.

Example 2.4. Take $G = \mathfrak{A}_5$, the alternative group of letter 5, and $p = 2$. By easy observation, we find five Sylow 2-subgroups of order 4, and each Sylow 2-subgroup contains three cyclic groups of order 2. Thus the poset diagram for $S_2(\mathfrak{A}_5)$ is given by:

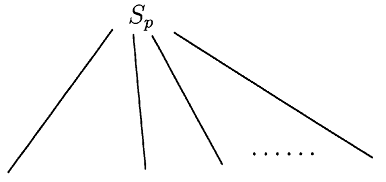


Here each V_4^i $(1 \leq i \leq 5)$ is a distinct Klein group, each C_2^j $(1 \leq j \leq 15)$ is a distinct cyclic group of order 2. Then $B_2(\mathfrak{A}_5)) = \{V_4^1, V_4^2, V_4^3, V_4^4, V_4^5\}$.

3 Proof of Theorem A

We address to the article written by Barmak and cited in Bibliography. Stong studied equivariant homotopy theory for finite T_0-spaces [6]. Let G be a finite group. A finite T_0-space with a G-action is called a *finite T_0-G-space.* Any finite T_0-G-space X has a core which is G-invariant and an equivariant strong deformation retract of X. Such a core is called a *G-core*. See our general reference Barmak [1, p106] for details. Note that a finite T_0-G-space is contractible if and only if its G-core consists of a point. We remark that $B_p(G)$ is a finite T_0-G-space by conjugation.

Proof of Theorem A If G is a finite nilpotent group, then G has a unique Sylow p-subgroup S_p. The poset diagram for $B_p(G)$ is given by:



By this diagram, the G-core of $B_p(G)$ is $\{S_p\}$, and so $B_p(G)$ is contractible. By McCord's theorem (Proposition 1.1), there exists the following commutative diagram:

$$\begin{array}{ccc} |\Delta(B_p(G))| & \xrightarrow{|\Delta(f)|} & |\Delta(\{S_p\})| \\ {\scriptstyle \mu_{B_p(G)}}\downarrow & & \downarrow{\scriptstyle \mu_{\{S_p\}}} \\ B_p(G) & \xrightarrow{f} & \{S_p\} \end{array}$$

where $f : B_p(G) \to \{S_p\}$ is homotopy equivalent. By Corollary 1.2, map $|\Delta(f)| : |\Delta(B_p(G))| \to |\Delta(\{S_p\})|$ is also homotopy equivalent. Therefore $|\Delta(B_p(G))|$ is contractible, that is, $\Delta(B_p(G))$ is contractible. □

Corollary B. *Let pq be the order of G such that p and q are distinct primes with $p > q$. Then $\Delta(B_p(G))$ is contractible.*

Proof. The number of Sylow p-subgroups of G is equivalent to 1 mudulo p. Moreover it is also the devisor of pq. Therefore the number of Sylow p-subgroups of G is 1, and so the Sylow p-subgroup is normal. □

For example, take $G = \mathfrak{S}_3$, the symmetric group of letter 3. Then $\Delta(B_3(\mathfrak{S}_3))$ is contractible.

4 Concluding remarks

Lemma 4.1. *A contractible finite T_0-G-space has a point which is fixed by the action of G.*

Proof. A contractible finite T_0-G-space has a G-core, i.e. a point, which is G-invariant. □

We showed the followig result in [2].

Lemma 4.2. *Let X be a finite T_0-G-space. Then $|\Delta(X)|/G$ is homotopy equivalent to $|\Delta(X)/G|$.*

Suppose that $B_p(G)$ is contractible. Then lemma 4.1 claims that G has a normal p-subgroup. Moreover the orbit space $B_p(G)/G$ of $B_p(G)$ is a finite T_0-space and also contractible.

Proposition 4.3. *Let $|\Delta(B_p(G))/G|$ be the geometric realization of $\Delta(B_p(G))/G$. If $B_p(G)$ is contractible, $|\Delta(B_p(G))/G|$ is also contractible.*

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[6] Stong, R.E., *Group actions on finite spaces*, Discrete Math. **49** (1984), 95-100.