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A STUDY OF BORSUK-ULAM TYPE INEQUALITIES

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ABSTRACT. After recalling Borsuk-Ulam type theorems, we shall provide some families of Borsuk-Ulam groups. As an application, we shall show variants of the (isovariant) Borsuk-Ulam theorem for such families.

1. Borsuk-Ulam type theorems

K. Borsuk [2] proved the following result, called the Borsuk-Ulam theorem.

Theorem 1.1. *Let S^n be the unit sphere centered at the origin of $\mathbb{R}^{n+1}$, $n \geq 1$.*

(1) *For any continuous map $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.*
(2) *If there exists an antipodal map $f : S^m \to S^n$, then $m \leq n$.*

This result was generalized in following way.

Theorem 1.2 (Free version). (1) *Suppose that a finite group G acts freely on S^m, S^n. If there exists a G-map $f : S^m \to S^n$ then $m \leq n$.*
(2) *Let X be a path-connected free C_k-space and Y a Hausdorff free C_k-space, $k \geq 2$. If $H_q(X; \mathbb{Z}_k) = 0$ for $1 \leq q \leq m$ and $H_{m+1}(Y/C_k; \mathbb{Z}_k) = 0$ for some $m \geq 1$, then there is no continuous C_k-map $f : X \to Y$ ([7]).*

Theorem 1.3 (Fixed-point-free version [5]). *Suppose that $G = {C_p}^k$ acts fixed-point-freely acts on S^m, S^n. If there exists a G-map $f : S^m \to S^n$, then $m \leq n$.*

Moreover, T. Bartsch [1] researched Borsuk-Ulam type theorems for G-maps between fixed-point-free representation spheres for any compact Lie group G. Consequently,

Theorem 1.4 ([1]). *Let G be a finite group. The weak Borsuk-Ulam theorem holds for G if and only if G is a p-group.*

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Here "weak" means that a weaker inequality for dimensions holds, see [1] for the detail.

2. Isovariant version

A G-map $f : X \to Y$ is called *G-isovariant* if $G_{f(x)} = G_x$ for any $x \in X$. Wasserman [9] studied an isovariant version of the Borsuk-Ulam theorem. Let V, W be orthogonal representations of G, and $S(V)$, $S(W)$ their unit spheres. The Borsuk-Ulam theorem for a free action leads to the following.

Proposition 2.1. *Let C_p be a cyclic group of prime order p. If there is a C_p-isovariant map $f : S(V) \to S(W)$, then*

$$\dim V - \dim V^{C_p} \leq \dim W - \dim W^{C_p}.$$

Definition. G is called a *Borsuk-Ulam group* (BUG) if the isovariant Borsuk-Ulam theorem holds for G-representations; namely, if there is a G-isovariant map $f : S(V) \to S(W)$, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G \text{ (Borsuk-Ulam inequality)}$$

holds.

A main problem is

Problem (unsolved). *Which groups are Borsuk-Ulam groups?*

There are some partial results on this problem.

Theorem 2.2 ([9]). *If G satisfies the prime condition, then G is a Borsuk-Ulam group.*

Here we say that G satisfies the prime condition if the following condition (PC):
Let $1 = G_0 \lhd \cdots \lhd G_r = G$ be a composition series. For every simple factor $F_i := G_{i+1}/G_i$,

$$p(g) := \sum_{p\text{: prime factor of } |g|} \frac{1}{p} \leq 1$$

for any $g \in F_i$.

Example 2.3 ([9]). (1) *Solvable groups.*
(2) A_5, A_6, $\ldots$, A_{11} *(but A_n, $n \geq 12$, do not satisfy (PC).)*

There is another type of example.

Theorem 2.4 ([8]). *$PSL(2,q)$ is a Borsuk-Ulam group, where q is a prime power.*

Remark. $PSL(2,59)$, $PSL(2,61)$, etc do not satisfy (PC).

This result is proved as the following way. Let $G = PSL(2,q)$ and $f : S(V) \to S(W)$ be G-isovariant. Then $\mathrm{res}_C f : S(V) \to S(W)$ is C-isovariant for any $C \in Cy(G)$, where $Cy(G)$ is the set of cyclic subgroups of G. Thus

$$\dim V - \dim V^C \leq \dim W - \dim W^C, \quad \forall C \in Cy(G)$$

$$(\because C \text{ is a BUG.})$$

$$\Downarrow \ (Q)$$

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

Question. *Which groups satisfy implication* (Q) *as above?*

We here call such a group G a (Q)-*group.*

Remark. A (Q)-group is a Borsuk-Ulam group.

Proposition 2.5 ([6]). *An abelian* (Q)*-group is cyclic or* ${C_p}^k$ *for some prime* p *and* $k \geq 0$.

3. Algebraic description of the Borsuk-Ulam inequality

We first recall the Möbius function. Let $S(G)$ be the set of subgroups of G.

Definition (Möbius function $\mu : S(G) \times S(G) \to \mathbb{Z}$). For H, $K \in S(G)$ with $H \leq K$, μ is inductively defined by

$$\sum_{H \leq L \leq K} \mu(H, L) = \begin{cases} 0 & H < K \\ 1 & H = K. \end{cases}$$

For $H \not\leq K$, set $\mu(H,K) = 0$.

As is well-known,

Proposition 3.1 (Möbius inversion).

$$f(K) = \sum_{H \leq K} g(H) \ (\forall K \in S(G))$$

$$\Updownarrow$$

$$g(K) = \sum_{H \leq K} \mu(H,K) f(H) \ (\forall K \in S(G))$$

Let V, W be G-representations and χ_V, χ_W their characters. For $H \in S(G)$,

$$\dim W^H - \dim V^H = \frac{1}{|H|} \sum_{g \in H} (\chi_W(g) - \chi_V(g))$$

$$\begin{aligned}\dim W - \dim V &= \chi_W(1) - \chi_V(1)\\ &= \frac{1}{|H|}\sum_{g\in H}(\chi_W(1) - \chi_V(1))\end{aligned}$$

Set

$$e(g) = \chi_W(1) - \chi_W(g) - \chi_V(1) + \chi_V(g)$$

and

$$h(H) = \sum_{g\in H} e(g).$$

Lemma 3.2.

$$h(H) = |H|(\dim W - \dim W^H - \dim V + \dim V^H).$$

In particular, $h(1) = 0$ and the Borsuk-Ulam inequality is equivalent to $h(G) \geq 0$. For $H \in S(G)$, we see

$$h(H) = \sum_{D\in Cy(H)} \sum_{g\in D^*} e(g),$$

where D^* is the set of generators of D. Set

$$k(D) = \sum_{g\in D^*} e(g).$$

For $C \in Cy(G)$,

$$h(C) = \sum_{D\leq C} k(D).$$

The Möbius inversion (on $Cy(G)$) says

$$k(C) = \sum_{D\leq C} \mu(D, C)h(D).$$

On the other hand,

$$h(G) = \sum_{g\in G} e(g) = \sum_{C\in Cy(G)} k(C).$$

Thus we have

$$\begin{aligned}h(G) &= \sum_{C\in Cy(G)} \sum_{D\leq C} \mu(D, C)h(D)\\ &= \sum_{1\neq D\in Cy(G)} \Big(\sum_{D\leq C\in Cy(G)} \mu(D, C)\Big)h(D).\end{aligned}$$

For $D \in Cy(G)$, set

$$m(D) = \sum_{D\leq C\in Cy(G)} \mu(D, C).$$

Clearly we see

Lemma 3.3. *If $m(D) \geq 0$ for $1 \neq D \in Cy(G)$, then (Q) holds.*

The following are examples of (Q)-groups: C_n, D_n, ${C_p}^k$, $PSL(2,q)$. On the other hand, the quaternion group Q_8 is not a (Q)-group. Furthermore, there is a class of finite groups with $m(D) \geq 0$ for any cyclic $D \neq 1$.

Definition. A (P)-group G is defined to be a finite group such that any nontrivial element has prime order.

Lemma 3.4. *A (P)-group is a (Q)-group.*

Proof. $m(D) = 1$ for cyclic $D \neq 1$. □

Clearly, if a (P)-group G is abelian, then G is ${C_p}^k$ for some p, k, but there are many nonabelian examples.

There are three types of (P)-groups, see [3] for the detail:

(1) p-groups with exponent p.
ex. ${C_p}^k$, $P_3 = \langle x, y, z | x^p = y^p = z^p = 1,\ [x,y] = z,\ [x,z] = [y,z] = 1\rangle$. ($|P_3| = p^3$.)

(2) $1 \to P \to G \to C_q \to 1$ (ex) with some condition, where P is of type 1 above. (p, q are distinct primes.)
ex. $Z_{p,q}$ metacyclic group of order pq, A_4.

(3) A_5. This is only one nonsolvable (P)-group.

4. Variants of the Borsuk-Ulam theorem

Proposition 4.1. *Let G be a (Q)-group and V, W G-representations. If there is a C-isovariant map*

$$f_C : S(V) \smallsetminus S(V)^C \to S(W) \smallsetminus S(W)^C$$

for every $C \in Cy(G)$, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

Proof. Note that $S(V) \smallsetminus S(V)^C$ is C-isovariantly homotopy equivalent to $S(V - V^C)$. Thus there exists a C-isovariant map

$$\bar{f}_C : S(V - V^C) \to S(W - W^C).$$

By the isovariant Borsuk-Ulam theorem for C, we obtain

$$\dim V - \dim V^C \leq \dim W - \dim W^C$$

for every $C \in Cy(G)$. The property (Q) shows

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

□

Remark. In this case, there is not necessarily a G-isovariant map $f : S(V) \to S(W)$.

Corollary 4.2. *Let G be a (P)-group and V, W G-representations. If there is a C-map*

$$f_C : S(V) \smallsetminus S(V)^C \to S(W) \smallsetminus S(W)^C$$

for every $C \in Cy(G)$, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

Proof. Indeed, since C has a prime order, f_C is automatically C-isovariant. □

We finally consider the nonlinear case.

Theorem 4.3. *Let G be a (P)-group of type 1, i.e., a p-group of exponent p. Let Σ_1 and Σ_2 be mod p homology spheres with smooth G-actions. If there is a C-map*

$$f_C : \Sigma_1 \smallsetminus \Sigma_1^C \to \Sigma_2 \smallsetminus \Sigma_2^C$$

for every $C \in Cy(G)$, then

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G.$$

Proof (Sketch). Note the following facts.

Facts. (1) *Σ_i^H is a mod p homology sphere (possibly empty) for any $H \leq G$ by Smith theory.*

(2) *$\Sigma_i \smallsetminus \Sigma_i^C$ is a mod p homology sphere with homological dimension* $\dim \Sigma_i - \dim \Sigma_i^C - 1$.

(3) *Σ_i has a linear dimension function, i.e., there is a G-representation V_i such that* $\dim \Sigma_i^H = \dim S(V_i)^H$ *for any $H \leq G$* ([4]).

Since $C \cong C_p$ acts freely on $\Sigma_i \smallsetminus \Sigma_i^C$, it follows that

$$\dim \Sigma_1 - \dim \Sigma_1^C \leq \dim \Sigma_2 - \dim \Sigma_2^C.$$

by a Borsuk-Ulam type theorem.

By Fact (3), we obtain

$$\dim V_1 - \dim V_1^C \leq \dim V_2 - \dim V_2^C.$$

This implies

$$\dim V_1 - \dim V_1^G \leq \dim V_2 - \dim V_2^G.$$

Thus

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G.$$

□

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