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RIGHT-ANGLED COXETER QUANDLES AND POLYHEDRAL PRODUCTS

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1. Introduction

This report is a survey of the paper [K].

A quandle is a set with a binary operation satisfying three conditions which axiomatize conjugation. In particular, there are mutually adjoint constructions of quandles and groups through conjugation: (1) we associate a quandle to a conjugation closed subset of a group; (2) we associate a group to a quandle. Thus by composing these two constructions, we can construct a new group out of a given conjugation closed subset of a group. However, the interest of the resulting group entirely depends on the choice of a conjugation closed subset of a group. One of good choices is the set of reflection of a Coxeter system (W, S) [N], and we call the resulting group the adjoint group of W. The adjoint group of W is given by an explicit presentation, but its structure has not been known except for the case when W is the symmetric group [AFGV, E]. Recently, Akita [A] investigates the natural connection between the adjoint group of W and the underlying Coxeter group W to describe the structures of the adjoint group of W. We will study the classifying space of the adjoint group of W based on Akita's result [A] in terms of spaces called polyhedral products when W is right-angled.

2. Coxeter quandles and Akita's result

2.1. **Coxeter systems.** A pair (W, S) of a group W and a finite set S is called a *Coxeter system* if it is equipped with a map $m\colon S\times S\to\{1,2,\ldots,\infty\}$ satisfying the following three conditions:

(1) $m(s,t)=m(t,s)$;
(2) $m(s,t)=1$ if and only if $s=t$;
(3) the group W has a presentation

$$W=\langle s\in S\,|\,(st)^{m(s,t)}=1\text{ for }m(s,t)<\infty\rangle.$$

Hereafter, (W, S) will denote a Coxeter system. Our reference of Coxeter groups is [D].

The Artin group associated with a Coxeter system (W, S) is defined by

$$A_W=\langle a_s\,(s\in S)\,|\,\underbrace{a_sa_ta_s\cdots}_{m(s,t)}=\underbrace{a_ta_sa_t\cdots}_{m(t,s)}\text{ for }1<m(s,t)<\infty\rangle.$$

Then the Coxeter group W is obtained by adding the relations $a_s^2 = 1$ to A_W and replacing a_s with s for $s \in S$ so that there is a natural projection

$$\pi\colon A_W \to W, \quad a_s \mapsto s.$$

2.2. **Coxeter quandles.** Recall that a *quandle* X is a set equipped with a binary operation $*\colon X \times X \to X$ satisfying the following three conditions:

(1) $x * x = x$;
(2) $(x * y) * z = (x * z) * (y * z)$;
(3) the map $X \to X$, $x \mapsto x * w$ is bijective for any $w \in X$.

We consider the following two mutually adjoint constructions producing groups from quandles and vice versa. To a quandle X there is associated a group

$$\mathrm{Ad}\,(X) = \langle e_x \ (x \in X) \,|\, e_{x*y} = e_y^{-1} e_x e_y \rangle \tag{2.1}$$

which is called the *adjoint group*, and a conjugation closed subset R of a group G is regarded as a quandle by the binary operation

$$*\colon R \times R \to R, \quad g * h = h^{-1} g h.$$

Then in particular, from a given conjugation closed subset X of a group one gets a new group $\mathrm{Ad}\,(X)$. and moreover, it is easy to verify that the map

$$\phi\colon \mathrm{Ad}\,(X) \to G, \quad e_x \mapsto x$$

is a well-defined surjection.

An element of the form $w^{-1} s w$ for $w \in W, s \in S$ is called a reflection of W, and we denote by X_W the set of all reflections of W. Then X_W is a quandle by the above construction and we call it the *Coxeter quandle* associated with a Coxeter system (W, S). Thus we get a group $\mathrm{Ad}\,(X_W)$, the classifying space of which is our object to study.

2.3. **Akita's results.** In [A], Akita proved several properties of $\mathrm{Ad}\,(X_W)$, and these results are summarized as follows.

Theorem 2.1. *The commutative square*

$$\begin{tikzcd} \mathrm{Ad}\,(X_W) \arrow[r, "\mathrm{ab}"] \arrow[d, "\phi"] & \mathbb{Z}^{c(W)} \arrow[d, "\mathrm{proj}"] \\ W \arrow[r, "\mathrm{ab}"] & (\mathbb{Z}/2)^{c(W)} \end{tikzcd}$$

is a pullback, where $c(W)$ is the integer explicitly defined by W.

Since there is a commutative diagram

$$\begin{tikzcd} A_W \arrow[r, "\mathrm{ab}"] \arrow[d, "\pi"] & \mathbb{Z}^{c(W)} \arrow[d, "\mathrm{proj}"] \\ W \arrow[r, "\mathrm{ab}"] & (\mathbb{Z}/2)^{c(W)}, \end{tikzcd}$$

we have:

Corollary 2.2 (Akita [A]). *The map $\pi\colon A_W \to W$ factors as the composite of surjections*

$$A_W \to \mathrm{Ad}\,(X_W) \xrightarrow{\phi} W.$$

Theorem 2.1 passes to a homotopy pullback by taking the classifying spaces.

Theorem 2.3. *There is a homotopy pullback*

$$\begin{tikzcd} B\mathrm{Ad}\,(X_W) \arrow[r] \arrow[d, "\phi"] & (S^1)^{c(W)} \arrow[d, "\mathrm{incl}"] \\ BW \arrow[r] & (\mathbb{R}P^\infty)^{c(W)}. \end{tikzcd}$$

3. Right-angled Coxeter groups and polyhedral products

3.1. **Right-angled Coxeter groups.** A Coxeter system (W, S) is called *right-angled* if $m(s,t) = 1, 2, \infty$ for any $s, t \in S$. We characterize the right-angularity of a Coxeter system (W, S) in terms of graph products of groups. Let G be a group and Γ be a simple graph with the vertex set V. The graph product of G_Γ is the quotient of the free product of $G_v = G$ for $v \in V$ by the commutator relations $[G_u, G_v] = 1$ whenever the vertices u, v are adjacent in Γ. Graph products of groups are natural in the sense that for a homomorphism $G \to H$ of groups and a subgraph inclusion $\Theta \to \Gamma$, there are homomorphisms

$$G_\Gamma \to H_\Gamma \quad \text{and} \quad G_\Theta \to G_\Gamma.$$

We regard a set S to be a discrete graph so that the graph product $\mathbb{Z}_S$ is identified with the free group generated by S. We then set

$$\theta\colon \mathbb{Z}_S \to W, \quad \tilde{\theta}\colon \mathbb{Z}_S \to A_W, \quad \bar{\theta}\colon \mathbb{Z}_S \to \mathrm{Ad}\,(X_W)$$

to be the projections induced from the inclusions of generators.

Define a graph Γ_W for a Coxeter system (W, S) such that its vertex set is S and $s, t \in S$ are adjacent in Γ_W whenever $m(s,t) = 2$. Then if (W, S) is right-angled, we have

$$W = \langle s \in S \,|\, s^2 = 1,\ [s,t] = 1 \text{ for } m(s,t) = 2\rangle \cong (\mathbb{Z}/2)_{\Gamma_W}.$$

Moreover, taking the generating set S into account, we get:

Proposition 3.1. *A Coxeter system (W, S) is right-angled if and only if there is an isomorphism*

$$W \cong (\mathbb{Z}/2)_{\Gamma_W}$$

such that the map $\theta\colon \mathbb{Z}_S \to W$ is identified with the projection $\mathbb{Z}_S \to (\mathbb{Z}/2)_{\Gamma_W}$.

If a Coxeter system (W, S) is right-angled, we also have

$$A_W = \langle s \in S \mid [s,t] = 1 \text{ for } m(s,t) = 2\rangle \cong \mathbb{Z}_{\Gamma_W}.$$

Then we similarly get:

Proposition 3.2. *If a Coxeter system (W, S) is right-angled, there is an isomorphism*

$$A_W \cong \mathbb{Z}_{\Gamma_W}$$

such that the map $\tilde{\theta}\colon \mathbb{Z}_S \to A_W$ is identified with the projection $\mathbb{Z}_S \to \mathbb{Z}_{\Gamma_W}$.

3.2. **Polyhedral products.** We translate Proposition 3.1 into homotopy theory by using spaces called polyhedral products. Let (X, A) be a pair of spaces and K be an abstract simplicial complex on the vertex set $[m]$. We associate to $\sigma \in K$, possibly empty, a subspace $D(\sigma)$ of X^m such that

$$D_{(X,A)}(\sigma) = Y_1 \times \cdots \times Y_m, \quad Y_i = \begin{cases} X & i \in \sigma \\ A & i \notin \sigma. \end{cases} \tag{3.1}$$

The *polyhedral product* of (X, A) associated with K is now defined by

$$Z(K; (X, A)) = \bigcup_{\sigma \in K} D_{(X,A)}(\sigma).$$

Although polyhedral products are defined more generally for a sequence of pairs of spaces, a single pair of spaces is enough for our purpose here. We refer to [BBCG, IK] for the basic homotopy theory of polyhedral products.

We express the classifying space of a right-angled Coxeter group by a polyhedral product. A simplicial complex is called *flag* if every collection of pairwise adjacent vertices is its simplex. For a simple graph Γ, we denote by $\Delta(\Gamma)$ the flag complex whose 1-skeleton is Γ.

Proposition 3.3. *A Coxeter system (W, S) is right-angled if and only if there is a homotopy equivalence*

$$BW \simeq Z(\Delta(\Gamma_W); (\mathbb{R}P^\infty, *))$$

such that the map $\theta\colon B\mathbb{Z}_S \to BW$ is identified with the inclusion $Z(S; (S^1, *)) \to Z(\Delta(\Gamma_W); (\mathbb{R}P^\infty, *))$.

4. Main theorem and its applications

4.1. **Main theorem.** By an old trick of homotopy theory, we can prove:

Lemma 4.1 (cf. [DS, Lemma 2.3.1]). *Let $(F, F') \to (E, E') \to B$ be a pair of homotopy fibrations where $(F, F'), (E, E')$ are NDR pairs. Then the sequence*

$$Z(K; (F, F')) \to Z(K; (E, E')) \to B^m$$

is a homotopy fibration.

Using this lemma, we can prove the following.

Proposition 4.2. *Let $F \to E \to B$ be a homotopy fibration such that the fiber inclusion is a cofibration. Then the commutative square*

$$\begin{array}{ccc} Z(K; (E, F)) & \xrightarrow{\text{incl}} & E^m \\ \downarrow & & \downarrow \\ Z(K; (B, *)) & \xrightarrow{\text{incl}} & B^m \end{array} \tag{4.1}$$

is a homotopy pullback, where m is the number of vertices of K.

We now prove the main theorem which indicates a homotopical inheritance of the right-angularity of a Coxeter system (W, S) by the adjoint group $\mathrm{Ad}\,(X_W)$ as in Proposition 3.3. Let I_2 be the mapping cylinder of the degree 2 map $S^1 \to S^1$, and we consider the pair (I_2, S^1) such that the inclusion $S^1 \to I_2 \simeq S^1$ is of degree 2. Note that we also have an inclusion $S^1 \to I_2$ which is a homotopy equivalence.

Theorem 4.3. *If a Coxeter system (W, S) is right-angled, then there is a homotopy equivalence*

$$B\mathrm{Ad}\,(X_W) \simeq Z(\Delta(\Gamma_W); (I_2, S^1))$$

such that the map $\bar{\theta}\colon B\mathbb{Z}_S \to B\mathrm{Ad}\,(X_W)$ is identified with the inclusion $Z(S; (S^1, *)) \to Z(\Delta(\Gamma_W); (I_2, S^1))$ induced from a homotopy equivalence $S^1 \xrightarrow{\simeq} I_2$.

4.2. **Applications of Theorem 4.3.** We first consider a torsion in the adjoint group $\mathrm{Ad}\,(X_W)$. By definition, W always has a torsion whereas one of the biggest problems on Artin groups to show whether they are torsion free or not [P]. Since $\mathrm{Ad}\,(X_W)$ is an intermediate object between W and A_W by Theorem ??, one may ask:

Problem 4.4. *Is $\mathrm{Ad}\,(X_W)$ torsion free?*

Suppose (W, S) is right-angled. Since $BA_W \simeq Z(\Delta(\Gamma_W); (S^1, *))$ is finite dimensional by Proposition 3.2, A_W is torsion free. So one may expect that $\mathrm{Ad}\,(X_W)$ is also torsion free. Indeed, by Theorem 4.3 $B\mathrm{Ad}\,(X_W) \simeq Z(\Delta(\Gamma_W); (I_2, S^1))$ is finite dimensional, so we get:

Corollary 4.5. *If a Coxeter system (W,S) is right-angled, then* $\mathrm{Ad}\,(X_W)$ *is torsion free.*

We next consider a property of the map $\Phi\colon A_W \to \mathrm{Ad}\,(X_W)$ when (W,S) is right-angled. We assume (W,S) is right-angled and $|S| = m$ so that we may identify S with $[m]$. Note that $\mathbb{Z}^m$ is identified with the graph product $\mathbb{Z}_{\Delta^{m-1}}$. Then by Theorem 3.2, the map ab: $BA_W \to B\mathbb{Z}^m$ is identified with the inclusion $Z(\Delta(\Gamma_W);(S^1,*)) \to Z(\Delta^{m-1};(S^1,*)) = (S^1)^m$, so by Proposition ??, the map ab: $BA_W \to B\mathbb{Z}^m$ is the projection

$$\Lambda_R(v_1,\ldots,v_m) \to \Lambda_R(\Delta(\Gamma_W))$$

in cohomology. In particular, since $\Lambda_R(\Delta(\Gamma_W))$ is a free R-module, the map $\mathrm{ab}^*\colon H^*(\mathbb{Z}^m;R) \to H^*(A_W;R)$ has a section as R-modules. On the other hand, there is a commutative diagram

$$\begin{tikzcd} A_W \arrow[r, "\mathrm{ab}"] \arrow[d, "\Phi"] & \mathbb{Z}^{c(W)} \arrow[d, equal] \\ \mathrm{Ad}\,(X_W) \arrow[r, "\mathrm{ab}"] & \mathbb{Z}^{c(W)} \end{tikzcd}$$

even when (W,S) is not right-angled. Thus by considering the induced commutative diagram in cohomology, we obtain:

Proposition 4.6. *If a Coxeter system (W,S) is right-angled, the map $\Phi^*\colon H^*(\mathrm{Ad}\,(X_W);R) \to H^*(A_W;R)$ has a section as R-modules.*

We will show that this property of Φ can be recovered by a homotopical property of the induced map $\Phi\colon BA_W \to B\mathrm{Ad}\,(X_W)$. To this end, we recall the natural homotopy decomposition of a suspension of a polyhedral product. Let (X,A) be a pair of spaces and K be a simplicial complex on the vertex set $[m]$. For any $\sigma \in K$, we put

$$\widehat{D}_{(X,A)}(\sigma) = Y_1 \wedge \cdots \wedge Y_m, \quad Y_i = \begin{cases} X & i \in \sigma \\ A & i \notin \sigma \end{cases}$$

and define the polyhedral smash product of (X,A) with respect to K by

$$\widehat{Z}(K;(X,A)) = \bigcup_{\sigma \in K} \widehat{D}_{(X,A)}(\sigma) \quad (\subset X^{\wedge n})$$

where $X^{\wedge n}$ is the smash product of n-copies of X. For a subset $I \subset [m]$, we put

$$K_I = \{\sigma \in K \mid \sigma \subset I\}.$$

Theorem 4.7 (Bahri, Bendersky, Cohen, and Gitler [BBCG]). *There is a homotopy equivalence*

$$\Sigma Z(K;(X,A)) \simeq \Sigma \bigvee_{\emptyset \neq I \subset [m]} \widehat{Z}(K_I;(X,A))$$

which is natural with respect to (X,A).

Proposition 4.6 is recovered by the following homotopical property of the map $\Phi\colon A_W \to \mathrm{Ad}\,(X_W)$.

Theorem 4.8. *If a Coxeter system (W, S) is right-angled, the map $\Sigma\Phi\colon \Sigma BA_W \to \Sigma B\mathrm{Ad}\,(X_W)$ has a left homotopy inverse.*

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