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MONOTONICALLY RETRACTABLE SPACES AND ROJAS-HERNÁNDEZ'S QUESTION

MASAMI SAKAI
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
KANAGAWA UNIVERSITY

1. Introduction

In this article, we give a short survey of monotonically retractable spaces which was recently introduced in Rojas-Hernández [5], and answer a question posed in Rojas-Hernández [5].

We assume that all spaces are Tychonoff topological spaces. For a space X, let $\boldsymbol{C_p(X)}$ be the space of all real-valued continuous functions on X with the topology of pointwise convergence. A space is said to be ***cosmic*** if it has a countable network.

The class of monotonically retractable spaces is useful to study the D-property of Lindelöf function spaces $C_p(X)$.

Definition 1.1 (E.K. van Douwen). A space (X, τ) is a ***D-space*** if for any neighborhood assignment $\phi : X \to \tau$, there exists a closed and discrete subspace $A \subset X$ such that $\bigcup\{\phi(x) : x \in A\} = X$.

Problem 1.2 ([2]). Is every regular Lindelöf space a D-space?

This problem is still open even for a Lindelöf function space $C_p(X)$.

In this paragraph, let X be a first-countable countably compact subspace of an ordinal (for example, $X = \omega_1$). Buzyakova [1] showed that $C_p(X)$ is Lindelöf, and asked whether $C_p(X)$ is a D-space. Later Peng [4] showed that the answer to Buzyakova's question is in the affirmative. On the other hand, Tkachuk [6] showed that the iterated function space $C_{p,2n+1}(X)$ is Lindelöf for all $n \in \omega$, and asked whether $C_{p,2n+1}(X)$ is a D-space. Finally, introducing the class of monotonically retractable spaces, Rojas-Hernández answered to Tkachuk's question in the affirmative.

2. A short survey of monotonically retractable spaces

Definition 2.1 (R. Rojas-Hernández, [5]). A space X is ***monotonically retractable*** if we can assign to any $A \in [X]^{\leq\omega}$ a set $K(A) \subset X$, a continuous retraction $r_A : X \to K(A)$ and a countable family $\mathcal{N}(A)$ of subsets of X such that:

(r1) $A \subset K(A)$;

(r2) If W is open in $K(A)$, then $r_A^{-1}(W) = \bigcup \mathcal{N}$ for some $\mathcal{N} \subset \mathcal{N}(A)$;

(r3) If $A, B \in [X]^{\leq\omega}$ and $A \subset B$, then $\mathcal{N}(A) \subset \mathcal{N}(B)$;

(r4) If $A_n \in [X]^{\leq\omega}$ for each $n \in \omega$, $A_n \subset A_{n+1}$ and $A = \bigcup\{A_n : n \in \omega\}$, then $\mathcal{N}(A) = \bigcup\{\mathcal{N}(A_n) : n \in \omega\}$.

Fact 2.2. *Every cosmic space is monotonically retractable. In particular, a second countable space, or a countable space is monotonically retractable.*

Proof. Let $\mathcal{N}$ be a countable network for X. For each $A \in [X]^{\leq\omega}$, let $K(A) = X, r_A = id_X$ and $\mathcal{N}(A) = \mathcal{N}$. □

We give another simple example of a monotonically retractable space. Fix a point $b = (b_\alpha) \in \prod_{\alpha<\kappa} X_\alpha$, and for each $x \in \prod_{\alpha<\kappa} X_\alpha$, let $supp(x) = \{\alpha < \kappa : x_\alpha \neq b_\alpha\}$. We put

$$\Sigma = \{x \in \prod_{\alpha<\kappa} X_\alpha : |supp(x)| \leq \omega\}.$$

This Σ is called a ***Σ-product*** with a base point b.

Fact 2.3. *Every Σ-product Σ of cosmic spaces is monotonically retractable.*

Proof. For each $A \in [\Sigma]^{\leq\omega}$, let $C(A) = \bigcup_{x\in A} supp(x)$, and let $pr_{C(A)}$ be the projection from Σ onto $\prod_{\alpha\in C(A)} X_\alpha$. We put $K(A) = \prod_{\alpha\in C(A)} X_\alpha$, $r_A = pr_{C(A)}$ and $\mathcal{N}(A) = \{r_A^{-1}(B) : B \in \mathcal{B}(A)\}$, where $\mathcal{B}(A)$ is the standard countable network for $\prod_{\alpha\in C(A)} X_\alpha$. □

We recall topological properties of monotonically retractable spaces without proofs.

Proposition 2.4 ([5]). *The following hold.*

1. *Every closed subspace of a Σ-product of cosmic spaces is monotonically retractable.*
2. *Every monotonically retractable space is collectionwise normal, and has the countable extent.*
3. *If X is monotonically retractable, then the tightness of X^ω is countable.*
4. *For a compact space X, X is monotonically retractable if and only if it is Corson compact (Cuth and Kalenda, 2015).*

Theorem 2.5 ([5, Theorem 3.18]). *If X is monotonically retractable , then $C_p(X)$ is a Lindelöf D-space.*

Theorem 2.6 ([5, Theorem 3.25]). *If X is monotonically retractable , then so is $C_pC_p(X)$.*

Hence, these theorems yields the following.

Corollary 2.7 ([5, Corollary 3.27]). *If X is monotonically retractable , then $C_{p,2n+1}(X)$ is a Lindelöf D-space for all $n \in \omega$.*

Theorem 2.8 ([5, Theorem 3.28]). *If X is a first countable countably compact subspace of an ordinal, then it is monotonically retractable.*

Hence Rojas-Hernández answered to Tkachuk's question in the affirmative.

Corollary 2.9 ([5, Corollary 3.29]). *If X is a first countable countably compact subspace of an ordinal, then $C_{p,2n+1}(X)$ is a Lindelöf D-space for all $n \in \omega$.*

3. Rojas-Hernández's question

Definition 3.1. A space is ***realcompact*** if it is homeomorphic to a closed subset of $\mathbb{R}^\kappa$ for some κ.

Fact 3.2 ([3]). *Every Lindelöf space is realcompact.*

Every Lindelöf space is collectionwise normal, and has the countable extent. Recall that every monotonically retractable space is collectionwise normal, and has the countable extent. Hence, Rojas-Hernández asked the following.

Question 3.3 (R. Rojas-Hernández [5], 2014). Suppose that X is a monotonically retractable realcompact space. Must X be Lindelöf?

Lemma 3.4 ([3]). *The following statements hold.*

1. *If Y is hereditarily realcompact and there exists a continuous map $\tau : X \to Y$ such that $\tau^{-1}(y)$ is compact for each $y \in Y$, then X is realcompact;*
2. *If X is realcompact and each point of X is a G_δ-set, then X is hereditarily realcompact.*

Theorem 3.5. *There exists a monotonically retractable, hereditarily realcompact space X which is not Lindelöf.*

Proof. Fix any second countable space Y with $|Y| = \omega_1$ and let $Z = Y \times \omega_1$. For each $\alpha < \omega_1$, we define $Z_\alpha, r_\alpha, \mathcal{B}_\alpha$ and $\mathcal{N}_\alpha$. Let $Z_\alpha = Y \times [0, \alpha]$. We define a map $r_\alpha : Z \to Z_\alpha$ as follows: for each $(y, \beta) \in Z$, $r_\alpha((y, \beta)) = (y, \beta)$ if $\beta \leq \alpha$; $r_\alpha((y, \beta)) = (y, \alpha)$ if $\beta > \alpha$. Then r_α is a continuous retraction. Let $\mathcal{B}_Y$ be a countable base for Y, and let

$$\mathcal{B}_\alpha = \{B \times (\beta, \gamma] : B \in \mathcal{B}_Y, \beta < \gamma \leq \alpha\}.$$

Then $\mathcal{B}_\alpha$ is a countable base for Z_α and $\alpha < \alpha'$ implies $\mathcal{B}_\alpha \subset \mathcal{B}_{\alpha'}$. Let

$$\mathcal{N}_\alpha = \{r_\alpha^{-1}(B) : B \in \mathcal{B}_\alpha\}.$$

By the definition of $\mathcal{B}_\alpha$,

$$\mathcal{N}_\alpha = \{B \times (\beta, \gamma] : B \in \mathcal{B}_Y, \beta < \gamma < \alpha\} \cup \{B \times (\beta, \omega_1) : B \in \mathcal{B}_Y, \beta < \alpha\}.$$

Then $\mathcal{N}_\alpha$ is a countable open cover of Z and satisfies the following:

(a) if W is an open set in Z_α, then $r_\alpha^{-1}(W) = \bigcup \mathcal{N}$ for some $\mathcal{N} \subset \mathcal{N}_\alpha$;
(b) if $\alpha \leq \alpha'$, then $\mathcal{N}_\alpha \subset \mathcal{N}_{\alpha'}$;
(c) if $\alpha_n < \omega_1$ for each $n \in \omega$, $\alpha_n \leq \alpha_{n+1}$ and $\alpha = \sup\{\alpha_n : n \in \omega\}$, then $\mathcal{N}_\alpha = \bigcup\{\mathcal{N}_{\alpha_n} : n \in \omega\}$.

The conditions (a) and (b) can be easily checked. We observe (c). If $\alpha = \alpha_n$ for some $n \in \omega$, then the conclusion obviously holds. Assume $\alpha_n < \alpha$ for each $n \in \omega$. Let $N \in \mathcal{N}_\alpha$. If N is of the form $N = B \times (\beta, \gamma]$, where $B \in \mathcal{B}_Y, \beta < \gamma < \alpha$, take an $n \in \omega$ with $\gamma < \alpha_n < \alpha$, then we have $N \in \mathcal{N}_{\alpha_n}$. If N is of the form $N = B \times (\beta, \omega_1)$, where $B \in \mathcal{B}_Y, \beta < \alpha$, take an $n \in \omega$ with $\beta < \alpha_n < \alpha$, then we have $N \in \mathcal{N}_{\alpha_n}$.

Now fix an onto map $\varphi : Y \to \omega_1$. Let

$$X = \{(y, \alpha) \in Z : \alpha \leq \varphi(y)\}.$$

By Lemma 3.4, X is hereditarily realcompact. However, it is not Lindelöf, because ω_1 is a continuous image of X. We see that X is monotonically retractable. Let $A \in [X]^{\leq\omega}$ and if $A = \{(y_n, \alpha_n) \in X : n \in \omega\}$, we put $\alpha(A) = \sup\{\alpha_n : n \in \omega\}$.

We define $K(A), r_A$ and $\mathcal{N}(A)$ naturally. Let $K(A) = X \cap Z_{\alpha(A)}$. Obviously $A \subset K(A)$, the condition (r1) holds. Recall the retraction $r_{\alpha(A)} : Z \to Z_{\alpha(A)}$. By the definitions of $r_{\alpha(A)}$ and X, the inclusion $r_{\alpha(A)}(X) \subset K(A)$ can be easily checked. Hence, the restricted map $r_A = r_{\alpha(A)} \restriction X : X \to K(A)$ is a retraction. Let

$$\mathcal{N}(A) = \{X \cap N : N \in \mathcal{N}_{\alpha(A)}\}.$$

This family is a countable open cover of X. We examine the condition (r2). Let W be an open set in $K(A)$, and take an open set W' in $Z_{\alpha(A)}$ such that $W = K(A) \cap W'$. By (a) above, $r^{-1}_{\alpha(A)}(W') = \bigcup \mathcal{N}$ for some $\mathcal{N} \subset \mathcal{N}_{\alpha(A)}$. Hence,

$$r_A^{-1}(W) = X \cap r^{-1}_{\alpha(A)}(W') = X \cap \left(\bigcup \mathcal{N}\right) = \bigcup \{X \cap N : N \in \mathcal{N}\}.$$

The condition (r3) easily follows from (b) above. Finally, we examine the condition (r4). Suppose $A_n \in [X]^{\leq \omega}$ for each $n \in \omega$, $A_n \subset A_{n+1}$ and $A = \bigcup \{A_n : n \in \omega\}$. Then, $\alpha(A) = \sup\{\alpha(A_n) : n \in \omega\}$ holds. By (c) above, we have $\mathcal{N}_{\alpha(A)} = \bigcup \{\mathcal{N}_{\alpha(A_n)} : n \in \omega\}$. This implies $\mathcal{N}(A) = \bigcup \{\mathcal{N}(A_n) : n \in \omega\}$. Thus X is monotonically retractable. □

4. Questions

We recall some interesting questions posed in [5].

Question 4.1 (R. Rojas-Hernández [5], 2014).

(1) Suppose that X is hereditarily monotonically retractable. Must X have a countable network? (Comment: Tkachuk showed that if X^2 is hereditarily monotonically retractable, then X has a countable network.)

(2) Suppose that X is a space such that $X^2 \setminus \Delta_X$ is monotonically retractable. Must X have a countable network?

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Department of Mathematics, Kanagawa University, Hiratsuka 259-1293, Japan
E-mail address: sakaim01@kanagawa-u.ac.jp