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The Keller-Segel system on networks

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Abstract

This paper is a review of the results contained in [3] where the well posedness of the Keller-Segel system defined on a weighted network is studied. Completing the system with appropriate transmission conditions at the vertices, we prove global in time existence and uniqueness of a continuous solution of the problem. An important ingredient of the proof is a formula for the Gaussian kernel on networks due to Roth [14] that we briefly recall.

1 Introduction

Given a a network Γ with vertices $\{v_i\}_{i=1}^n$ and edges $\{e_j\}_{j=1}^m$, we consider the Keller-Segel system

$$\begin{cases} \partial_t u_j = \partial_{xx} u_j - \partial_x(u_j\,\partial_x c_j) & \text{on } (0,\infty)\times e_j\,,\ j=1,\dots,m,\\ \varepsilon\,\partial_t c_j = \partial_{xx} c_j + u_j - \alpha\, c_j & \text{on } (0,\infty)\times e_j\,,\ j=1,\dots,m,\\ \sum\limits_{j\in E(v_i)} \kappa(e_j)\frac{\partial u_j}{\partial n}(t,v_i) = 0\,, & t>0\,,\ i=1,\dots,n,\\ \sum\limits_{j\in E(v_i)} \kappa(e_j)\frac{\partial c_j}{\partial n}(t,v_i) = 0, & t>0\,,\ i=1,\dots,n,\\ u_j(t,v_i) = u_k(t,v_i) \text{ if } j,k\in E(v_i)\,, & t>0\,,\ i=1,\dots,n,\\ c_j(t,v_i) = c_k(t,v_i) \text{ if } j,k\in E(v_i)\,, & t>0\,,\ i=1,\dots,n.\\ u_j(0,x) = u_j^0(x), \quad c_j(0,x) = c_j^0(x)\,, & \text{on } e_j\,,\ j=1,\dots,m, \end{cases} \tag{1.1}$$

($E(v_i)$ denotes the index set of edges incident a vertex v_i). The previous system has been recently considered in [2] to describe the evolution of the ameboid organism *Physarum polycephalum.* Indeed there are experimental evidences that during its evolution this slime mold arranges a network of thin tubes where nutrients and chemical signals are transmitted to the different parts of the organism according to a chemotactical process (see [12, 15]). In [2], the system (1.1) is analyzed numerically and it is shown that a discrete system obtained via a finite differences approximation of (1.1) is well posed.

The goal of this paper is to prove the existence of a time global and spatially continuous solution (u, c) to (1.1) in the case $\varepsilon > 0$ (for the case $\varepsilon = 0$, see [3]). Observe that system (1.1) is formally equivalent to m Keller-Segel systems, one on each of the m edges, coupled via the transition conditions at the vertices of the network. Coherently with the parabolic nature of the problem (see [11]), we look for a continuous solution on the whole network and consequently we prescribe the continuity of u and c at the vertices. Moreover, we require for u the flux conservation at the vertices, while for c a Kirchhoff type condition which guarantees the validity of the maximum principle for diffusion equations on networks. For simplicity reason, the network has no boundary nodes, but the results can be easily extended to the case of Dirichlet or mixed boundary conditions.

We consider solution of (1.1) in the following integral sense

$$u(t,y) = P_t u^0(y) - \int_0^t P_{(t-s)}\partial_x(u(s)\partial_x c(s))(y)ds, \tag{1.2}$$

$$c(t,y) = e^{-(\alpha/\varepsilon)t}P_{(t/\varepsilon)}c^0(y) + \frac{1}{\varepsilon}\int_0^t e^{-(\alpha/\varepsilon)(t-s)}P_{((t-s)/\varepsilon)}\,u(s)(y)ds, \tag{1.3}$$

where $(P_t)_{t\geq 0}$ is the semigroup generated by the laplacian $-\Delta_\Gamma$ on Γ. The interest in considering the formulation (1.2)-(1.3) lies in the fact that $(P_t)_{t\geq 0}$ is given explicitly through the fundamental solution $H = H(t,x,y)$ of the heat equation on networks (see [14]). Then, with this integral formula at hand, the proofs for local and global existence follow the corresponding arguments in the Euclidean case, with however some specific modifications due to the network structure. In particular, it is not possible to use known results about the existence of solutions of (1.1) on a bounded interval $[0, L]$ with homogeneous Neumann boundary conditions (see [8] for instance) since the transition conditions at the vertices involve functions defined on different edges. Moreover, in order to get appropriate time bounds on the norm of u and c we need to prove optimal L^p bounds for the heat kernel H on Γ, which improve earlier results in [14, 4, 5].

The Keller-Segel has been introduced in the early seventies in [9] in order to model the aggregation phenomenon undergone by the slime mold *Dictyostelium discoideum.* In this biological context, u represents the cell concentration of the organism and satisfies the continuity equation in (1.1), while c is the chemoattractant concentration and solves the diffusion equation in (1.1). In the Euclidean case, i.e. when (1.1) is considered on a domain of $\mathbb{R}^d$, there is a vast literature: depending on the space dimension d, $\varepsilon > 0$ (double parabolic case) or $\varepsilon = 0$ (parabolic-elliptic case) and the initial data u^0, c^0, different phenomena can occur, such as global existence, finite or infinite time blow-up, peaks formation, threshold phenomena, etc. We refer to [6, 7, 13] and the references therein for more details on that problem.

The paper is organized as follows. In Section 2 we introduce some basic definitions concerning the network Γ and the functions defined on it. In Section 3 we recall the fundamental solution of the heat equation on Γ and we deduce the optimal L^p-estimates for the heat kernel. Section 4 is devoted to the study of the Keller-Segel system on Γ.

2 Differential equations on networks

In this section we introduce notations and basic definitions for the study of differential equations networks.
Let consider a finite, connected and non-oriented (or undirected) *network* Γ. This means that the underlying *graph* $\mathcal{G} = (V, E)$ is defined through a non empty finite set of n *vertices* or *nodes*, $V := \{v_1, \dots, v_n\}$, a non empty finite set of m non-oriented open *edges* (or *links*), $E := \{e_1, \dots, e_m\}$, and that between every pair of nodes $v_i, v_j \in V$ there exists a path with edges in E. Furthermore, we assume that the graph has no self-loops (no edge connecting a vertex to itself). On the other hand, the graph can contain multiple links, i.e. the map $E \mapsto V \times V$ associating to each non-oriented edge its endpoints can be not injective.

Every edge may have a different length. However, we parametrize and normalize each $e_j \in E$ so that to identify $\overline{e}_j$ with the interval $[0, 1]$. Since the network is undirected, every edge $e_j \in E$ can be parametrized in two different ways giving rise to two oriented edges $e_j^{\pm}$, i.e. there exist two homeomorphism $\Pi_j^{\pm} : [0, 1] \mapsto (\overline{e}_j)^{\pm}$, such that $\Pi_j^{+}(0) = \Pi_j^{-}(1)$ and $\Pi_j^{+}(1) = \Pi_j^{-}(0)$. We shall call an oriented edge an *arc*, and we shall denote by a_j any of the two edges $e_j^{\pm}$. Moreover, we shall denote by $-a_j$ the arc opposite to a_j and the initial and terminal endpoints of a_j by $I(a_j)$ and $T(a_j)$, respectively. We also denote by $E(v_i)$ the set of the index j such that the edge e_j has an endpoint at the vertex $v_i \in V$ and by $d(v_i)$ the *degree* of v_i, i.e. the cardinality of $E(v_i)$.

Next, we define a *path* C on the network Γ as a finite sequence of (at least two) arcs, $(a_{j_1}, \dots, a_{j_k})$, $k \geq 2$, such that $T(a_{j_l}) = I(a_{j_{l+1}})$, $l = 1, \dots, k-1$. Thus a path is always oriented. We associate to each path $C = (a_{j_1}, \dots, a_{j_k})$ its length $|C|$ as the number of the arcs composing C. Then, given two points x and y on Γ, we shall note $C_k(x, y)$ the set of the paths of length k such that x belongs to the first arc of the path and y belongs to the last arc of the path, i.e.

$$C_k(x, y) := \{C = (a_{j_1}, \dots, a_{j_k}) \ : \ x \in a_{j_1} \text{ and } y \in a_{j_k}\}, \qquad k = 2, 3, \dots .$$

A *geodesic path* joining x to y on Γ is any path of minimum length in $\cup_{k \geq 2} C_k(x, y)$. We shall denote $\mathcal{L}(x, y)$ the common length of any geodesic path joining x to y

and we also define

$$\rho(x,y) := \mathcal{L}(x,y) - 2\,.$$

For every x and y belonging to the same $\overline{e}_j$, we define the distance $d(x,y)$ as

$$d(x,y) := |(\Pi_j^{\pm})^{-1}(x) - (\Pi_j^{\pm})^{-1}(y)|\,, \qquad x,\, y \in \overline{e}_j\,.$$

Then, for every x and y on Γ, we define the distance of x to y along $C = (a_{j_1}, \dots, a_{j_k}) \in C_k(x,y)$ as

$$d_C(x,y) := d(x, T(a_{j_1})) + d(y, I(a_{j_k})) + |C| - 2\,.$$

Obviously, $d_C(x,y)$ is symmetric with respect to x and y, i.e. $d_C(x,y) = d_{-C}(y,x)$.

Finally, to each non-oriented edge $e_j \in E$ we associate a positive weight $\kappa(e_j)$ and we assume that

$$0 < \kappa_0 \le \kappa(e_j) \le \kappa_1\,, \qquad \forall\, j = 1, \dots, m\,.$$

The weights $\kappa(e_j)$ shall influence the transmission or the reflection of u through the nodes. Indeed, for each couple of arcs (a_i, a_j), we introduce the *transfer/reflection coefficient* from a_i to a_j as

$$\epsilon_{(a_i \to a_j)} := \begin{cases} \dfrac{2\kappa(e_i)}{\sum_{l \in E(T(a_i))} \kappa(e_l)} & \text{if } T(a_i) = I(a_j) \text{ and } a_j \neq -a_i \text{ (transmission)} \\ \dfrac{2\kappa(e_i)}{\sum_{l \in E(T(a_i))} \kappa(e_l)} - 1 & \text{if } a_j = -a_i \text{ (reflection)} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

The *weight* $\epsilon(C)$ of a path $C = (a_{j_1}, \dots, a_{j_k})$ is then the product of the transfer/reflection coefficients of all the pairs of consecutive arcs composing C, i.e.

$$\epsilon(C) := \prod_{l=1}^{k-1} \epsilon_{(a_{j_l} \to a_{j_{l+1}})}\,. \tag{2.2}$$

It is worth noticing that in case of reflection, the coefficient $\epsilon_{(a_i \to -a_i)}$ may be negative, and so also the weight $\epsilon(C)$ of all path C passing through a_i and $-a_i$ consecutively. Moreover, $\epsilon_{(a_i \to a_j)} \neq \epsilon_{(a_j \to a_i)}$ and $\epsilon(C) \neq \epsilon(-C)$, in general.

A function u defined on the network Γ is a collection of m functions $(u_j)_{j=1}^m$ such that $u_j := u_{|\overline{e}_j}$. To every function u on Γ we associate the vector valued function $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_m)$ defined on $[0,1]$ such that $\tilde{u}_j := u \circ \Pi_j^{\pm}$. Then, we denote $u_j'(x)$ and $u_j''(x)$, $x \in e_j$, the derivatives $\tilde{u}_j'(\xi)$ and $\tilde{u}_j''(\xi)$ with respect to $\xi \in (0,1)$, $\xi = (\Pi_j^{\pm})^{-1}(x)$. We also define the exterior normal derivative of u_j at the endpoints of the arc a_j as

$$\frac{\partial u_j}{\partial n}(I(a_j)) = -\lim_{h \to 0^+} \frac{\tilde{u}_j(h) - \tilde{u}_j(0)}{h} \quad \text{and} \quad \frac{\partial u_j}{\partial n}(T(a_j)) = \lim_{h \to 0^-} \frac{\tilde{u}_j(1+h) - \tilde{u}_j(1)}{h}\,.$$

Next, we define the space of continuous functions on Γ

$$C^0(\Gamma) := \{u = (u_j)_{j=1}^m \ : \ u_j(v_i) = u_k(v_i) \text{ if } j,k \in E(v_i)\,,\ i = 1,\dots,n\}\,,$$

the integral of a function u over Γ

$$\int_\Gamma u(x)dx := \sum_{j=1}^m \kappa(e_j) \int_0^1 \tilde{u}_j(\xi)\, d\xi\,,$$

the Lebesgue spaces

$$L^p(\Gamma) := \{u = (u_j)_{j=1}^m \ : \ \|u\|_{L^p(\Gamma)}^p := \sum_{j=1}^m \kappa(e_j)\|\tilde{u}_j\|_{L^p(0,1)}^p < \infty\}\,, \quad p \in [1,\infty)\,,$$

$$L^\infty(\Gamma) := \{u = (u_j)_{j=1}^m \ : \ \|u\|_{L^\infty(\Gamma)} := \max_{1\le j\le m} \kappa(e_j)\|\tilde{u}_j\|_{L^\infty(0,1)}\}\,,$$

and the Sobolev spaces

$$W^{1,\infty}(\Gamma) := \{u \in C^0(\Gamma) \ : \ u' \in L^\infty(\Gamma)\}\,,$$

$$H^r(\Gamma) := \{u \in C^0(\Gamma) \ : \ \|u\|_{H^r(\Gamma)}^2 := \sum_{j=1}^m \kappa(e_j)\|\tilde{u}_j\|_{H^r(0,1)}^2 < \infty\}\,.$$

3 The heat equation on networks

Aim of this section is to review some of the main properties of the fundamental solution of the heat equation on networks. The fundamental solution has been computed by Roth [14] for a finite network and generalized to the case of an infinite homogeneous tree and a countable graph by Cattaneo [4, 5]. However, some properties, such as the optimal L^1 and L^∞ time decay, are not contained in the cited papers. We start introducing the operator $(D(-\Delta_\Gamma), -\Delta_\Gamma)$, where the domain $D(-\Delta_\Gamma)$ is the set of the function u in $H^2(\Gamma)$ satisfying the transmission conditions of Kirchhoff type at every vertex $v_i \in V$,

$$D(-\Delta_\Gamma) := \{u \in H^2(\Gamma) \ : \ \sum_{j\in E(v_i)} \kappa(e_j)\frac{\partial u_j}{\partial n}(v_i) = 0\,,\ i = 1,\dots,n\}\,,$$

and, for all $u \in D(-\Delta_\Gamma)$, the laplacian Δ_Γ on Γ is naturally defined as $\Delta_\Gamma u = u''$. Then, $-\Delta_\Gamma$ is densely defined in the Hilbert space $L^2(\Gamma)$ endowed with the scalar product

$$(u,v)_{L^2(\Gamma)} = \sum_{j=1}^m \kappa(e_j) \int_0^1 \tilde{u}_j'(\xi)\tilde{v}_j'(\xi)\, d\xi\,.$$

It is also symmetric and positive and consequently accretive. Thanks to the transmission conditions, it can be also proved that $-\Delta_\Gamma$ is m-accretive and therefore self-adjoint (see for instance [10]). Hence, we can associate to $-\Delta_\Gamma$ a semigroup of contractions on $L^2(\Gamma)$, say $(\mathcal{T}(t))_{t\geq 0}$, whose generator is Δ_Γ. To conclude, given any $f \in L^2(\Gamma)$, the function $u(t) = \mathcal{T}(t)f$ is the unique solution of the heat equation

$$\begin{cases} \partial_t u = \Delta_\Gamma u & \text{on } (0,\infty)\times\Gamma\,, \\ u(0) = f & \text{on } \Gamma\,, \end{cases} \tag{3.1}$$

in the space $C([0,\infty), L^2(\Gamma)) \cap C((0,\infty), D(-\Delta_\Gamma)) \cap C^1((0,\infty), L^2(\Gamma))$.

Problem (3.1) can be also written as a system of m heat equations coupled through the continuity and transmission conditions at the vertex, i.e.

$$\begin{cases} \partial_t u_j = \partial_{xx} u_j & \text{on } (0,\infty)\times e_j\,,\ j=1,\dots,m \\ u_j(0) = f_j & \text{on } e_j\,,\ j=1,\dots,m \\ u_j(t,v_i) = u_k(t,v_i) \text{ if } j,k\in E(v_i)\,,\ i=1,\dots,n & t>0 \quad \text{(continuity)} \\ \sum\limits_{j\in E(v_i)} \kappa(e_j)\frac{\partial u_j}{\partial n}(t,v_i) = 0\,,\ i=1,\dots,n & t>0 \quad \text{(transmission condition)} \end{cases} \tag{3.2}$$

These transmission conditions together provide, for each node v_i, a system of $d(v_i)$ equations for the $d(v_i)$ components u_j of the solution u such that $j \in E(v_i)$. Moreover, they reduces at the vertex v_i of degree 1, $d(v_i) = 1$, to the homogeneous Neumann boundary condition.

Finally, it is worth noticing that the choice of the orientation of the edges e_j has no consequences, since the heat equation (3.1)-(3.2) and the problem (1.1) are invariant under the transformation $\xi \to (1-\xi)$ that commute Π_j^+ into Π_j^- and vice versa, as well as all the definitions given above. On the other hand, orientation appears to be necessary for the construction of the fundamental solution of the heat equation on Γ below, that we shall use for the resolution of (1.1). For the analysis of (3.1)-(3.2) through the abstract semigroup method see [10] and the references therein.

Let $G(t,z) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}}$ denote the heat kernel on $(0,\infty)\times\mathbb{R}$, and consider the function defined on $(0,\infty)\times\Gamma\times\Gamma$ as

$$H(t,x,y) = \delta_{i,j}\,\kappa^{-1}(e_i)\,G(t,d(x,y)) + L(t,x,y)\,, \tag{3.3}$$

for $x \in \bar{e}_i\,,\ y \in \bar{e}_j\,,\ i,j \in \{1,\dots,m\}$, where δ_{ij} is the usual Kronecker's delta function and

$$L(t,x,y) = \sum_{k\geq\rho(x,y)} \sum_{C\in C_{k+2}(x,y)} \kappa^{-1}(e_i)\,\epsilon(C)\,G(t,d_C(x,y))\,. \tag{3.4}$$

The first term in (3.3) is simply the restriction of the fundamental solution of the heat equation on each edge of the network. The second term is determined in such a way that the function H satisfies the continuity and transmission conditions in (3.2) with respect to y, for any fixed $x \in \Gamma$. More specifically, since the network is composed of m edges, it holds, for all $k \in \mathbb{N}$, that

$$\operatorname{card}(C_{k+2}(x,y)) \leq 2\left(\max_{i=1,\dots,n} d(v_i)\right)^{k+1} \leq 2\,m^{k+1}\,. \tag{3.5}$$

Hence, the sum with respect to the paths $C \in C_{k+2}(x,y)$ in (3.4) is finite. We also have that the coefficients (2.1) are uniformly bounded : $|\epsilon_{(a_i \to a_j)}| \leq 2\kappa_1 \kappa_0^{-1} := \bar{\epsilon}$, $i,j = 1,\ldots,m$. Therefore, by (3.5), we get

$$|L(t,x,y)| \leq \kappa_0^{-1} \sum_{k=0}^{+\infty} (\bar{\epsilon}\, m)^{k+1} \frac{e^{-k^2/4t}}{\sqrt{\pi\, t}} < \infty\,.$$

The latter estimate implies that the series giving $L(t,x,y)$ is normally convergent over $[t_1,t_2] \times \Gamma \times \Gamma$, for any fixed $t_1, t_2 > 0$. Therefore, the associated vector valued function $\tilde{H} = \tilde{H}(t,\xi,\eta)$ is continuous with respect to $(t,\xi,\eta) \in (0,\infty) \times [0,1] \times [0,1]$, component by component. Similarly, for any fixed $\xi \in (0,1)$, the derivatives $\partial_t \tilde{H}$, $\partial_\eta \tilde{H}$ and $\partial_{\eta\,\eta} \tilde{H}$ exist and are continuous with respect to $(t,\xi,\eta) \in (0,\infty) \times (0,1) \times (0,1)$. They can be computed differentiating under the sum sign and $\tilde{H}$ satisfies the heat equation $\partial_t \tilde{H} = \partial_{\eta\,\eta} \tilde{H}$, component by component. These and other properties of the function H are resumed below.

Theorem 3.1 ([14]). *Let H be the function defined in* (3.3). *Then,*

(i) H is continuous on $(0,\infty) \times \Gamma \times \Gamma$;

(ii) $\partial_t H(t,x,y)$ exists for all $(t,x,y) \in (0,\infty) \times \Gamma \times \Gamma$ and it is continuous on $(0,\infty) \times \Gamma \times \Gamma$;

(iii) the derivatives $\partial_\eta \tilde{H}(t,\xi,\eta)$ and $\partial_{\eta\,\eta} \tilde{H}(t,\xi,\eta)$, exist for all $(t,\xi,\eta) \in (0,\infty) \times (0,1) \times (0,1)$ and are continuous on $(0,\infty) \times (0,1) \times (0,1)$;

(iv) $H(t,x,\cdot) \in D(-\Delta_\Gamma)$ for all $(t,x) \in (0,\infty) \times \Gamma$;

(v) $\partial_t H(t,x,y) = \partial_{yy} H(t,x,y)$ for all $(t,x,y) \in (0,\infty) \times \Gamma \times \Gamma$;

(vi) for all $f \in C^0(\Gamma)$, $\int_\Gamma H(t,x,y) f(x) dx \to f(y)$ for $t \to 0^+$, uniformly with respect to $y \in \Gamma$;

(vii) for all $f \in C^0(\Gamma)$, the function

$$P_t f(y) := \int_\Gamma H(t,x,y) f(x) dx, \qquad (t,y) \in (0,\infty) \times \Gamma$$

with $P_0 f = f$ is the unique continuous solution of the initial valued problem (3.1).

Moreover, H is symmetric with respect to $x,y \in \Gamma$, i.e. $H(t,x,y) = H(t,y,x)$ for all $t \in (0,\infty)$ and the properties above hold true with respect to x, for any fixed y.

The function H is also the unique function satisfying properties (i)-(vii) in Theorem 3.1. As observed in [14], $(P_t)_{t \geq 0}$ is a strongly continuous semigroup on $L^2(\Gamma)$, whose infinitesimal generator is the closure of $-\Delta_\Gamma$ in $L^2(\Gamma)$. It is obviously the same semigroup determined in [10] by variational methods.

It is worth noticing that H is not a priori positive since the weights $\epsilon(C)$ could be negative. Furthermore, the spatial symmetry of H is due to the symmetry of G and to fact that changing x with y in (3.3)-(3.4), the path C changes into $-C$ and $\kappa^{-1}(e_i)\,\epsilon(C) = \kappa^{-1}(e_j)\,\epsilon(-C)$, (see (2.1)-(2.2)). The construction of H has been done in [14] in the case $\kappa(e_j) = 1$, $\forall j$. The generalization to the case of a weighted graph has been considered in [4, 5].

We close this section showing the optimal decay in time of H and its derivatives. For the proof we refer to [3, Appendix B]

Proposition 3.2. *Let H be defined as in* (3.3)*. Then,*

$$\int_\Gamma H(t,x,y)dy = 1\,, \qquad \forall\ (t,x) \in (0,\infty)\times\Gamma\,, \tag{3.6}$$

and there exist constants $C_i > 0$, $i = 1,\dots,4$, such that for all $t > 0$ it holds

$$\sup_{x\in\Gamma} \|H(t,x,\cdot)\|_{L^1(\Gamma)} \le C_1\,, \tag{3.7}$$

$$\|H(t)\|_{L^\infty(\Gamma\times\Gamma)} \le C_2(1+t^{-1/2})\,, \tag{3.8}$$

$$\sup_{x\in\Gamma} \|\partial_y H(t,x,\cdot)\|_{L^1(\Gamma)} + \sup_{y\in\Gamma} \|\partial_y H(t,\cdot,y)\|_{L^1(\Gamma)} \le C_3(1+t^{-1/2})\,, \tag{3.9}$$

$$\|\partial_y H(t)\|_{L^\infty(\Gamma\times\Gamma)} \le C_4(1+t^{-1})\,. \tag{3.10}$$

Moreover, since H is symmetric with respect to x and y, all the above properties hold true changing x with y.

4 The Keller-Segel system on the network

According to the notations and definitions of the previous section, system (1.1), endowed with the natural continuity and transmission conditions, can be written on the network Γ as following

$$\begin{aligned}
&\partial_t u_j = \partial_{yy} u_j - \partial_y(u_j\,\partial_y c_j) && \text{on } (0,\infty)\times e_j\,,\ j=1,\dots,m, && (4.1)\\
&\varepsilon\,\partial_t c_j = \partial_{yy} c_j + u_j - \alpha\, c_j && \text{on } (0,\infty)\times e_j\,,\ j=1,\dots,m, && (4.2)\\
&u_j(0,y) = u_j^0(y) \quad \text{and} \quad c_j(0,y) = c_j^0(y)\,, && y\in\Gamma\,,\ j=1,\dots,m, && (4.3)\\
&\sum_{j\in E(v_i)} \kappa(e_j)\frac{\partial u_j}{\partial n}(t,v_i) = 0\,, && t>0\,,\ i=1,\dots,n, && (4.4)\\
&\sum_{j\in E(v_i)} \kappa(e_j)\frac{\partial c_j}{\partial n}(t,v_i) = 0, && t>0\,,\ i=1,\dots,n, && (4.5)\\
&u_j(t,v_i) = u_k(t,v_i) \text{ if } j,k\in E(v_i)\,, && t>0\,,\ i=1,\dots,n, && (4.6)\\
&c_j(t,v_i) = c_k(t,v_i) \text{ if } j,k\in E(v_i)\,, && t>0\,,\ i=1,\dots,n. && (4.7)
\end{aligned}$$

As for problem (3.2), there is no coupling among the m systems (4.1)-(4.2)-(4.3) on each edge e_j. The systems are coupled only through the transmission conditions (4.4) and (4.5), which express the conservation of the flux at the vertices for u and c (Kirchhoff condition), and through the continuity conditions (4.6) and (4.7). Again, conditions (4.4)-(4.7) give exactly $2d(v_i)$ equations for the $2d(v_i)$ functions u_j, c_j such that $j \in E(v_i)$. Furthermore, conditions (4.4) and (4.5), together with the continuity of u, guarantee the conservation of the initial mass

$$\int_\Gamma u(t,y)\,dy = \int_\Gamma u^0(y)\,dy =: M\,, \qquad t>0\,. \tag{4.8}$$

Indeed

$$\begin{aligned}
\frac{d}{dt}\int_\Gamma u(t,y)\,dy &= \sum_{j=1}^{m} \kappa(e_j)\frac{d}{dt}\int_0^1 \tilde{u}_j(t,\eta)\,d\eta \\
&= \sum_{j=1}^{m} \kappa(e_j)\int_0^1 (\partial_{\eta\eta}\tilde{u}_j(t,\eta) - \partial_\eta(\tilde{u}_j(t,\eta)\partial_\eta\tilde{c}_j(t,\eta)))d\eta \\
&= \sum_{j=1}^{m} \kappa(e_j)[\partial_\eta\tilde{u}_j(t,\eta) - \tilde{u}_j(t,\eta)\partial_\eta\tilde{c}_j(t,\eta)]_0^1 \\
&= \sum_{i=1}^{n}\sum_{j\in E(v_i)} \kappa(e_j)[\frac{\partial u_j}{\partial n} - u_j\frac{\partial c_j}{\partial n}](t,v_i) = 0\,.
\end{aligned}$$

Remark 4.1. *A different but again natural condition that one can impose at the vertices instead of* (4.4) *is the conservation of the total flux*

$$\sum_{j\in E(v_i)} \kappa(e_j)[\frac{\partial u_j}{\partial n} - u_j\frac{\partial c_j}{\partial n}](t,v_i) = 0\,, \qquad t>0,\ i=1,\ldots,n\,.$$

However, the latter together with the continuity of u at the vertices and the Kirchhoff condition (4.5) *imply* (4.4).

Remark 4.2 (Energy). *As for the euclidian case, solutions of the Keller-Segel system* (1.1) *on Γ that satisfy the continuity and transmission conditions* (4.4)-(4.7)*, satisfy also the energy dissipation equation*

$$\frac{d}{dt}\mathcal{E}(u(t),c(t)) = -\int_\Gamma u(t,x)|\partial_x(\log u - c)|^2(t,x)\,dx - \varepsilon\int_\Gamma(\partial_t c(t,x))^2 dx\,,$$

where $\mathcal{E}$ is the usual free energy associated to the Keller-Segel system, i.e.

$$\mathcal{E}(u,v) := \int_\Gamma u\log u\,dx - \int_\Gamma u\,c\,dx + \frac{1}{2}\int_\Gamma |\partial_x c|^2 dx + \frac{\alpha}{2}\int_\Gamma c^2\,dx\,.$$

In this section we consider solutions of the Keller-Segel system in the integral form (1.2)-(1.3). Then, for f integrable over Γ, we introduce the notation

$$(H(t)*f)(y) := \int_\Gamma H(t,x,y)f(x)dx\,, \qquad y\in\Gamma\,.$$

Thanks to the continuity of the heat kernel H on Γ, if u is continuous on Γ and c satisfies the Kirchhoff condition (4.5), equations (1.2)-(1.3) read equivalently as

$$u(t,y) = (H(t) * u^0)(y) + \int_0^t (\partial_x H(t-s) * (u(s)\partial_x c(s)))(y)ds\,, \tag{4.9}$$

$$c(t,y) = e^{-\frac{\alpha}{\varepsilon}t}(H(t\varepsilon^{-1}) * c^0)(y) + \frac{1}{\varepsilon}\int_0^t e^{-\frac{\alpha}{\varepsilon}(t-s)}(H((t-s)\varepsilon^{-1}) * u(s))(y)ds\,. \tag{4.10}$$

It is worth noticing that thanks to property (3.6), any integral solutions (4.9)-(4.10) satisfies the mass conservation (4.8).

Theorem 4.3 (Local existence). *Let $\varepsilon > 0$, $\alpha \geq 0$ and assume $u^0 \in L^\infty(\Gamma)$, $c^0 \in W^{1,\infty}(\Gamma)$. Then, there exist $T = T(\|u^0\|_{L^\infty(\Gamma)}, \|\partial_x c^0\|_{L^\infty(\Gamma)}, \varepsilon) > 0$ and a unique integral solution* (4.9)-(4.10) *of the Keller-Segel system with*

$$u \in L^\infty((0,T); C^0(\Gamma))\,, \quad c \in L^\infty((0,T); W^{1,\infty}(\Gamma))\,,$$

satisfying the transmission conditions (4.4) *and* (4.5) *and the mass conservation* (4.8).

Proof. For u_0 given, $A := \|u^0\|_{L^\infty(\Gamma)}$, $K := \sup_{t>0, y\in\Gamma} \|H(t,\cdot,y)\|_{L^1(\Gamma))} > 1$ and $T > 0$ to be chosen later, let

$$B := \{u \in L^\infty((0,T)\times\Gamma) : u(0,y) = u^0(y) \text{ and } \sup_{0\leq t<T} \|u(t)\|_{L^\infty(\Gamma)} \leq K\,A + 1\}$$

and $d(u_1,u_2) := \sup_{0\leq t<T} \|u_1(t) - u_2(t)\|_{L^\infty(\Gamma)}$.

Next, for $u \in B$ fixed, c_0 given and c defined through u by (4.10), we define on B the map

$$\Psi(u)(t,y) := (H(t)*u^0)(y) + \int_0^t (\partial_x H(t-s)*(u(s)\partial_x c(s)))(y)ds\,, \qquad (t,y) \in (0,T)\times\Gamma\,. \tag{4.11}$$

Since (B,d) is a non empty complete metric space, we shall prove the claimed local existence using the Banach fixed point theorem.

First step : $\Psi(B) \subset B$. From (4.11) we have for $y \in \Gamma$

$$|\Psi(u)(t,y)| \leq K\,A + \kappa_0^{-2} \sup_{0\leq s<T} \|u(s)\|_{L^\infty(\Gamma)} \int_0^t \|\partial_x H(t-s,\cdot,y)\|_{L^1(\Gamma)} \|\partial_x c(s)\|_{L^\infty(\Gamma)} ds. \tag{4.12}$$

Next, owing to the property $\partial_y H(t,x,y) = -\partial_x H(t,x,y)$, by (4.10) we get for any $y \in \Gamma$

$$\begin{aligned}\partial_y c(t,y) &= e^{-(\alpha/\varepsilon)t}(\partial_y H(t\varepsilon^{-1}) * c^0)(y) + \frac{1}{\varepsilon}\int_0^t e^{-(\alpha/\varepsilon)(t-s)}(\partial_y H((t-s)\varepsilon^{-1}) * u(s))(y)ds \\ &= -e^{-(\alpha/\varepsilon)t}(\partial_x H(t\varepsilon^{-1}) * c^0)(y) + \frac{1}{\varepsilon}\int_0^t e^{-(\alpha/\varepsilon)(t-s)}(\partial_y H((t-s)\varepsilon^{-1}) * u(s))(y)ds\,.\end{aligned} \tag{4.13}$$

Furthermore, we observe that embedding the non-oriented network Γ into the oriented one $(V, \{e_j^\pm; j = 1, \dots, m\})$, denoted by 2Γ, by construction the fundamental solution H of the heat equation (3.2) on Γ is also solution on 2Γ and $\int_{2\Gamma} H(t,x,y)f(x)dx = 2\int_\Gamma H(t,x,y)f(x)dx$, for any f integrable on Γ (and so on 2Γ). Therefore, for any $y \in \Gamma$,

$$\begin{aligned}
\int_\Gamma \partial_x H(t\,\varepsilon^{-1}, x, y) c^0(x)\, dx &= \frac{1}{2}\int_{2\Gamma} \partial_x H(t\,\varepsilon^{-1}, x, y) c^0(x)\, dx \\
&= \frac{1}{2}\sum_{j=1}^m \kappa(e_j)\int_0^1 \partial_\xi \tilde{H}_j(t\,\varepsilon^{-1}, \xi, \eta)\tilde{c}_j^0(\xi)\, d\xi \\
&+ \frac{1}{2}\sum_{j=1}^m \kappa(e_j)\int_0^1 \partial_\xi \tilde{H}_j(t\,\varepsilon^{-1}, 1-\xi, \eta)\tilde{c}_j^0(1-\xi)\, d\xi \\
&= \frac{1}{2}\sum_{j=1}^m \kappa(e_j)[\tilde{H}_j(t\,\varepsilon^{-1}, \xi, \eta)\tilde{c}_j^0(\xi)]_0^1 - \frac{1}{2}\sum_{j=1}^m \kappa(e_j)\int_0^1 \tilde{H}_j(t\,\varepsilon^{-1}, \xi, \eta)\partial_\xi \tilde{c}_j^0(\xi)\, d\xi \\
&+ \frac{1}{2}\sum_{j=1}^m \kappa(e_j)[\tilde{H}_j(t\,\varepsilon^{-1}, 1-\xi, \eta)\tilde{c}_j^0(1-\xi)]_0^1 \\
&+ \frac{1}{2}\sum_{j=1}^m \kappa(e_j)\int_0^1 \tilde{H}_j(t\,\varepsilon^{-1}, 1-\xi, \eta)\partial_\xi \tilde{c}_j^0(1-\xi)\, d\xi = -(H(t\,\varepsilon^{-1}) * \partial_x c^0)(y)\,,
\end{aligned}$$

and (4.13) becomes

$$\partial_y c(t,y) = -e^{-(\alpha/\varepsilon)t}(H(t\,\varepsilon^{-1}) * \partial_x c^0)(y) + \frac{1}{\varepsilon}\int_0^t e^{-(\alpha/\varepsilon)(t-s)}(\partial_y H((t-s)\varepsilon^{-1}) * u(s))(y) ds\,.$$

Using (3.7) we arrive at the following estimate for the spatial derivative of c

$$|\partial_y c(t,y)| \le \frac{K}{\kappa_0}\|\partial_x c^0\|_{L^\infty(\Gamma)} + (\varepsilon\,\kappa_0)^{-1}\sup_{0\le s<T}\|u(s)\|_{L^\infty(\Gamma)}\int_0^t \left\|\partial_y H\left(\frac{t-s}{\varepsilon}, \cdot, y\right)\right\|_{L^1(\Gamma)} ds\,,$$

and by (3.9)

$$\|\partial_y c(t)\|_{L^\infty(\Gamma)} \le \kappa_0^{-1}K\|\partial_x c^0\|_{L^\infty(\Gamma)} + C(K\,A+1)(\varepsilon^{-1}t + \varepsilon^{-\frac{1}{2}}t^{\frac{1}{2}})\,, \qquad (4.14)$$

where $C > 0$ does not depend on ε. Finally, plugging (4.14) into (4.12) and using the decaying properties of H again, we get for $t \in (0,T)$

$$\begin{aligned}
\|\Psi(u)(t)\|_{L^\infty(\Gamma)} &\le K\,A + C(K\,A+1)\int_0^t (1 + (t-s)^{-1/2})\|\partial_x c(s)\|_{L^\infty(\Gamma)} ds \\
&\le K\,A + \widetilde{C}(t + t^{1/2})(1 + \varepsilon^{-1}t + \varepsilon^{-\frac{1}{2}}t^{\frac{1}{2}})\,,
\end{aligned}$$

where $\widetilde{C} = \widetilde{C}(K, A, \Gamma, \|\partial_x c^0\|_{L^\infty(\Gamma)})$. Therefore, for $T = T(\|u^0\|_{L^\infty(\Gamma)}, \|\partial_x c^0\|_{L^\infty(\Gamma)}, \varepsilon)$ positive and sufficiently small, it holds

$$\sup_{0\le t<T}\|\Psi(u)(t)\|_{L^\infty(\Gamma)} \le K\,A + 1\,.$$

To obtain the claim, we also observe that $\Psi(u)(0,y) = u^0(y)$ since by definition $H(0) * u^0 = u^0$.

Second step : Ψ is a contraction map on B. Let $u_1, u_2 \in B$. By (4.11) and arguing as in the previous step, we get for all $(t,y) \in (0,T) \times \Gamma$

$$\begin{aligned}|\Psi(u_1) - \Psi(u_2)|(t,y) &\le \int_0^t |\partial_x H(t-s) * [(u_1 - u_2)\partial_x c_1 + u_2(\partial_x c_1 - \partial_x c_2)](s)|(y)ds \\ &\le d(u_1,u_2)\kappa_0^{-2} \int_0^t \|\partial_x H(t-s,\cdot,y)\|_{L^1(\Gamma)} \|\partial_x c_1(s)\|_{L^\infty(\Gamma)} ds \\ &\quad + (K\,A+1)\kappa_0^{-2} \int_0^t \|\partial_x H(t-s,\cdot,y)\|_{L^1(\Gamma)} \|(\partial_x c_1 - \partial_x c_2)(s)\|_{L^\infty(\Gamma)} ds\,,\end{aligned} \tag{4.15}$$

and for all $t \in (0,T)$

$$\|(\partial_y c_1 - \partial_y c_2)(t)\|_{L^\infty(\Gamma)} \le C\, d(u_1,u_2)(\varepsilon^{-1}t + \varepsilon^{-\frac{1}{2}} t^{\frac{1}{2}})\,. \tag{4.16}$$

Plugging (4.14) and (4.16) into (4.15) and using (3.9), we arrive at

$$\|(\Psi(u_1) - \Psi(u_2))(t)\|_{L^\infty(\Gamma)} \le \widetilde{C}\, d(u_1,u_2)(t + t^{1/2})(1 + \varepsilon^{-1}t + \varepsilon^{-\frac{1}{2}} t^{\frac{1}{2}})\,.$$

Hence, for T sufficiently small again, Ψ is a contraction on B.

Third step : conclusion. By the previous steps, it follows that there exists a unique fixed point $u \in B$ of Ψ and that (u,c) satisfies the integral system (4.9)-(4.10). Furthermore, c is continuous on Γ and satisfies the transmission condition (4.5) because H is continuous on Γ and satisfies the same condition. Again because of the regularity of H, u is differentiable on Γ and c is twice differentiable. Consequently, performing an integration by part on each edge in the second term of the r.h.s. of (4.9), u can be also written as

$$\begin{aligned}u(t,y) = (H(t) * u^0)(y) &+ \int_0^t \sum_{i=1}^n H(t-s,v_i,y) \sum_{j \in E(v_i)} \kappa(e_j) u_j(s,v_i) \frac{\partial c_j}{\partial n}(s,v_i)\, ds \\ &- \int_0^t (H(t-s) * \partial_x(u(s)\partial_x c(s)))(y) ds\,,\end{aligned}$$

implying that $u(t) \in C^0(\Gamma)$ holds true for all $t \in (0,T)$. Finally, the continuity of u together with (4.5) gives that

$$u(t,y) = (H(t) * u^0)(y) - \int_0^t (H(t-s) * \partial_x(u(s)\partial_x c(s)))(y) ds\,.$$

So that u satisfies (4.4) and the proof is complete. □

We conclude this section showing the existence of a classical solution of system (4.1)-(4.7) in $(0,T)$ for any $T > 0$, i.e. we do not exclude that the solution blow-up for $T \to +\infty$. More specifically, we shall prove the following.

Theorem 4.4 (Global existence and positivity). *Under the hypothesis of Theorem 4.3, for all $T > 0$ there exists a solution (u, c) of the Keller-Segel system on the time interval $[0, T]$. Moreover, if the initial data u^0 and c^0 are nonnegative, the solution (u, c) is nonnegative.*

Proof. The global existence result is obtained by extending the local in time solution obtained in Theorem 4.3. Indeed, let T_{max} be the maximal time of existence of the obtained local solution. Then, the limits as $t \to T_{max}^-$ of u and c exist and depend only on $\|u^0\|_{L^\infty(\Gamma)}$, $\|\partial_x c^0\|_{L^\infty(\Gamma)}$ and ε. Therefore, it is possible to extend $(u(t), c(t))$ behind T_{max}, iteratively as many time as it is necessary to reach $T > 0$.

In order to obtain the positivity of the solution (u, c) when the initial data are positive, we analyze the time evolution of $\int_\Gamma \phi(u(t, y))\, dy$, where ϕ is a smooth function on $\mathbb{R}$ such that $\phi(z) > 0$ if $z < 0$, $\phi(z) = 0$ if $z \geq 0$ and there exists $C > 0$ such that $0 \leq \phi''(z)z^2 \leq C\phi(z)$, for all $z \in \mathbb{R}$. Owing to (4.1) and to the Kirchhoff conditions (4.4) and (4.5), we have for any $\delta > 0$

$$
\begin{aligned}
\frac{d}{dt}\int_\Gamma \phi(u(t,y))\,dy &= \sum_{j=1}^{m} \kappa(e_j) \int_0^1 \phi'(\tilde{u}_j(t,\eta))(\partial_{\eta\eta}\tilde{u}_j - \partial_\eta(\tilde{u}_j\partial_\eta\tilde{c}_j))(t,\eta)d\eta \\
&\leq -\sum_{j=1}^{m} \kappa(e_j) \int_0^1 \phi''(\tilde{u}_j)(\partial_\eta\tilde{u}_j)^2(t,\eta)d\eta \\
&\quad + \sum_{j=1}^{m} \kappa(e_j)\|\partial_\eta\tilde{c}_j(t)\|_{L^\infty(0,1)} \int_0^1 \phi''(\tilde{u}_j)|\tilde{u}_j||\partial_\eta\tilde{u}_j|(t,\eta)d\eta \\
&\leq \left(\frac{\delta}{2} - 1\right) \int_\Gamma \phi''(u)(\partial_y u)^2(t,y)dy + \frac{\kappa_0^{-1}}{2\,\delta}\|\partial_y c(t)\|^2_{L^\infty(\Gamma)} \int_\Gamma \phi''(u)u^2(t,y)\,dy\,.
\end{aligned}
$$

Choosing $\delta < 2$, by the properties of ϕ we get the differential inequality

$$
\frac{d}{dt}\int_\Gamma \phi(u(t,y))\,dy \leq \frac{C\,\kappa_0^{-1}}{2\,\delta}\|\partial_y c(t)\|^2_{L^\infty(\Gamma)} \int_\Gamma \phi(u(t,y))\,dy.
$$

Applying the Gronwall lemma, we obtain that $\phi(u(t, y)) = 0$, so that $u(t, y)) \geq 0$.

The positivity of c does not follow from (4.10), since H is not a priori positive, as observed before. Instead, it follows from the maximum principle for parabolic equations on network [1], taking also into account that u is positive. □

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