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Boundedness and convergence to steady states in a two-species chemotaxis system with logistic source

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1. Introduction

We consider the two-species chemotaxis system

$$(1.1)\qquad \begin{cases} u_t = \Delta u - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u), & x\in\Omega,\ t>0,\\ v_t = \Delta v - \nabla\cdot(v\chi_2(w)\nabla w) + \mu_2 v(1-v), & x\in\Omega,\ t>0,\\ w_t = d\Delta w + h(u,v,w), & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = \nabla w\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n\in\mathbb{N})$ with smooth boundary $\partial\Omega$ and ν is the outward normal vector to $\partial\Omega$. The initial data u_0, v_0 and w_0 are assumed to be nonnegative functions. The unknown functions $u(x,t)$ and $v(x,t)$ represent the population densities of two species and $w(x,t)$ shows the concentration of the substance at place x and time t.

In a mathematical view, global existence and behavior of solutions are fundamental theme. However, the problem (1.1) has some difficult points caused by the logistic term and by generalization of χ_i and h. For example, we cannot use the Lyapunov function. To overcome the difficulty, Negreanu–Tello [**9, 10**] built a technical way to prove global existence and asymptotic behavior of solutions to (1.1). In [**10**] they dealt with (1.1) when $d=0$, $\mu_i>0$ under the condition

$$\exists\,\overline{w}\ge w_0;\ h(\overline{u},\overline{v},\overline{w})\le 0,$$

where $\overline{u}$, $\overline{v}$ satisfy some representations determined by $\overline{w}$. In [**9**] they studied (1.1) when $0<d<1$, $\mu_i=0$ under similar conditions as in [**10**] and

$$(1.2)\qquad \chi_i' + \frac{1}{1-d}\chi_i^2 \le 0 \quad (i=1,2).$$

They supposed in [**9, 10**] that the functions h, χ_i for $i=1,2$ generalize of the prototypical case $\chi_i(w)=\frac{\chi_{0,i}}{(1+w)^{\sigma_i}}$ $(\chi_{0,i}>0,\sigma_i\ge 1)$, $h(u,v,w)=u+v-w$. As to the special case that $d=1$ and $h(u,v,w)=u+v-w$, Zhang–Li [**13**] proved global existence of solutions to (1.1) under the assumption that μ_i is small and $\chi_i(w)\le\frac{\chi_{0,i}}{(1+w)^{\sigma_i}}$ for $\sigma_i>1$, $\chi_{0,i}>0$ being small enough.

The purpose of the present paper is to obtain global existence and asymptotic stability of solutions to (1.1) without the restriction of $0 \le d < 1$. We shall suppose throughout this paper that h, χ_i $(i = 1, 2)$ satisfy the following conditions:

$$\chi_i \in C^{1+\theta}([0,\infty)) \cap L^1(0,\infty) \ (0 < \exists\, \theta < 1), \quad \chi_i > 0 \quad (i = 1,2), \tag{1.3}$$

$$h \in C^1([0,\infty) \times [0,\infty) \times [0,\infty)), \quad h(0,0,0) \ge 0, \tag{1.4}$$

$$\exists\, \gamma > 0;\ \frac{\partial h}{\partial u}(u,v,w) \ge 0, \quad \frac{\partial h}{\partial v}(u,v,w) \ge 0, \quad \frac{\partial h}{\partial w}(u,v,w) \le -\gamma, \tag{1.5}$$

$$\exists\, \delta > 0,\ \exists\, M > 0;\ |h(u,v,w) + \delta w| \le M(u + v + 1), \tag{1.6}$$

$$\exists\, k_i > 0;\ -\chi_i(w) h(0,0,w) \le k_i \quad (i = 1,2). \tag{1.7}$$

We also assume that

$$\exists\, p > n;\ 2d\chi_i'(w) + \left((d-1)p + \sqrt{(d-1)^2p^2 + 4dp}\right)[\chi_i(w)]^2 \le 0 \quad (i = 1,2). \tag{1.8}$$

The above conditions cover the prototypical example $\chi_i(w) = \frac{\chi_{0,i}}{(1+w)^{\sigma_i}}$ $(\chi_{0,i} > 0,\ \sigma_i > 1)$, $h(u,v,w) = u + v - w$. We assume that the initial data u_0, v_0, w_0 satisfy

$$0 \le u_0 \in C(\overline{\Omega}) \setminus \{0\}, \ 0 \le v_0 \in C(\overline{\Omega}) \setminus \{0\}, \ 0 \le w_0 \in W^{1,q}(\Omega) \ (\exists\, q > n). \tag{1.9}$$

Now the main results read as follows. The first theorem is concerned with global existence and boundedness in (1.1).

Theorem 1.1. *Let $d \ge 0$, $\mu_i > 0$ $(i = 1, 2)$. Assume that h, χ_i satisfy* (1.3)–(1.8). *Then for any u_0, v_0, w_0 satisfying* (1.9) *for some $q > n$, there exists an exactly one pair (u, v, w) of nonnegative functions*

$$u,\ v,\ w \in C(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \quad \text{when } d > 0,$$
$$u,\ v,\ w \in C([0,\infty); W^{1,q}(\Omega)) \cap C^1((0,\infty); W^{1,q}(\Omega)) \quad \text{when } d = 0,$$

which satisfy (1.1). *Moreover, the solution (u, v, w) is uniformly bounded, i.e., there exists a constant $C_1 > 0$ such that*

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \le C_1 \quad \text{for all } t \ge 0.$$

Remark 1.1. When $0 < d < 1$, we note that the condition (1.8) in Theorem 1.1 relaxes (1.2) assumed in [**9**], because the following relation holds:

$$\frac{(d-1)p + \sqrt{(d-1)^2p^2 + 4dp}}{2d} < \frac{1}{1-d}.$$

Now the second one, which gives asymptotic stability in (1.1), read as follows. We first introduce some notation. Since Theorem 1.1 guarantees that u, v and w exist globally and are bounded and nonnegative, it is possible to define nonnegative numbers α, β by

$$\alpha := \max_{(u,v,w) \in I} h_u(u,v,w), \qquad \beta := \max_{(u,v,w) \in I} h_v(u,v,w), \tag{1.10}$$

where $I = (0, C_1)^3$ and C_1 is defined in Theorem 1.1.

Theorem 1.2. *Let $d > 0$, $\mu_i > 0$ $(i = 1, 2)$. Under the conditions* (1.3)–(1.9) *and*

$$\alpha > 0, \quad \beta > 0, \quad \chi_1(0)^2 < \frac{16\mu_1 d\gamma}{\alpha^2 + \beta^2 + 2\alpha\beta}, \quad \chi_2(0)^2 < \frac{16\mu_2 d\gamma}{\alpha^2 + \beta^2 + 2\alpha\beta}, \tag{1.11}$$

the unique global solution (u, v, w) of (1.1) *satisfies that there exist $C > 0$ and $\lambda > 0$ such that*

$$\|u(t) - 1\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \widetilde{w}\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \quad (t > 0),$$

where $\widetilde{w} \ge 0$ such that $h(1, 1, \widetilde{w}) = 0$.

Remark 1.2. From (1.4)–(1.6) there exists $\tilde{w}$ such that $h(1, 1, \tilde{w}) = 0$. Indeed, if we choose $\overline{w} \ge 3M/\delta$, then (1.6) yields that $h(1, 1, \overline{w}) \le 3M - \delta\overline{w} \le 0$. On the other hand, (1.4) and (1.5) imply that $h(1, 1, 0) \ge h(0, 0, 0) \ge 0$. Hence, by the intermediate value theorem there exists $\tilde{w} \ge 0$ such that

$$h(1, 1, \tilde{w}) = 0.$$

The strategy for the proof of Theorem 1.1 is to construct estimates for $\int_\Omega u^p$ and $\int_\Omega v^p$. One of the keys for this strategy is to derive inequality

$$\frac{d}{dt}\int_\Omega u^p[f_1(w)]^{-r} \le a\int_\Omega u^p[f_1(w)]^{-r} - b\left(\int_\Omega u^p[f_1(w)]^{-r}\right)^{\frac{p+1}{p}} \tag{1.12}$$

for some positive constants a, b, where

$$f_1(w) := \exp\left\{\int_0^w \chi_1(s)\, ds\right\}.$$

Negreanu–Tello [**9, 10**] proved a similar differential inequality for "all" $p \ge 1$ and $r := \frac{(p-1)p}{p-d(p-1)}$. In this work we derive (1.12) for "some" $p > n$ and some $r = r(d, p) > 0$ by modifying the proof in [**9, 10**]. This enables us to improve the previous work and to remove the restriction of $0 \le d < 1$. On the other hand, the strategy for the proof of Theorem 1.2 is to modify an argument in [**8**]. The key for this strategy is to construct the following energy estimate:

$$\frac{d}{dt}E(t) \le -\varepsilon\left(\int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\tilde{w})^2\right)$$

with some function $E(t) \ge 0$ and some $\varepsilon > 0$. This strategy enables us to improve the conditions assumed in [**7**].

This paper is organized as follows. In Section 2 we collect basic facts which will be used later. In Section 3 we prove global existence and boundedness (Theorem 1.1). Section 4 is devoted to the proof of asymptotic stability (Theorem 1.2).

2. Preliminaries

In this paper we need the following well-known facts concerning the Laplacian in Ω supplemented with homogeneous Neumann boundary conditions (for details, see [**4, 5**]).

Lemma 2.1. *Suppose $k > 0$. Let Δ denote the realization of the Laplacian in $L^s(\Omega)$ with domain $\{z \in W^{2,s}(\Omega) \mid \nabla z \cdot \nu = 0$ on $\partial\Omega\}$ for $s \in (1,\infty)$. Then the operator $-\Delta + k$ is sectorial and possesses closed fractional powers $(-\Delta + k)^\eta$, $\eta \in (0,1)$, with dense domain $D((-\Delta + k)^\eta)$. Moreover, the following holds.*

(i) *If $m \in \{0,1\}$, $p \in [1,\infty]$ and $q \in (1,\infty)$, then there exists a constant $c_1 > 0$ such that for all $z \in D((-\Delta + k)^\eta)$,*

$$\|z\|_{W^{m,p}(\Omega)} \le c_1 \|(-\Delta + k)^\eta z\|_{L^q(\Omega)},$$

provided that $m < 2\eta$ and $m - n/p < 2\eta - n/q$.

(ii) *Suppose $p \in [1,\infty)$. Then the associated heat semigroup $(e^{t\Delta})_{t\ge 0}$ maps $L^p(\Omega)$ into $D((-\Delta + k)^\eta)$ in any of the space $L^q(\Omega)$, $q \ge p$, and there exist $c_2 > 0$ and $\lambda > 0$ such that for all $z \in L^p(\Omega)$,*

$$\|(-\Delta + k)^\eta e^{t(\Delta - k)} z\|_{L^q(\Omega)} \le c_2 t^{-\eta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\lambda t} \|z\|_{L^p(\Omega)} \quad (t > 0).$$

(iii) *Let $p \in (1,\infty)$. Then there exists $\lambda > 0$ such that for every $\varepsilon > 0$ there exists $c_3 > 0$ such that for all $\mathbb{R}^n$-valued $\omega \in C_0^\infty(\Omega)$,*

$$\left\|(-\Delta + k)^\eta e^{t\Delta} \nabla \cdot \omega\right\|_{L^p(\Omega)} \le c_3 t^{-\eta - \varepsilon - \frac{1}{2}} e^{-\lambda t} \|\omega\|_{L^p(\Omega)} \quad (t > 0). \tag{2.1}$$

Accordingly, the operator $(-\Delta + k)^\eta e^{t\Delta}\nabla\cdot$ admits a unique extension to all of $L^p(\Omega)$ which, again denoted by $(-\Delta + k)^\eta e^{t\Delta}\nabla\cdot$, satisfies (2.1) for all $\mathbb{R}^n$-valued $w \in L^p(\Omega)$.

Lemma 2.2. *Let $d \ge 0$, $\mu_i \ge 0$ $(i = 1,2)$. Assume that h, χ_i satisfy (1.3), (1.4), (1.6). Then for any u_0, v_0, w_0 satisfying (1.9) for some $q > n$, there exist $T_{\max} \in (0,\infty]$ and an exactly one pair (u,v,w) of nonnegative functions*

$$u,\ v,\ w \in C(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})) \quad \text{when } d > 0,$$
$$u,\ v,\ w \in C([0,T_{\max}); W^{1,q}(\Omega)) \cap C^1((0,T_{\max}); W^{1,q}(\Omega)) \quad \text{when } d = 0,$$

which satisfy (1.1). Moreover,

$$\text{either } T_{\max} = \infty \quad \text{or} \quad \lim_{t\to T_{\max}} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)}) = \infty.$$

Proof. We first consider the case $d > 0$. The proof of local existence of classical solutions to (1.1) is based on a standard contraction mapping argument, which can be found in [**11, 12**]. The case $d = 0$ is show in [**10**]. Finally the maximum principle is applied to yield $u > 0$, $v > 0$, $w \ge 0$ in $\Omega \times (0,T_{\max})$. □

3. Global existence and boundedness

Let (u, v, w) be the solution to (1.1) on $[0, T_{\max})$ as in Lemma 2.2. We introduce the functions $f_1 = f_1(w)$ and $f_2 = f_2(w)$ by

$$f_i(w) := \exp\left\{\int_0^w \chi_i(s)\,ds\right\} \quad \text{for } i = 1, 2 \tag{3.1}$$

to prove the following lemma.

Lemma 3.1. *Let $d \geq 0$, $\mu_i \geq 0$ $(i = 1, 2)$. Assume that χ_i satisfy* (1.3) *and* (1.8) *with some $p > n$. Then there exists $r = r(d, p) > 0$ such that*

$$\frac{d}{dt}\int_\Omega u^p f_1^{-r} \leq p\mu_1 \int_\Omega u^p f_1^{-r}(1-u) - r\int_\Omega u^p f_1^{-r}\chi_1(w)h(u,v,w), \tag{3.2}$$

$$\frac{d}{dt}\int_\Omega v^p f_2^{-r} \leq p\mu_2 \int_\Omega v^p f_2^{-r}(1-v) - r\int_\Omega v^p f_2^{-r}\chi_2(w)h(u,v,w). \tag{3.3}$$

Proof. We let $p \geq 1$ be fixed later. From the first and third equations in (1.1) we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p f_1^{-r} = {}& p\int_\Omega u^{p-1} f_1^{-r}\nabla\cdot(\nabla u - u\chi_1(w)\nabla w) + p\mu_1\int_\Omega u^p f_1^{-r}(1-u)\\ &- rd\int_\Omega u^p f_1^{-r}\chi_1(w)\Delta w - r\int_\Omega u^p f_1^{-r}\chi_1(w)h(u,v,w).\end{aligned}$$

Denoting by I_1 and I_2 the first and third terms on the right-hand side as

$$\begin{aligned} I_1 &:= p\int_\Omega u^{p-1} f_1^{-r}\nabla\cdot(\nabla u - u\chi_1(w)\nabla w),\\ I_2 &:= -rd\int_\Omega u^p f_1^{-r}\chi_1(w)\Delta w,\end{aligned}$$

we can write as

$$\frac{d}{dt}\int_\Omega u^p f_1^{-r} = I_1 + I_2 + p\mu_1\int_\Omega u^p f_1^{-r}(1-u) - r\int_\Omega u^p f_1^{-r}\chi_1(w)h(u,v,w). \tag{3.4}$$

We shall show that the following inequality:

$$\exists p > n,\ \exists r > 0;\ I_1 + I_2 \leq 0.$$

Noting that

$$f_1\nabla\left(\frac{u}{f_1}\right) = \nabla u - u\chi_1(w)\nabla w,$$

we obtain

$$\begin{aligned} I_1 &= p\int_\Omega u^{p-1} f_1^{-r}\nabla\cdot\left(f_1\nabla\left(\frac{u}{f_1}\right)\right)\\ &= p\int_\Omega \left(\frac{u}{f_1}\right)^{p-1} f_1^{-r+p-1}\nabla\cdot\left(f_1\nabla\left(\frac{u}{f_1}\right)\right)\\ &= -p(p-1)\int_\Omega\left(\frac{u}{f_1}\right)^{p-2} f_1^{-r+p}\left|\nabla\left(\frac{u}{f_1}\right)\right|^2\\ &\quad - p(-r+p-1)\int_\Omega\left(\frac{u}{f_1}\right)^{p-1} f_1^{-r+p}\chi_1(w)\nabla\left(\frac{u}{f_1}\right)\cdot\nabla w.\end{aligned}$$

Similarly, we see that

$$\begin{aligned} I_2 &= -rd \int_\Omega \left(\frac{u}{f_1}\right)^p f_1^{-r+p} \chi_1(w) \Delta w \\ &= rdp \int_\Omega \left(\frac{u}{f_1}\right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left(\frac{u}{f_1}\right) \cdot \nabla w \\ &\quad + rd \int_\Omega \left(\frac{u}{f_1}\right)^p f_1^{-r+p} \left((-r+p)[\chi_1(w)]^2 + \chi_1'(w)\right) |\nabla w|^2. \end{aligned}$$

Therefore it follows that

$$\begin{aligned} &I_1 + I_2 \\ &= -p(p-1) \int_\Omega \left(\frac{u}{f_1}\right)^{p-2} f_1^{-r+p} \left|\nabla \left(\frac{u}{f_1}\right)\right|^2 \\ &\quad - (p(p-1) - (1+d)pr) \int_\Omega \left(\frac{u}{f_1}\right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left(\frac{u}{f_1}\right) \cdot \nabla w \\ &\quad + \int_\Omega \left(\frac{u}{f_1}\right)^p f_1^{-r+p} \left(dr(-r+p)[\chi_1(w)]^2 + dr\chi_1'(w)\right) |\nabla w|^2 \\ &= -p(p-1) \int_\Omega \left(\frac{u}{f_1}\right)^{p-2} f_1^{-r+p} \left|\nabla \left(\frac{u}{f_1}\right) + \frac{p(p-1) - (1+d)pr}{2p(p-1)} \chi_1(w) \frac{u}{f_1} \nabla w\right|^2 \\ &+ \int_\Omega \left(\frac{u}{f_1}\right)^p f_1^{-r+p} \left[\left(\frac{(p(p-1) - (1+d)pr)^2}{4p(p-1)} + dr(-r+p)\right) [\chi_1(w)]^2 + dr\chi_1'(w)\right] |\nabla w|^2. \end{aligned}$$

Here we write as

$$\begin{aligned} &\left(\frac{(p(p-1) - (1+d)pr)^2}{4p(p-1)} + dr(-r+p)\right) [\chi_1(w)]^2 + dr\chi_1'(w) \\ &\qquad\qquad = \frac{1}{4p(p-1)}(a_1 r^2 + 2a_2 r + a_3), \end{aligned}$$

where a_1, a_2, a_3 are given by

$$\begin{aligned} a_1 &:= \left((d-1)^2 p + 4d\right) [\chi_1(w)]^2, \\ a_2 &:= (p-1)\left(p(d-1)[\chi_1(w)]^2 + 2d\chi_1'(w)\right), \\ a_3 &:= p(p-1)^2 [\chi_1(w)]^2. \end{aligned}$$

Then there exists $p > n$ such that the discriminant

$$D_r = 4(p-1)^2 \left[(p\chi_1^2(d-1) + 2d\chi_1')^2 - p\chi_1^4(p(d-1)^2 + 4d)\right]$$

is nonnegative in view of (1.8). Therefore we have that there exists $r > 0$ such that

$$I_1 + I_2 \leq 0.$$

Hence (3.4) implies

$$\frac{d}{dt} \int_\Omega u^p f_1^{-r} \leq p\mu_1 \int_\Omega u^p f_1^{-r}(1-u) - r \int_\Omega u^p f_1^{-r} \chi_1 h(u, v, w).$$

This means that (3.2) holds. In the same way, we obtain (3.3). □

Lemma 3.2. *Let $d \geq 0$, $\mu_i > 0$ $(i = 1,2)$. Assume that h, χ_i satisfy* (1.3)–(1.5), (1.7), *and* (1.8) *with some positive constants k_i $(i = 1,2)$ and $p > n$, then*

$$\|u(t)\|_{L^p(\Omega)} \leq \left(e^{\|\chi_1\|_{L^1(0,\infty)}}\right)^{r/p} \max\left\{\|u_0\|_{L^p(\Omega)}, \frac{p\mu_1 + rk_1}{p\mu_1}|\Omega|^{1/p}\right\}, \tag{3.5}$$

$$\|v(t)\|_{L^p(\Omega)} \leq \left(e^{\|\chi_2\|_{L^1(0,\infty)}}\right)^{r/p} \max\left\{\|v_0\|_{L^p(\Omega)}, \frac{p\mu_2 + rk_2}{p\mu_2}|\Omega|^{1/p}\right\}. \tag{3.6}$$

Proof. From the mean value theorem, the condition (1.5) and the fact that u, $v > 0$, it follows that for some ξ_1, ξ_2 satisfying $0 \leq \xi_1 \leq u$ and $0 \leq \xi_2 \leq v$,

$$\begin{aligned} h(u,v,w) &= \frac{\partial h}{\partial u}(\xi_1, v, w)u + \frac{\partial h}{\partial v}(0, \xi_2, w)v + h(0,0,w) \\ &\geq h(0,0,w). \end{aligned}$$

This together with the condition (1.7) leads to

$$\begin{aligned} -r\int_\Omega u^p f_1^{-r}\chi_1(w)h(u,v,w) &\leq -r\int_\Omega u^p f_1^{-r}\chi_1(w)h(0,0,w) \\ &\leq k_1 r\int_\Omega u^p f_1^{-r}. \end{aligned} \tag{3.7}$$

Combining (3.2) with (3.7), we obtain

$$\frac{d}{dt}\int_\Omega u^p f_1^{-r} \leq (\mu_1 p + k_1 r)\int_\Omega u^p f_1^{-r} - \mu_1 p\int_\Omega u^{p+1} f_1^{-r}.$$

Hence the Hölder inequality gives

$$\frac{d}{dt}\int_\Omega u^p f_1^{-r} \leq (\mu_1 p + k_1 r)\int_\Omega u^p f_1^{-r} - \mu_1 p|\Omega|^{-1/p}\left(\int_\Omega u^p f_1^{-r}\right)^{(p+1)/p}.$$

Solving this differential inequality, we infer

$$\left(\int_\Omega u^p f_1^{-r}\right)^{1/p} \leq \max\left\{\left(\int_\Omega u_0^p f_1^{-r}\right)^{1/p}, \frac{p\mu_1 + rk_1}{p\mu_1}|\Omega|^{1/p}\right\}.$$

Recalling the definition (3.1), we notice the relation $1 \leq f_1(w) \leq e^{\|\chi_1\|_{L^1(0,\infty)}}$, which yields (3.5). In the same way, we obtain (3.6). □

Remark 3.1. When $d = 0$, (3.2), (3.3), (3.5) and (3.6) still hold for all $p \geq 1$. Indeed, we have only to choose $r = 1 - p$ in the above proof.

Proof of Theorem 1.1. First consider the case $d > 0$. We let $\tau \in (0, T_{\max})$. In view of Lemma 2.2 it is sufficient to make sure that

$$\|u(t)\|_{L^\infty(\Omega)} \leq C_u(\tau), \quad \|v(t)\|_{L^\infty(\Omega)} \leq C_v(\tau), \quad \|w(t)\|_{L^\infty(\Omega)} \leq C_w(\tau), \quad t \in (\tau, T_{\max})$$

holds with some $C_u(\tau)$, $C_v(\tau)$, $C_w(\tau) > 0$. We let $\rho \in \left(\frac{p+n}{2p}, 1\right)$. This means $1 < 2\rho - \frac{n}{p}$. Writing as

$$w_t = d(\Delta - \delta/d)w + h(u, v, w) + \delta w,$$

and applying the variation of constants formula for w, we have

$$w(t) = e^{dt(\Delta-\delta/d)}w_0 + \int_0^t e^{d(t-s)(\Delta-\delta/d)}(h(u(s), v(s), w(s)) + \delta w(s))\, ds.$$

From Lemma 2.1 and (1.6) we obtain that for all $t \in (\tau, T_{\max})$,

$$\begin{aligned}
\|w(t)\|_{W^{1,\infty}(\Omega)} &\leq c_1 \left\|(-\Delta + \delta/d)^\rho\, w(t)\right\|_{L^p(\Omega)} \\
&\leq c_1c_2t^{-\rho}e^{-\lambda t}\|w_0\|_{L^p(\Omega)} \\
&\quad + c_1c_2 \int_0^t (t-s)^{-\rho}e^{-\lambda(t-s)}\|h(u(s), v(s), w(s)) + \delta w(s)\|_{L^p(\Omega)}\, ds \\
&\leq c_1c_2\tau^{-\rho}e^{-\lambda\tau}\|w_0\|_{L^p(\Omega)} + c_1c_2c_4 \int_0^t (t-s)^{-\rho}e^{-\lambda(t-s)}\, ds,
\end{aligned}$$

where $c_4 := \sup_{0\leq s<T_{\max}} \{M(\|u(s)\|_{L^p(\Omega)} + \|v(s)\|_{L^p(\Omega)} + 1)\}$ ($< \infty$ by Lemma 3.2). Noting that

$$\int_0^t (t-s)^{-\rho}e^{-\lambda(t-s)}ds \leq \int_0^\infty r^{-\rho}e^{-\lambda r}dr < \infty,$$

we deduce that

$$\|w(t)\|_{W^{1,\infty}(\Omega)} \leq c_1c_2 \left(\tau^{-\rho}e^{-\lambda\tau} + c_4 \int_0^\infty r^{-\rho}e^{-\lambda r}dr\right) =: C_w(\tau). \tag{3.8}$$

Since (1.8) implies $\chi_1' < 0$, it follows from (3.5) and (3.8) that for all $t \in (\tau/2, T_{\max})$,

$$\begin{aligned}
\|u(t)\chi_1(w(t))\nabla w(t)\|_{L^p(\Omega)} &\leq \chi_1(0)\|u(t)\|_{L^p(\Omega)}\|\nabla w(t)\|_{L^\infty(\Omega)} \\
&\leq \chi_1(0) \sup_{0\leq t<T_{\max}} \|u(t)\|_{L^p(\Omega)}C_w(\tau/2) =: c_5.
\end{aligned} \tag{3.9}$$

Employing the variation of constants formula for u yields

$$\begin{aligned}
u(t) &= e^{(t-\tau/2)(\Delta-1)}u\left(\frac{\tau}{2}\right) - \int_{\tau/2}^t e^{(t-s)(\Delta-1)}\nabla\cdot(u(s)\chi_1(w(s))\nabla w(s))\, ds \\
&\quad + \int_{\tau/2}^t e^{(t-s)(\Delta-1)}[(\mu_1+1)u(s) - \mu_1 u(s)^2]\, ds \\
&=: J_1 + J_2 + J_3, \quad t \in (\tau, T_{\max}).
\end{aligned}$$

Let $\eta \in \left(\frac{n}{2p}, \frac{1}{2}\right)$ and $\varepsilon \in \left(0, \frac{1}{2} - \eta\right)$. Then we observe that $0 < 2\eta - \frac{n}{p}$ and $\eta + \varepsilon + \frac{1}{2} < 1$. By Lemmas 2.1 and 3.2 we see that for all $t \in (\tau, T_{\max})$,

$$\begin{aligned}
\|J_1\|_{L^\infty(\Omega)} &= \left\| e^{(t-\tau/2)(\Delta-1)} u\left(\frac{\tau}{2}\right) \right\|_{L^\infty(\Omega)} \\
&\le c_1 \left\| (-\Delta+1)^\eta e^{(t-\tau/2)(\Delta-1)} u\left(\frac{\tau}{2}\right) \right\|_{L^p(\Omega)} \\
&\le c_1 c_2 \left(t - \frac{\tau}{2}\right)^{-\eta} e^{-\lambda t} \left\| u\left(\frac{\tau}{2}\right) \right\|_{L^p(\Omega)} \\
&\le 2^\eta c_1 c_2 \tau^{-\eta} e^{-\eta\tau} \sup_{0 \le t < T_{\max}} \|u(t)\|_{L^p(\Omega)}.
\end{aligned}$$

Using Lemma 2.1 and (3.9), we obtain

$$\begin{aligned}
\|J_2\|_{L^\infty(\Omega)} &\le \int_{\tau/2}^{t} \|e^{(t-s)(\Delta-1)} \nabla \cdot (u(s)\chi_1(w(s))\nabla w(s))\|_{L^\infty(\Omega)}\, ds \\
&\le c_1 \int_{\tau/2}^{t} \|(-\Delta+1)^\eta e^{(t-s)(\Delta-1)} \nabla \cdot (u(s)\chi_1(w(s))\nabla w(s))\|_{L^p(\Omega)}\, ds \\
&\le c_1 c_3 \int_{\tau/2}^{t} (t-s)^{-\eta-\varepsilon-1/2} e^{-(\nu+1)(t-s)} \|u(s)\chi_1(w(s))\nabla w(s)\|_{L^p(\Omega)}\, ds \\
&\le c_1 c_3 c_5 \int_0^\infty r^{-(\eta+\varepsilon+1/2)} e^{-(\nu+1)r}\, dr.
\end{aligned}$$

Since the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ has the order preserving property, we infer

$$\begin{aligned}
J_3 &= \int_{\tau/2}^{t} e^{(t-s)(\Delta-1)} \left[-\mu_1 \left(u(s) - \frac{\mu_1+1}{2\mu_1} \right)^2 + \frac{(\mu_1+1)^2}{4\mu_1} \right] ds \\
&\le \frac{(\mu_1+1)^2}{4\mu_1} \int_{\tau/2}^{t} e^{(t-s)\Delta} e^{-(t-s)}\, ds,
\end{aligned}$$

and moreover, by the maximum principle we have

$$\begin{aligned}
J_3 &\le \frac{(\mu_1+1)^2}{4\mu_1} \int_{\tau/2}^{t} e^{-(t-s)}\, ds \\
&\le \frac{(\mu_1+1)^2}{4\mu_1} (1 - e^{-\tau/2}).
\end{aligned}$$

Therefore we obtain that there exists $C_u(\tau) > 0$ such that

$$\begin{aligned}
u(t) &\le \|J_1\|_{L^\infty(\Omega)} + \|J_2\|_{L^\infty(\Omega)} + J_3 \\
&\le C_u(\tau), \quad t \in (\tau, T_{\max}).
\end{aligned}$$

The positivity of u yields that

$$\|u(t)\|_{L^\infty(\Omega)} \le C_u(\tau), \quad t \in (\tau, T_{\max}).$$

The same argument as for u gives the $L^\infty(\Omega)$ bound for v. This completes the proof in the case $d > 0$.

Next consider the case $d = 0$. From Remark 3.1 we have

$$\|u(t)\|_{L^p(\Omega)} \le \exp\{\|\chi_1\|_{L^1(0,\infty)}\}^{(p-1)/p} \max\left\{\|u_0\|_{L^p(\Omega)}, \frac{p\mu_1 + (p-1)k_1}{p\mu_1}|\Omega|^{1/p}\right\}$$

for all $p \ge 1$. Taking the limits as $p \to \infty$, we obtain the $L^\infty(\Omega)$ bound for u, and similarly for v. The L^∞ bound for w follows from

$$w(t) = e^{-\delta t} w_0 + \int_0^t e^{-\delta(t-s)}(h(u, v, w) + \delta w).$$

This completes the proof when $d = 0$. □

4. Asymptotic behavior

In this section we will establish asymptotic stability of solutions to (1.1). For the proof of Theorem 1.2, we shall prepare some elementary results.

Lemma 4.1 ([**1**, Lemma 3.1]). *Suppose that $f : (1, \infty) \to \mathbb{R}$ is a uniformly continuous nonnegative function satisfying $\int_1^\infty f(t)\,dt < \infty$. Then $f(t) \to 0$ as $t \to \infty$.*

Lemma 4.2. *Let $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$. Suppose that*

$$a_1 > 0, \quad a_3 > 0, \quad a_5 - \frac{a_2^2}{4a_1} - \frac{a_4^2}{4a_3} > 0. \tag{4.1}$$

Then

$$a_1x^2 + a_2xz + a_3y^2 + a_4yz + a_5z^2 \ge 0 \tag{4.2}$$

holds for all $x, y, z \in \mathbb{R}$.

Proof. From straightforward calculations we obtain

$$\begin{aligned} &a_1x^2 + a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ &= a_1\left(x + \frac{a_3z}{2a_1}\right)^2 + a_3\left(y + \frac{a_4z}{2a_3}\right)^2 + \left(a_5 - \frac{a_3^2}{4a_1} - \frac{a_4^2}{4a_3}\right)z^2. \end{aligned}$$

In view of the above equation, (4.1) leads to (4.2). □

Now we will prove the key estimate for the proof of Theorem 1.2.

Lemma 4.3. *Let (u, v, w) be a solution to* (1.1). *Under the conditions* (1.3)–(1.9) *and* (1.11), *there exist $\delta_1, \delta_2 > 0$ and $\varepsilon > 0$ such that the nonnegative functions E_1 and F_1 defined by*

$$E_1(t) := \int_\Omega (u - 1 - \log u) + \delta_1\frac{\mu_1}{\mu_2}\int_\Omega (v - 1 - \log v) + \frac{\delta_2}{2}\int_\Omega (w - \widetilde{w})^2$$

and

$$F_1(t) := \int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2$$

satisfy

$$\frac{d}{dt}E_1(t) \le -\varepsilon F_1(t) \qquad (t>0). \tag{4.3}$$

Proof. Thanks to (1.11), we can choose $\delta_1 = \frac{\beta}{\alpha} > 0$ and $\delta_2 > 0$ satisfying

$$\max\left\{\frac{\chi_1(0)^2(1+\delta_1)}{4d}, \frac{\mu_1\chi_2(0)^2(1+\delta_1)}{4\mu_2 d}\right\} < \delta_2 < \frac{4\mu_1\gamma\delta_1}{\alpha^2\delta_1+\beta^2}. \tag{4.4}$$

We denote by $A_1(t)$, $B_1(t)$, $C_1(t)$ the functions defined as

$$\begin{aligned} A_1(t) &:= \int_\Omega (u-1-\log u), \quad B_1(t) = \int_\Omega (v-1-\log v), \\ C_1(t) &:= \frac{1}{2}\int_\Omega (w-\widetilde{w})^2, \end{aligned}$$

and we write as

$$E_1(t) = A_1(t) + \delta_1\frac{\mu_1}{\mu_2}B_1(t) + \delta_2 C_1(t).$$

The Taylor formula applied to $H(s) = s - \log s$ $(s \ge 0)$ yields $A_1(t) = \int_\Omega (H(u) - H(1))$ is a nonnegative function for $t > 0$ (more detail, see [**1**, Lemma 3.2]). Similarly, we have that $B_1(t)$ is a positive function. By straightforward calculations we infer

$$\begin{aligned} \frac{d}{dt}A_1(t) &= -\mu_1\int_\Omega (u-1)^2 - \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega \frac{\chi_1(w)}{u}\nabla u\cdot\nabla w, \\ \frac{d}{dt}B_1(t) &= -\mu_2\int_\Omega (v-1)^2 - \int_\Omega \frac{|\nabla v|^2}{v^2} + \int_\Omega \frac{\chi_2(w)}{v}\nabla v\cdot\nabla w, \\ \frac{d}{dt}C_1(t) &= \int_\Omega h_u(u-1)(w-\widetilde{w}) + \int_\Omega h_v(v-1)(w-\widetilde{w}) + \int_\Omega h_w(w-\widetilde{w})^2 \\ &\quad - d\int_\Omega |\nabla w|^2 \end{aligned}$$

with some derivatives h_u, h_v and h_w. Hence we have

$$\frac{d}{dt}E_1(t) = I_3(t) + I_4(t), \tag{4.5}$$

where

$$\begin{aligned} I_3(t) &:= -\mu_1\int_\Omega (u-1)^2 - \delta_1\mu_1\int_\Omega (v-1)^2 + \delta_2\int_\Omega h_u(u-1)(w-\widetilde{w}) \\ &\quad + \delta_2\int_\Omega h_v(v-1)(w-\widetilde{w}) + \delta_2\int_\Omega h_w(w-\widetilde{w})^2 \end{aligned}$$

and

$$I_4(t) := -\int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega \frac{\chi_1(w)}{u}\nabla u\cdot\nabla w - \delta_1\frac{\mu_1}{\mu_2}\int_\Omega \frac{|\nabla v|^2}{v^2} + \delta_1\frac{\mu_1}{\mu_2}\int_\Omega \frac{\chi_2(w)}{v}\nabla v\cdot\nabla w - d\delta_2\int_\Omega |\nabla w|^2. \tag{4.6}$$

At first, we shall show from Lemma 4.2 that there exists $\varepsilon_1 > 0$ such that

$$I_3(t) \le -\varepsilon_1\left(\int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2\right). \tag{4.7}$$

To see this, we put

$$g_1(\varepsilon) := \mu_1 - \varepsilon, \qquad g_2(\varepsilon) := \delta_1\mu_1 - \varepsilon,$$
$$g_3(\varepsilon) := (-\delta_2 h_w - \varepsilon) - \frac{h_u^2}{4(\mu_1-\varepsilon)}\delta_2^2 - \frac{h_v^2}{4(\delta_1\mu_1-\varepsilon)}\delta_2^2.$$

Since $\mu_1 > 0$ and $\delta_1 = \frac{\beta}{\alpha} > 0$, we have $g_1(0) = \mu_1 > 0$ and $g_2(0) = \delta_1\mu_1 > 0$. In light of (1.5) and the definitions of $\delta_2, \alpha, \beta > 0$ (see (1.10) and (4.4)) we obtain

$$\begin{aligned} g_3(0) &= \delta_2\left(-h_w - \left(\frac{h_u^2}{4\mu_1} + \frac{h_v^2}{4\delta_1\mu_1}\right)\delta_2\right) \\ &\ge \delta_2\left(\gamma - \left(\frac{\alpha^2}{4\mu_1} + \frac{\beta^2}{4\delta_1\mu_1}\right)\delta_2\right) \\ &\ge \delta_2\left(\gamma - \left(\frac{\alpha^2\delta_1+\beta}{4\delta_1\mu_1}\right)\delta_2\right) > 0. \end{aligned}$$

Combination of the above inequalities and the continuity of g_i for $i = 1,2,3$ yield that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ hold for $i = 1,2,3$. Thanks to Lemma 4.2 with

$$\begin{aligned} &a_1 = \mu_1 - \varepsilon_1, \quad a_2 = -\delta_2 h_u, \quad a_3 = \delta_1\mu_1 - \varepsilon_1, \\ &a_4 = -\delta_2 h_v, \quad a_5 = -\delta_2 h_w - \varepsilon_1, \\ &x = u(t) - 1, \quad y = v(t) - 1, \quad z = w(t) - \widetilde{w}, \end{aligned}$$

we obtain (4.7) with $\varepsilon_1 > 0$. Lastly we will prove

$$I_4(t) \le 0. \tag{4.8}$$

Noting that $\chi_i' < 0$ (from (1.8)) and then using the Young inequality, we have

$$\begin{aligned} \int_\Omega \frac{\chi_1(w)}{u}\nabla u\cdot\nabla w &\le \chi_1(0)\int_\Omega \frac{|\nabla u\cdot\nabla w|}{u} \\ &\le \frac{\chi_1(0)^2(1+\delta_1)}{4d\delta_2}\int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{d\delta_2}{1+\delta_1}\int_\Omega |\nabla w|^2 \end{aligned}$$

and

$$\delta_1\frac{\mu_1}{\mu_2}\int_\Omega \frac{\chi_2(w)}{v}\nabla v\cdot\nabla w \le \chi_2(0)\delta_1\frac{\mu_1}{\mu_2}\int_\Omega \frac{|\nabla v\cdot\nabla w|}{v}$$
$$\le \frac{\chi_2(0)^2\delta_1(1+\delta_1)}{4d\delta_2}\left(\frac{\mu_1}{\mu_2}\right)^2\int_\Omega\frac{|\nabla v|^2}{v^2}+\frac{d\delta_1\delta_2}{1+\delta_1}\int_\Omega|\nabla w|^2.$$

Plugging these into (4.6) we infer

$$I_4(t)\le -\left(1-\frac{\chi_1(0)^2(1+\delta_1)}{4d\delta_2}\right)\int_\Omega\frac{|\nabla u|^2}{u^2}$$
$$-\delta_1\frac{\mu_1}{\mu_2}\left(1-\frac{\mu_1\chi_2(0)^2(1+\delta_1)}{4d\mu_2\delta_2}\right)\int_\Omega\frac{|\nabla v|^2}{v^2}.$$

We note from the definition of $\delta_2>0$ that

$$1-\frac{\chi_1(0)^2(1+\delta_1)}{4d\delta_2}>0,$$
$$1-\frac{\mu_1\chi_2(0)^2(1+\delta_1)}{4d\mu_2\delta_2}>0.$$

Thus we have (4.8). Combination of (4.5), (4.7) and (4.8) implies the end of the proof. □

Lemma 4.4. *Let (u,v,w) be a solution to* (1.1). *Under the conditions* (1.3)–(1.9) *and* (1.11), *(u,v,w) has the following asymptotic behavior*:

$$\|u(t)-1\|_{L^\infty(\Omega)}\to 0,\quad \|v(t)-1\|_{L^\infty(\Omega)}\to 0,\quad \|w(t)-\widetilde{w}\|_{L^\infty(\Omega)}\to 0\quad (t\to\infty).$$

Proof. Firstly the boundedness of u, v, ∇w and a standard parabolic regularity theory ([**6**]) yield that there exist $\theta\in(0,1)$ and $C>0$ such that

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[1,t])}+\|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[1,t])}+\|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[1,t])}\le C\quad\text{for all } t\ge 1.$$

Therefore in view of the Gagliardo–Nirenberg inequality

$$\|\varphi\|_{L^\infty(\Omega)}\le c\|\varphi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|\varphi\|_{L^2(\Omega)}^{\frac{2}{n+2}}\quad (\varphi\in W^{1,\infty}(\Omega)), \tag{4.9}$$

it is sufficient to show that

$$\|u(t)-1\|_{L^2(\Omega)}\to 0,\quad \|v(t)-1\|_{L^2(\Omega)}\to 0,\quad \|w(t)-\widetilde{w}\|_{L^2(\Omega)}\to 0\quad (t\to\infty).$$

We let

$$f_1(t):=\int_\Omega(u-1)^2+\int_\Omega(v-1)^2+\int_\Omega(w-\widetilde{w})^2.$$

We have that $f_1(t)$ is a nonnegative function, and thanks to the regularity of u,v,w we can see that $f_1(t)$ is uniformly continuous. Moreover, integrating (4.3) over $(1,\infty)$, we infer from the positivity of $E_1(t)$ that

$$\int_1^\infty f_1(t)\,dt\le\frac{1}{\varepsilon}E_1(1)<\infty.$$

Therefore we conclude from Lemma 4.1 that $f_1(t) \to 0$ $(t \to \infty)$, which means

$$\int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2 \to 0 \quad (t \to \infty).$$

This implies the end of the proof. □

Lemma 4.5. *Let* (u, v, w) *be a solution to* (1.1). *Under the conditions* (1.3)–(1.9) *and* (1.11), *there exist* $C > 0$ *and* $\lambda > 0$ *such that*

$$\|u(t)-1\|_{L^\infty(\Omega)} + \|v(t)-1\|_{L^\infty(\Omega)} + \|w(t)-\widetilde{w}\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \quad (t > 0).$$

Proof. From the L'Hôpital theorem applied to $H_1(s) := s - \log s$ we can see

$$\lim_{s\to 1} \frac{H_1(s) - H_1(1)}{(s-1)^2} = \lim_{s \to 1} \frac{H_1''(s)}{2} = \frac{1}{2}. \tag{4.10}$$

In view of the combination of (4.10) and $\|u-1\|_{L^\infty(\Omega)} \to 0$ from Lemma 4.4 we obtain that there exists $t_0 > 0$ such that

$$\frac{1}{4}\int_\Omega (u-1)^2 \le A_1(t) = \int_\Omega (H(u) - H(1)) \le \int_\Omega (u-1)^2 \quad (t > t_0). \tag{4.11}$$

A similar argument yields that there exists $t_1 > t_0$ such that

$$\frac{1}{4}\int_\Omega (v-1)^2 \le B_1(t) \le \int_\Omega (v-1)^2 \quad (t > t_1). \tag{4.12}$$

We infer from (4.11) and the definitions of $E_1(t)$, $F_1(t)$ that

$$E_1(t) \le c_6 F_1(t)$$

for all $t > t_1$ with some $c_6 > 0$. Plugging this into (4.3), we have

$$\frac{d}{dt}E_1(t) \le -\varepsilon F_1(t) \le -\frac{\varepsilon}{c_6}E_1(t) \qquad (t > t_1),$$

which implies that there exist $c_7 > 0$ and $\ell > 0$ such that

$$E_1(t) \le c_7 e^{-\ell t} \qquad (t > t_1).$$

Thus we obtain from (4.11) and (4.12) that

$$\int_\Omega (u-1)^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-\widetilde{w})^2 \le c_8 E_1(t) \le c_7 c_8 e^{-\ell t}$$

for all $t > t_1$ with some $c_8 > 0$. From the Gagliardo–Nirenberg inequality (4.9) with the regularity of u, v, w, we achieve that there exist $C > 0$ and $\lambda > 0$ such that

$$\|u(t)-1\|_{L^\infty(\Omega)} + \|v(t)-1\|_{L^\infty(\Omega)} + \|w(t)-\widetilde{w}\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \quad (t > 0).$$

This completes the proof of Lemma 4.5. □

Proof of Theorem 1.2. Theorem 1.2 follows directly from Lemma 4.5. □

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