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Complementarity of subspaces of ℓ_∞ revisited

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1 Introduction

This note is a survey of [9]. Let X be a Banach space. A closed subspace M of X is said to be *complemented* in X if there exists a closed subspace N of X such that $X = M \oplus N$ (that is, $X = M + N$ and $M \cap N = \{0\}$), or equivalently, there exists a bounded linear projection from X onto M. The study on complementarity of closed subspaces of Banach spaces has played a central role in the isomorphic theory; and is still of interest for many mathematicians working around Banach space theory since some long-standing problems was solved in 1990s by using (hereditarily) indecomposable Banach spaces.

The first example of an uncomplemented closed subspace of a Banach space is the (null) convergent sequence space c (or c_0) in the bounded sequence space ℓ_∞. This appeared as a consequence of the study on representation of linear operators on certain Banach spaces by Phillips [8]. After a quarter century later, Whitley [10] gave a simplified proof which based on an idea due to Nakamura and Kakutani [7]. Namely, he showed that $(\ell_\infty/c_0)^*$ has no countable total subsets, where a subset F of the dual space X^* of a Banach space X is said to be *total* if $f(x) = 0$ for each $f \in F$ implies that $x = 0$. Since the property that X^* has a countable total subset is preserved under taking subspaces or by linear isomorphisms, Whitley's argument is sufficient for denying the complementarity of c_0 in ℓ_∞.

In 1967, Lindenstrauss [5] characterized complemented subspaces of ℓ_∞ by showing that ℓ_∞ is a prime Banach space, where an infinite dimensional Banach space X is said to be *prime* if every infinite dimensional complemented subspace of X is isomorphic to X. From this and the fact that ℓ_∞ is injective, an infinite dimensional closed subspace of ℓ_∞ is complemented in ℓ_∞ if and only if it is isomorphic to ℓ_∞. This powerful characterization concludes, at least, any separable subspace of ℓ_∞ cannot be complemented in ℓ_∞, which drastically improves the result of Phillips. However, we note that it is not always effective in determining the complementarity of concrete non-separable subspaces of ℓ_∞. To do this, we still have to investigate for case by case; because we do not know whether checking an infinite dimensional subspace of ℓ_∞ is (not) isomorphic to ℓ_∞ is easier than examining the complementarity of the subspace directly.

The aim of this note is to present a simple criterion for complementarity of subspaces of ℓ_∞ induced by bounded linear operators admitting matrix representations.

2 Matrix representations of operators on ℓ_∞

We begin with preliminary works on matrix representations of operators on ℓ_∞. In what follows, let (e_n) be the standard unit vector basis for the space c_{00} of all complex sequences with finitely nonzero coordinates, that is, let $e_n = (0, \dots, 0, 1, 0, \dots)$ and $e_n^* a = a_n$ for each $n \in \mathbb{N}$ and each $a = (a_n) \in \ell_\infty$, where 1 is in the n-th position.

A linear operator T on ℓ_∞ is said to *admits a matrix representation* if there exists an infinite matrix (t_{ij}) of complex numbers such that $e_i^* T a = \sum_{j=1}^{\infty} t_{ij} a_j$ for each $a = (a_n) \in \ell_\infty$ and each $i \in \mathbb{N}$. Some basic facts about linear operators on ℓ_∞ admitting matrix representations are collected in the following proposition. The proof is routine; so it is included only for the sake of completeness.

Proposition 2.1. *Let T be a linear operator on ℓ_∞.*

(i) *T admits a matrix representation if and only if*

$$e_i^* T a = \lim_n e_i^* T(a_1, \dots, a_n, 0, \dots)$$

for each $(a_n) \in \ell_\infty$ and each $i \in \mathbb{N}$.

(ii) *Suppose that T admits a matrix representation (t_{ij}). Then T is bounded if and only if $M = \sup\{\sum_{j=1}^{\infty} |t_{ij}| : i \in \mathbb{N}\} < \infty$. In that case, $\|T\| = M$.*

For a Banach spaces X, let $B(X)$ be the Banach space of all bounded linear operators on X.

Corollary 2.2. *Let $M(\ell_\infty)$ be the subspace of $B(\ell_\infty)$ consisting of all operators admitting matrix representations. Then $M(\ell_\infty)$ is isometrically isomorphic to $\ell_\infty(\ell_1)$.*

We next consider some special properties of elements T of $M(\ell_\infty)$ satisfying $T(c_0) \subset c_0$. For this, we need the following basic lemma.

Lemma 2.3. *Let $T \in B(c_0)$. Then there exists a unique weak*-to-weak* continuous operator T_∞ on ℓ_∞ with $\|T_\infty\| = \|T\|$ that extends T.*

For weak*-to-weak* continuous linear operators T on ℓ_∞, the condition $T(c_0) \subset c_0$ can be characterized by a simple way.

Lemma 2.4. *Let S be a weak*-to-weak* continuous linear operator on ℓ_∞. Then $S(c_0) \subset c_0$ if and only if $S = T_\infty$ for some $T \in B(c_0)$.*

The following result helps us to understanding a position of bounded linear operators on ℓ_∞ admitting matrix representations.

Proposition 2.5. *Let $T \in B(\ell_\infty)$.*

(i) *If T is weak*-to-weak* continuous then $T \in M(\ell_\infty)$.*

(ii) *If $T \in M(\ell_\infty)$ and $T(c_0) \subset c_0$, then T is weak*-to-weak* continuous.*

Now let $M_0(\ell_\infty) = \{T \in M(\ell_\infty) : T(c_0) \subset c_0\}$. Then, by the preceding proposition, $T \in M_0(\ell_\infty)$ if and only if T is a weak*-to-weak* continuous operator on ℓ_∞ satisfying $T(c_0) \subset c_0$.

The following provides a simple characterization of $M_0(\ell_\infty)$ in $M(\ell_\infty)$.

Proposition 2.6. *Let $T \in M(\ell_\infty)$ with a matrix representation (t_{ij}). Then $T \in M_0(\ell_\infty)$ if and only if $t_{ij} \to 0$ as $i \to \infty$ for each $j \in \mathbb{N}$.*

We conclude this section with another characterization of $M_0(\ell_\infty)$ which shows that all elements of $M_0(\ell_\infty)$ are induced by those of $B(c_0)$.

Corollary 2.7. *$M_0(\ell_\infty) = \{T_\infty : T \in B(c_0)\}$. Consequently, $M_0(\ell_\infty)$ is isometrically isomorphic to $B(c_0)$.*

3 Subspaces of ℓ_∞ induced by matrices

Let $B(\ell_\infty)$ denote the Banach space of bounded linear operators on ℓ_∞. Suppose that $T \in B(\ell_\infty)$. We consider the closed subspaces $c(T) := T^{-1}(c)$ and $c_0(T) := T^{-1}(c_0)$ of ℓ_∞, respectively. We note that $c(I) = c$ and $c_0(I) = c_0$ while $c(0) = c_0(0) = \ell_\infty$.

A linear operator T on ℓ_∞ is said to *admits a matrix representation* if there exists an infinite matrix (t_{ij}) of complex numbers such that $(Ta)_n = \sum_{j=1}^{\infty} t_{nj}a_j$ for each $a = (a_n) \in \ell_\infty$. If $T \in M(\ell_\infty)$, the spaces $c(T)$ and $c_0(T)$ are closely related to objects studied in the monograph [1]. In particular, $c(T)$ is called the *bounded summability field* of T; see also [2, 3].

We first consider some conditions equivalent to $c_0(T) = \ell_\infty$. The following is a key ingredient for the proof of the main theorem in this paper.

Theorem 3.1. *Let $T \in M_0(\ell_\infty)$ with a matrix representation (t_{ij}). Then the following are equivalent:*

(i) $c_0(T) = \ell_\infty$.

(ii) *T is a compact operator on ℓ_∞.*

(iii) $\lim_i \sum_{j=1}^{\infty} |t_{ij}| = 0$.

The following is the main theorem. The proof is based on a combination of a *gliding hump argument* and Whitley's method [10].

Theorem 3.2. *Let T be a non-compact element of $M_0(\ell_\infty)$ with a matrix representation (t_{ij}). If M is a closed subspace with $c_0 \subset M \subset c(T)$, then $(\ell_\infty/M)^*$ has no countable total subsets. Consequently, M is not complemented in ℓ_∞.*

As a consequence of Theorems 3.1 and 3.2, we have the following dichotomy.

Corollary 3.3. *Let $T \in M_0(\ell_\infty)$. Then one and only one of the following two statements holds:*

(i) $c_0(T) = c(T) = \ell_\infty$.

(ii) *All closed subspaces M of ℓ_∞ with $c_0 \subset M \subset c(T)$ are uncomplemented in ℓ_∞.*

The rest of this section is devoted to presenting some applications of Theorem 3.2. Recall that a sequence $a = (a_n) \in \ell_\infty$ is said to be *mean convergent* to α if the sequence $(n^{-1}\sum_{j=1}^{n} a_j)$ converges to α, and *almost convergent* to the *almost limit* α if $\varphi(a) = \alpha$ for each Banach limit φ on ℓ_∞. It is well-known as Lorentz's theorem [6] that $a = (a_n) \in \ell_\infty$ is almost convergent to α if and only if

$$\lim_m \sup_{n\in\mathbb{N}} \left| \frac{1}{m} \sum_{j=1}^{m} a_{n+j-1} - \alpha \right| = 0.$$

The spaces of all mean convergent, almost convergent and almost null sequences are denoted by $\mathcal{M}$, f and f_0, respectively. We note that $c_0 \subset f_0 \subset f \subset \mathcal{M}$ holds.

Corollary 3.4. *All the spaces $\mathcal{M}, f, f_0$ are closed and uncomplemented in ℓ_∞.*

Corollary 3.5. *Let d and d_0 be subspaces of ℓ_∞ given by*

$$d = \{a = (a_n) \in \ell_\infty : (a_n - a_{n+1}) \text{ converges}\}$$
$$d_0 = \{a = (a_n) \in \ell_\infty : (a_n - a_{n+1}) \text{ converges to } 0\}$$

Then d, d_0 are closed and uncomplemented in ℓ_∞.

4 A weak* closed subspace

In this section, we show the limit of Whitley's method. The following is a key ingredient.

Theorem 4.1. *There exists an uncomplemented weak* closed subspace W of ℓ_∞. Moreover, W contains an isometric copy of ℓ_∞.*

Moreover, weak* closed subspaces have a special property.

Proposition 4.2. *Let M be a weak* closed subspace of ℓ_∞. Then there exists a countable total subset of $(\ell_\infty/M)^*$.*

As a consequence, for a closed subspace M of ℓ_∞, the property that $(\ell_\infty/M)^*$ has a countable total subset is necessary but not sufficient for assuring the complementarity of M in ℓ_∞. We wonder what structural conditions are equivalent to this isomorphic property. We finally mention an impact of the property that $(\ell_\infty/M)^*$ has a countable total subset, where M is a closed subspace of ℓ_∞ containing c_0.

Proposition 4.3 (Jameson [4]). *Let M be a closed subspace of ℓ_∞ containing c_0. If $(\ell_\infty/M)^*$ has a countable total subset. Then $\ell_\infty(N) \subset M$ for some infinite subset N of $\mathbb{N}$, where $\ell_\infty(N) = \{a = (a_n) \in \ell_\infty : a_n = 0 \text{ for each } n \notin N\}$.*

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