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# Circular solutions to the elastic flow in hyperbolic space

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**Abstract**

In the recent work [6] the authors studied the elastic flow in the hyperbolic plane for closed curves. Subconvergence to elastica was established under an additional length penalization. In this paper we show that this penalization is not necessary for the convergence of closed geodesic circles by computing the evolution explicitly.
We supplement the paper with a comparison between the elastic flow in the hyperbolic plane and the Willmore flow of surfaces of revolution.

## 1 Introduction

Let $f\colon \mathbb{S}^1 \to M$ be a smooth immersion of a closed curve in a smooth Riemannian manifold $(M^n, g)$ of dimension $n \geq 2$. Similar to the Bernoulli model of an elastic rod in the Euclidean case we define its elastic energy as

$$\mathcal{E}_\lambda(f) = \frac{1}{2}\int_{\mathbb{S}^1} (|\vec{\kappa}|_g^2 + 2\lambda)\,\mathrm{d}s, \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\mathrm{d}s = |\partial_x f|_g\,\mathrm{d}x$ and the geodesic curvature $\vec{\kappa}$ is given as $\vec{\kappa} = \nabla_{\partial_s f}\partial_s f$, where $\partial_s f = \frac{1}{|\partial_x f|_g}\partial_x f$ is the unit velocity vector field along $f$ and $\nabla$ denotes the covariant derivative.

Critical points of the elastic energy are called *elastica* and have been studied for instance in [11]. Here, we let $(M, g)$ be the Poincaré half-plane as in [6], thus $M$ has constant sectional curvature $-1$. Then the elastica satisfy

$$\nabla_{L^2}\mathcal{E}_\lambda(f) = (\nabla_{\partial_s}^\perp)^2\vec{\kappa} + \frac{1}{2}|\vec{\kappa}|_g^2\vec{\kappa} - \lambda\vec{\kappa} - \vec{\kappa} = 0\,, \tag{1.2}$$

where $\nabla_{\partial_s}^\perp$ denotes the projection of the covariant derivative $\nabla_{\partial_s f}$ onto the subspace orthogonal to $\partial_s f$ (see [11],[6, Remark 2.5]). In this work we look for self-similar solutions to the gradient flow associated to the energy $\mathcal{E}_\lambda$. This evolution has been studied in Euclidean space for instance in [17, 9, 8, 13] and on the sphere in [5].

The main theorem from [6] is the following.

**Theorem 1.1** ([6, Theorem 1.1]). *Let $\mathbb{H}^2$ be the hyperbolic half space, $f_0 : \mathbb{S}^1 \to \mathbb{H}^2$ be a given smooth, regular and closed curve, and $\lambda \geq 0$.*

*(i) There exists a smooth global solution $f : \mathbb{S}^1 \times [0,\infty) \to \mathbb{H}^2$ of the initial boundary value problem*

$$\begin{cases} \partial_t f = -\nabla_{L^2}\mathcal{E}_\lambda(f) = -(\nabla_{\partial_s}^\perp)^2\vec{\kappa} - \frac{1}{2}|\vec{\kappa}|_g^2\vec{\kappa} + \vec{\kappa} + \lambda\vec{\kappa}, & \textit{in } \mathbb{S}^1 \times (0,T), \\ f(x,0) = f_0(x), & \textit{for } x \in \mathbb{S}^1. \end{cases} \tag{1.3}$$

*(ii) Moreover, if $\lambda > 0$, as $t_i \to \infty$ there exists real values $p_i \in \mathbb{R}$, $\alpha_i > 0$ such that the curves $\alpha_i(f(t_i, ) - (p_i, 0))$ subconverge, when reparametrised with constant speed, to a critical point of $\mathcal{E}_\lambda$, that is to a solution of (1.2).*

A crucial ingredient in the proof of [6, Theorem 1.1] is the following generalisation of the Theorem of Fenchel.

**Theorem 1.2** (see [16, 14]). *For any smooth, closed curve in hyperbolic space the total absolute curvature $\int |\vec{\kappa}|_g \, ds$ is bounded from below by $2\pi$.*

This was applied in [6] to show that the length of the curve is uniformly bounded from below during the elastic flow. To find an uniform upper bound on the length one penalises the growth of the length with a positive multiplier $\lambda > 0$.
Here, we consider geodesic circles in the hyperbolic plane and see that we can remove the assumption $\lambda > 0$ in Theorem 1.1(ii) in this case for the subconvergence. We show that a convergence result even holds for $\lambda > -\frac{1}{2}$. More precisely, we have the following result.

**Proposition 1.1.** *Let $\lambda > -\frac{1}{2}$ and $f_0$ be a geodesic circle in $\mathbb{H}^2$, i.e. $f_0$ is a regular, smooth parametrisation of $\partial B_r^{\mathbb{H}^2}(y) \subset \mathbb{H}^2$ for some $r > 0$ and $y \in \mathbb{H}^2$.*

*Then there exists a family of circles $f\colon [0,\infty) \times \mathbb{S}^1 \to \mathbb{H}^2$ solving the elastic flow (3.1). Moreover, $f$ converges to the limit circle*

$$f_\infty(x) = \begin{pmatrix} 0 \\ a_\infty \end{pmatrix} + r_\infty \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}, \quad \text{where } \frac{a_\infty}{r_\infty} = \sqrt{2(\lambda+1)}.$$

*In particular, for $\lambda = 0$ we see that the global solution with circular initial value converges to the circle satisfying $\frac{a_\infty}{r_\infty} = \sqrt{2}$.*

Note that the limit elastica $f_\infty$ for $\lambda = 0$ is called *circular free elastica* in [11] and is the global minimum of the elastic energy of closed curves in the hyperbolic plane (see Figure 1). This circle corresponds to the Clifford torus in $\mathbb{R}^3$ (as a surface of revolution, see [10]). This torus is the global minimum of the Willmore energy of closed surfaces of genus 1 ([12]). The proof of Proposition 1.1 is given in Section 3.

**Remark 1.1.** *Note that circles in the hyperbolic plane are uniquely determined up to scaling and translation in the direction of the first coordinate, which is why the limit circle of Proposition 1.1 is uniquely determined up to such isometries.*

*Writing a circle in $\mathbb{H}^2$ as*

$$\begin{pmatrix} 0 \\ a \end{pmatrix} + r \begin{pmatrix} \cos x \\ \sin x \end{pmatrix},$$

*with $a > r > 0$ (see Lemma 2.1), the quotient $\frac{a}{r}$ is the (absolute) curvature of the circle (see (3.5)). Proposition 1.1 is shown by studying the ODE for the curvature $\frac{a}{r}$, where $a$ and $r$ are time-dependent (see Proposition 3.1). Writing $\rho := a/r$ this ODE reads*

$$\frac{d}{dt}\rho = -\rho(\rho^2 - 1)(\frac{1}{2}\rho^2 - \lambda - 1),$$

see (3.4). *In a similar fashion one finds for the elastic flow in* Euclidean space *that circles of radius $r = r(t)$ satisfy*

$$\dot{r} = \frac{1}{r}\left(\frac{1}{2}\frac{1}{r^2} - \lambda\right),$$

*while for circles in the* sphere *we find*

$$\dot{r} = \frac{1-r^2}{r}\left(\frac{1}{2}\left(\frac{1}{r^2} - 1\right) - \lambda + 1\right),$$

*(see [5, Introduction]). Notice that the right hand sides of these three equations have the following structure*

$$\textit{some factor}\ (\frac{1}{2}|\vec{\kappa}|_g^2 - \lambda + \textit{sectional curvature}),$$

*since the sectional curvatures in $\mathbb{H}^2$, the Euclidean space and the sphere are respectively $-1$, $0$ and $1$.*

The article is structured as follows. In the next section we recall some basic facts on the geometry of the hyperbolic plane. In Section 3 Proposition 1.1 is proven. In the last section we compare the elastic flow in $\mathbb{H}^2$ for $\lambda = 0$ and the Willmore flow of surfaces of revolution and show, by a direct computation, that they differ only by the factor $2f_2^4$.

## 2 The geometry of the hyperbolic plane

In this article we consider the Poincaré half-plane model for the hyperbolic space, i.e. the Riemannian manifold $(\mathbb{H}^2, g)$ with

$$\mathbb{H}^2 = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 > 0\} \qquad \text{and} \qquad g_{(y_1,y_2)} = \frac{1}{y_2^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is well known that $(\mathbb{H}^2, g)$ has constant sectional curvature equal to $-1$. The Christoffel symbols of $(\mathbb{H}^2, g)$ are given by the following expressions

$$\Gamma_{11}^1 = \Gamma_{22}^1 = 0, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{y_2}, \quad \Gamma_{11}^2 = \frac{1}{y_2}, \quad \Gamma_{22}^2 = -\frac{1}{y_2} \text{ and } \Gamma_{12}^2 = \Gamma_{21}^2 = 0.$$

Identifying $\partial_{y_1}$ with $(1,0)^t$ and $\partial_{y_2}$ with $(0,1)^t$ one easily verifies the following formula for the covariant derivative in $\mathbb{H}^2$

$$\nabla_{\partial_x f} X = \begin{pmatrix} \partial_x X_1 - \frac{1}{f_2}(X_1 \partial_x f_2 + X_2 \partial_x f_1) \\ \partial_x X_2 + \frac{1}{f_2}(X_1 \partial_x f_1 - X_2 \partial_x f_2) \end{pmatrix} \tag{2.1}$$

for a vector field $X$ along a curve $f = (f_1, f_2)\colon \mathbb{S}^1 \to \mathbb{H}^2$. Since we will consider geodesic circles in $\mathbb{H}^2$ the following remark will be useful.

**Remark 2.1** ([2, Proposition 2.6]). *The geodesic distance between $(x_1, y_1)$, $(x_2, y_2) \in \mathbb{H}^2$ is given by*

$$\mathrm{dist}_{\mathbb{H}^2}((x_1, y_1), (x_2, y_2)) = \operatorname{arccosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2 y_2 y_1}\right).$$

*In particular, if $x_1 = x_2 = x$, then $\mathrm{dist}_{\mathbb{H}^2}((x, y_1), (x, y_2)) = \left|\log \frac{y_2}{y_1}\right|$.*

By this formula it is easy to see that a ball in $\mathbb{H}^2$ coincides with an Euclidean ball and vice versa, as shown in the following lemma. In Figure 1 (an isometry of) the circular limit curve from Proposition 1.1 is sketched.

**Lemma 2.1.** *Let $r, y_1 > 0$ and $x_1 \in \mathbb{R}$. Let $B_r^{\mathbb{H}^2}(x_1, y_1)$ be a ball in the hyperbolic plane defined by $B_r^{\mathbb{H}^2}(x_1, y_1) = \{(x, y) \in \mathbb{H}^2 :\ \mathrm{dist}_{\mathbb{H}^2}((x, y), (x_1, y_1)) < r\}$. Then*

$$B_r^{\mathbb{H}^2}(x_1, y_1) = B_{\sinh(r) y_1}^{\mathbb{R}^2}(x_1, y_1 \cosh(r)),$$

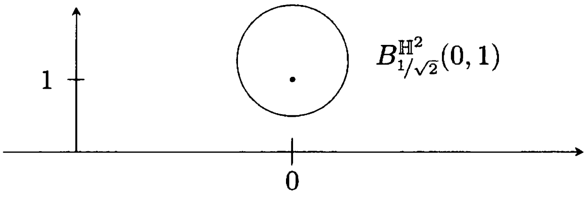


Figure 1: The ball of radius $1/\sqrt{2}$ and midpoint $(0,1)$ in the hyperbolic plane.

*where $B^{\mathbb{R}^2}_\rho(a)$ denotes the ball of radius $\rho$ and center $a$ in the Euclidean plane $\mathbb{R}^2 \supset \mathbb{H}^2$. Conversely for $y_1 > r > 0$ we find*

$$B^{\mathbb{R}^2}_r(x_1, y_1) = B^{\mathbb{H}^2}_{\tanh^{-1}(r/y_1)}(x_1, \sqrt{y_1^2 - r^2}).$$

*Proof.* Let $(x,y) \in \mathbb{H}^2$. Then, by Remark 2.1 above and the monotonicity of cosh one sees that $(x,y) \in B^{\mathbb{H}^2}_r(x_1, y_1)$ if and only if

$$\cosh(r) > 1 + \frac{(x-x_1)^2 + (y-y_1)^2}{2yy_1}.$$

By rearrenging the terms we see that this is equivalent to

$$2yy_1 \cosh(r) > (x-x_1)^2 + y^2 + y_1^2(\cosh^2(r) - \sinh^2(r)),$$

and hence to

$$\sinh^2(r) y_1^2 > (x-x_1)^2 + (y - \cosh(r)y_1)^2,$$

that gives $(x,y) \in B^{\mathbb{R}^2}_{\sinh(r)y_1}(x_1, y_1\cosh(r))$. The second part of the lemma follows using the first part together with the fact that if $B^{\mathbb{R}^2}_r(x_1,y_1) = B^{\mathbb{H}^2}_R(x_1,\eta)$, then $r = \sinh(R)\eta$ and $\eta\cosh(R) = y_1$, whence

$$\tanh(R) = \frac{r}{y_1} \in (0,1) \text{ and hence } R = \tanh^{-1}(r/y_1).$$

Similarly we find

$$\eta = \frac{y_1}{\cosh(R)} = y_1\sqrt{1-(r/y_1)^2} = \sqrt{y_1^2 - r^2}.$$

□

# 3 The elastic flow of circles in the hyperbolic plane

In this section, we study the evolution of circles under the elastic flow in the hyperbolic plane. Since we will consider a family of parametrised curves $f = f(x,t)$ whose time-derivative $\partial_t f$ is not necessarily normal to $\partial_x f$ we will work with the following flow instead of (1.3). We consider

$$\begin{cases} (\partial_t f)^\perp = -\nabla_{L^2}\mathcal{E}_\lambda(f), & \text{in } \mathbb{S}^1 \times (0,T), \\ f(x,0) = f_0(x), & \text{for } x \in \mathbb{S}^1, \end{cases} \tag{3.1}$$

where $^{\perp}$ denotes the normal component Note that one can always transform a solution to (3 1) to a solution of (1 3) using the flow of vector fields on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, i e via a reparametrisation (see e g [6 p 12])

The following proposition gives a necessary and suffient condition under which a family of circles is a solution to the elastic flow (3 1) Here and in the following we will always assume that $f_0$ parametrises a geodesic circle in $\mathbb{H}^2$ i e by Lemma 2 1 we may assume that

$$f_0(x) = \begin{pmatrix} 0 \\ a_0 \end{pmatrix} + r_0 \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix} \qquad \text{for some } a_0 > r_0 > 0 \tag{3 2}$$

**Proposition 3.1** *Let $\lambda \in \mathbb{R}$ and $0 < T \leq \infty$*

1. *Let $a(t)$ $[0,T) \to \mathbb{R}$, $r(t)$ $[0,T) \to \mathbb{R}$ be smooth functions satisfying $a(0) = a_0$ $r(0) = r_0$ and $a(t) > r(t) > 0$ for all $0 \leq t < T$ If the following family of curves $f$ $\mathbb{S}^1 \times [0,T) \to \mathbb{H}^2$*

$$f(x,t) = \begin{pmatrix} 0 \\ a(t) \end{pmatrix} + r(t) \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} \tag{3 3}$$

*is a solution to (3 1) with initial value $f_0$ as in (3 2) then $\rho(t) = \dfrac{a(t)}{r(t)}$ $[0,T) \to (0,\infty)$ is a solution to the ODE*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) & = H_\lambda(\rho) \\ \rho(0) & = \rho_0 \end{cases} \tag{3 4}$$

*where $H_\lambda(\rho) = -\dfrac{1}{2}\rho(\rho^2 - 1)(\rho^2 - 2(\lambda+1))$ and $\rho_0 = \dfrac{a_0}{r_0} > 1$*

2. *Moreover if $\rho$ $[0,T) \to (0,\infty)$ is a smooth solution to (3 4) with $\rho_0 > 1$, then writing $\rho_0 = a_0/r_0$ with $a_0 > r_0 > 0$ there exist unique smooth functions $a, r$ $[0,T) \to (0,\infty)$ satisfying $a(0) = a_0$ $r(0) = r_0$ and $a(t) > r(t) > 0$ for all $0 \leq t < T$, such that $f$ (given by (3 3)) is a solution to (3 1) with initial value $f_0$ from (3 2)*
3. *For all $\lambda \in \mathbb{R}$ and $\rho_0 > 1$ there exists a unique smooth solution $\rho$ $[0,\infty) \to (1,\infty)$ to (3 4)*
4. *If $\lambda > -\frac{1}{2}$ then the smooth solution $\rho$ $[0,\infty) \to (1,\infty)$ converges monotonically to $\mu_\lambda = \sqrt{2(\lambda+1)}$*

The last part of the statement yields the existence of circular self similar solutions to the elastic flow, and these circles converge monotonically to the circle with constant curvature $\mu_\lambda$ (see below)

**Corollary 3.1** *For any $\lambda \in \mathbb{R}$ and any initial value $f_0$ as in (3 2) there exists a family (3 3) of circles that satisfies the elastic flow (3 1) Moreover for $\lambda > -\frac{1}{2}$ this family converges to the circle which is given by*

$$\frac{a}{r} = \mu_\lambda = \sqrt{2(\lambda+1)}$$

*If $\lambda \leq -\frac{1}{2}$ then the circular self-similar solution still exist but the circles expand to infinity as $t \to \infty$*

Note that this corollary and Lemma 2 1 immediately imply Proposition 1 1

*Proof.* The first part of this corollary follows from the third and second claim of Proposition 3.1. The statement for $\lambda > -\frac{1}{2}$ is also given in Proposition 3.1 whereas for $\lambda \leq -\frac{1}{2}$ Lemma A.1 gives a solution $\rho$ of (3.4) that converges asymptotically from above to $\rho = 1$. For the associated circular solution $f$ this means that the quotient $a/r$ converges to 1 and hence the circular solutions converge to a circle touching tangentially the line $\{(x,0) : x \in \mathbb{R}\}$ (see Lemma 2.1), i.e. expanding to infinity. □

The proof of Proposition 3.1 is given after the next lemmata. Note that the condition $a > r > 0$ ensures that $f$ is a well defined circle in $\mathbb{H}^2$.

**Lemma 3.1.** *For the circular curve $f$ from* (3.3) *with $a > r > 0$ it holds*

$$|\vec{\kappa}(x)|_g = \frac{a}{r}, \tag{3.5}$$

$$-\nabla_{L^2}\mathcal{E}_\lambda(f) = -\left(\lambda + 1 - \frac{1}{2}\frac{a^2}{r^2}\right)(a + r\sin x)\,\frac{a}{r}\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}. \tag{3.6}$$

*In particular, $f$ is an elastic curve if and only if $\lambda > -\frac{1}{2}$ and $\frac{a}{r} = \sqrt{2(\lambda+1)}$.*

*Proof.* We have $\partial_x f = r\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}$, whence

$$|\partial_x f|_g = \frac{1}{f_2}|\partial_x f|_{euc} = \frac{r}{a + r\sin x}$$

(that is well defined since $a > r > 0$) and thus for the arc length derivative $\partial_s = \frac{1}{|\partial_x f|_g}\partial_x$ we find

$$\partial_s = \frac{1}{r}(a + r\sin x)\partial_x = \frac{f_2}{r}\partial_x.$$

The tangential vector is

$$\partial_s f = (a + r\sin x)\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}$$

and from (2.1) we find

$$\nabla_{\partial_s f} X = \partial_s X + \begin{pmatrix}-X_1\cos x + X_2\sin x\\ -X_1\sin x - X_2\cos x\end{pmatrix} = \partial_s X - \begin{pmatrix}X_1\cos x - X_2\sin x\\ X_1\sin x + X_2\cos x\end{pmatrix} \tag{3.7}$$

for vector fields $X = (X_1, X_2)$ along $f$. Using this formula we compute the curvature vector field

$$\begin{aligned}
\vec{\kappa}(x) &= \nabla_{\partial_s f}\partial_s f\\
&= \frac{a + r\sin x}{r}\partial_x\left[(a + r\sin x)\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}\right] + (a + r\sin x)\begin{pmatrix}\sin x\cos x + \cos x\sin x\\ \sin^2 x - \cos^2 x\end{pmatrix}\\
&= (a + r\sin x)\cos x\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix} + \frac{(a + r\sin x)^2}{r}\begin{pmatrix}-\cos x\\ -\sin x\end{pmatrix}\\
&\quad + (a + r\sin x)\begin{pmatrix}2\sin x\cos x\\ \sin^2 x - \cos^2 x\end{pmatrix}\\
&= (a + r\sin x)\begin{pmatrix}\sin x\cos x\\ \sin^2 x\end{pmatrix} - \frac{(a + r\sin x)^2}{r}\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}\\
&= -(a + r\sin x)\,\frac{a}{r}\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}.
\end{aligned}$$

Thus,

$$|\vec{\kappa}|_g = \frac{1}{a + r\sin x} \cdot \frac{a}{r} \cdot (a + r\sin x) = \frac{a}{r},$$

that is (3.5). Using (3.7) again we find for $\vec{\kappa} = (\vec{\kappa}_1, \vec{\kappa}_2)$ that

$$\begin{aligned}
\nabla_{\partial_s f}\vec{\kappa} &= \partial_s \vec{\kappa} - \begin{pmatrix} \vec{\kappa}_1 \cos x - \vec{\kappa}_2 \sin x \\ \vec{\kappa}_1 \sin x + \vec{\kappa}_2 \cos x \end{pmatrix} \\
&= -\frac{1}{r}(a + r\sin x)\,\partial_x \left[ (a + r\sin x)\,\frac{a}{r} \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} \right] + (a + r\sin x)\,\frac{a}{r} \begin{pmatrix} \cos^2 x - \sin^2 x \\ 2\cos x \sin x \end{pmatrix} \\
&= -\frac{1}{r}(a + r\sin x) \left[ a\cos x \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} + (a + r\sin x)\,\frac{a}{r} \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} \right] \\
&\quad + (a + r\sin x)\,\frac{a}{r} \begin{pmatrix} \cos^2 x - \sin^2 x \\ 2\cos x \sin x \end{pmatrix} \\
&= -(a + r\sin x)\,\frac{a}{r} \begin{pmatrix} \cos^2 x - (\frac{a}{r}\sin x + \sin^2 x) - (\cos^2 x - \sin^2 x) \\ \cos x \sin x + (\frac{a}{r}\cos x + \sin x \cos x) - 2\sin x \cos x \end{pmatrix} \\
&= -(a + r\sin x)\,\frac{a}{r} \begin{pmatrix} -\frac{a}{r}\sin x \\ \frac{a}{r}\cos x \end{pmatrix} = (a + r\sin x)\,\frac{a^2}{r^2} \begin{pmatrix} \sin x \\ -\cos x \end{pmatrix}
\end{aligned}$$

is tangential to $f$, i.e. a multiple of $\partial_x f = r \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}$. Thus

$$(\nabla_{\partial_s f})^{\perp}\vec{\kappa} = 0$$

and hence

$$((\nabla_{\partial_s f})^{\perp})^2 \vec{\kappa} = 0.$$

Summarising, we find

$$\begin{aligned}
-\nabla_{L^2}\mathcal{E}_\lambda(f) &= -(\nabla^{\perp}_{\partial_s f})^2 \vec{\kappa} - \frac{1}{2}|\vec{\kappa}|_g^2 \vec{\kappa} + \vec{\kappa} + \lambda \vec{\kappa} \\
&= \left( -\frac{1}{2}|\vec{\kappa}|_g^2 + \lambda + 1 \right) \vec{\kappa} \\
&= -\left( \lambda + 1 - \frac{1}{2}\frac{a^2}{r^2} \right) (a + r\sin x)\,\frac{a}{r} \begin{pmatrix} \cos x \\ \sin x \end{pmatrix},
\end{aligned}$$

that is (3.6).

Since $a > r > 0$ then $\nabla_{L^2}\mathcal{E}_\lambda(f) = 0$ only when the quotient $a/r$ is constant and equal to $\sqrt{2(\lambda+1)}$. Since the quotient has to be bigger than one, this can only be the case when $\lambda$ is strictly bigger than $-\frac{1}{2}$. This yields the claim. □

**Lemma 3.2.** *Let $\lambda \in \mathbb{R}$, $f_0$ given by* (3.2) *and $f$ be as in* (3.3). *Then $f \colon \mathbb{S}^1 \times [0,T) \to \mathbb{H}^2$ is a solution to the elastic flow* (3.1) *if and only if the functions $a, r \colon [0,T) \to \mathbb{R}$ satisfy*

$$\begin{cases} \dfrac{\dot{a}}{r}\sin x + \dfrac{\dot{r}}{r} = \dfrac{a}{r}\left(\dfrac{a}{r} + \sin x\right)\left(\dfrac{1}{2}\left(\dfrac{a}{r}\right)^2 - (\lambda+1)\right), & t > 0, x \in \mathbb{R}, \\ \dfrac{a(0)}{r(0)} = \dfrac{a_0}{r_0}, & \end{cases} \tag{3.8}$$

*where the dot represents the time derivative of $a$ and $r$.*

*Proof.* For $(\partial_t f)^\perp$ we calculate

$$\partial_t f = \begin{pmatrix} \dot r \cos x \\ \dot a + \dot r \sin x \end{pmatrix},$$

and thus, since $N := \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} \perp \partial_x f$ we find $\partial_t f = \begin{pmatrix} 0 \\ \dot a \end{pmatrix} + \dot r N$ and hence, using

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}^\perp = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, N \rangle_{f(x)} \cdot \frac{N}{|N|^2_{f(x)}} = \frac{(a + r\sin x)^2}{(a + r \sin x)^2} (\sin x) N = (\sin x) N$$

we find

$$(\partial_t f)^\perp = (\dot a \ \sin x + \dot r) N.$$

Thus, together with (3.6) this shows that $f$ from (3.3) is a solution to (3.1) if and only if

$$(\dot a \ \sin x + \dot r) N = -\left(\lambda + 1 - \frac{1}{2}\frac{a^2}{r^2}\right)(a + r \sin x)\frac{a}{r} N,$$

i.e.

$$\frac{\dot a}{r} \sin x + \frac{\dot r}{r} = -\frac{a}{r}\left(\frac{a}{r} + \sin x\right)\left(\lambda + 1 - \frac{1}{2}\frac{a^2}{r^2}\right).$$

Taking the initial values into account this yields the system (3.8). □

*Proof of Proposition 3.1.* We use here several times Lemma 3.2.

1. Let $f$ from (3.3) be a solution to the elastic flow with initial value $f_0$ as in (3.2). Then, by the previous lemma, the functions $0 < r < a$ are solutions to (3.8). Differentiating (3.8) with respect to $x$ yields

$$\frac{\dot a}{r} \cos x = \frac{a}{r}\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda + 1)\right) \cos x$$

for all $x$, in particular for $x = 0$ we find

$$\dot a = a\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda + 1)\right). \tag{3.9}$$

Plugging this equation into (3.8) yields

$$\begin{aligned} \dot r &= a(\frac{a}{r} + \sin x)\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda+1)\right) - a\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda+1)\right)\sin x \\ &= \frac{a^2}{r}\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda+1)\right). \end{aligned}$$

Thus the function $\rho := \frac{a}{r}$ satisfies $\rho(0) = \frac{a_0}{r_0}$ and

$$\begin{aligned} \dot\rho &= \frac{\dot a}{r} - \frac{a \dot r}{r^2} \\ &= \frac{a}{r}\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda+1)\right) - \frac{a^3}{r^3}\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda+1)\right) \\ &= -\frac{1}{2}\rho\left(\rho^2 - 2(\lambda+1)\right)\left(\rho^2 - 1\right) = H_\lambda(\rho). \end{aligned}$$

2. Conversely, let $\rho$ be the solution to $\dot\rho = H_\lambda(\rho)$ and $\rho(0) = \rho_0 = \dfrac{a_0}{r_0}$ for some $a_0 > r_0 > 0$. Then $\rho > 1$ by Lemma A.1. Moreover, let $a$ be the solution to the linear ODE
$$\begin{cases} \dot a = a \cdot \frac{1}{2}\left(\rho^2 - 2(\lambda+1)\right), & t > 0 \\ a(0) = a_0 \end{cases} \tag{3.10}$$
and define $r$ by $r(t) := \frac{a(t)}{\rho(t)}$. Then $a > 0$ since $a_0 > 0$, thus $r > 0$ and hence $\rho(t) = \frac{a(t)}{r(t)}$ for all $t \in [0, T)$. Moreover, $a$ satisfies (3.10), whence
$$\begin{aligned} \frac{\dot r}{r} &= \frac{\mathrm{d}}{\mathrm{d}t}\log r \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\log a - \frac{\mathrm{d}}{\mathrm{d}t}\log\rho \\ &= \frac{\dot a}{a} - \frac{\dot\rho}{\rho} \\ &= \frac{1}{2}\left(\left(\frac{a}{r}\right)^2 - 2(\lambda+1)\right) - \frac{r}{a}H_\lambda\left(\frac{a}{r}\right) \\ &= \frac{1}{2}\left(\left(\frac{a}{r}\right)^2 - 2(\lambda+1)\right) + \frac{1}{2}\left(\left(\frac{a}{r}\right)^2 - 2(\lambda+1)\right)\left(\left(\frac{a}{r}\right)^2 - 1\right) \\ &= \frac{1}{2}\left(\left(\frac{a}{r}\right)^2 - 2(\lambda+1)\right)\left(\frac{a}{r}\right)^2 \end{aligned}$$
which shows that
$$\begin{aligned} \frac{\dot a}{r}\sin x + \frac{\dot r}{r} &= \frac{a}{r}\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda+1)\right)\sin x + \frac{1}{2}\left(\left(\frac{a}{r}\right)^2 - 2(\lambda+1)\right)\left(\frac{a}{r}\right)^2 \\ &= \frac{a}{r}\left(\frac{a}{r} + \sin x\right)\left(\frac{1}{2}\left(\frac{a}{r}\right)^2 - (\lambda+1)\right), \end{aligned}$$
i.e. (3.8) holds. Lemma 3.2 implies the rest of the claim.
3. The last two assertions follow from Lemma A.1. □

# 4 Relationship to the Willmore flow of surfaces of revolution

It is well known (see e.g. [10, 11]) that there is a very interesting relation between the elastic energy $\mathcal{E}$ (that is $\mathcal{E}_\lambda$ with $\lambda = 0$) of curves in the hyperbolic half plane and the Willmore energy of surfaces of revolution. As a consequence the two gradient flows are also closely related as observed in [6]. Here, we give the detailed computations to see that the two differ only by a factor when we take into account that the Willmore flow keeps the rotationally symmetry. We shortly fix some notation to state the result.

Let $f : \mathbb{S}^1 \to \mathbb{R}^2_+ := \{(x,z)^t : z > 0\}$ be a closed curve parametrised by Euclidean arc-length, i.e. $(f_1')^2 + (f_2')^2 \equiv 1$. By rotating the curve around the $x$-axis we obtain a surface of revolution in $\mathbb{R}^3$
$$h_f : \mathbb{S}^1 \times [0, 2\pi] \ni (x, \varphi) \mapsto (f_1(x), f_2(x)\cos(\varphi), f_2(x)\sin(\varphi))^t \in \mathbb{R}^3 .$$

Its Willmore energy is given by

$$W(h_f) = \int H^2 \, \mathrm{d}S \, ,$$

with $H$ the mean curvature. The same curve $f$ can be considered as a curve in $\mathbb{H}^2$. In order to stress this fact we denote it as $f_{\mathbb{H}^2}$. Similarly, when needed, we write $f_{\mathbb{R}^2}$ to indicate that $f$ is now considered as a curve in $\mathbb{R}^2$.

**Theorem 4.1.** *One has*

1. $\mathcal{E}(f_{\mathbb{H}^2}) = \frac{2}{\pi} W(h_f)$;
2. *the $L^2$-gradient of $\mathcal{E}$ satisfy*

$$\nabla_{L^2}\mathcal{E}(f_{\mathbb{H}^2}) = -2f_2^4(\Delta H + 2H(H^2 - K))\vec{n}_{\mathbb{R}^2}$$

*where we recall that*

$$\nabla_{L^2} W(h_f) = (\Delta H + 2H(H^2 - K))\vec{N}$$

*with* $\vec{n}_{\mathbb{R}^2} = (f_2', -f_1')$ *and* $\vec{N} = (f_2', -f_1' \cos(\varphi), -f_1' \sin(\varphi))$;

3. *the elastic flow can be written as*

$$\begin{aligned}(\partial_t f_{\mathbb{H}^2})^{\perp} = -\nabla_{L^2}\mathcal{E}(f_{\mathbb{H}^2}) &= 2f_2^3(\Delta H + 2H(H^2 - K))\vec{n}_{\mathbb{H}^2} \\ &= -2f_2^4(\Delta H + 2H(H^2 - K))\vec{n}_{\mathbb{R}^2} \, .\end{aligned}$$

The last part of the statement gives the relation between the elastic flow and the Willmore flow of surfaces of revolution. Of course we cannot compare directly these two flows since one lives in $\mathbb{H}^2$ while the other takes place in $\mathbb{R}^3$. On the other hand, as we will explain below, when we start from a surface of revolution this symmetry is preserved by the Willmore flow and hence we can describe the Willmore flow simply considering the evolution of the generating curve in $\mathbb{R}^2$. This is the flow that differ by a factor $2f_2^4$ from the elastic flow in $\mathbb{H}^2$.

*Proof.* **Relation between the energies** This is well known and can be found for instance in [10, 11, 4, 3]. We repeat the arguments. The first fundamental form of the surface of revolution $h_f$ is given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & f_2^2 \end{pmatrix},$$

the induced area element is $f_2 \, \mathrm{d}x \, \mathrm{d}\varphi$, the normal vector is

$$\vec{N} = (f_2', -f_1' \cos(\varphi), -f_1' \sin(\varphi))^t,$$

the principal curvatures are $\lambda_1 = f_1'' f_2' - f_2'' f_1'$ and $\lambda_2 = \frac{f_1'}{f_2}$ and the mean and Gauss curvatures are respectively

$$H = \frac{1}{2}\left(f_1'' f_2' - f_2'' f_1' + \frac{f_1'}{f_2}\right) \text{ and } K = (f_1'' f_2' - f_2'' f_1')\,\frac{f_1'}{f_2} \, .$$

(see [7, page 161]) and the Willmore energy of this surface of revolution is given by

$$W(h_f) = \int H^2 \, \mathrm{d}S = \frac{\pi}{2} \int_{\mathbb{S}^1} \left(f_1'' f_2' - f_2'' f_1' + \frac{f_1'}{f_2}\right)^2 f_2 \, \mathrm{d}x \, .$$

Now we consider the same curve as a curve $f : \mathbb{S}^1 \to \mathbb{H}^2$, $f = f_{\mathbb{H}^2}$. Since this curve is parametrised by Euclidean arc-length we find

$$\partial_s f = \frac{1}{|\partial_x f|_g} \partial_x f = f_2 \partial_x f \quad \text{and} \quad \partial_s = f_2 \partial_x \,.$$

For the 'hyperbolic' curvature from (2.1) we find

$$\vec{\kappa} = \nabla_s \partial_s f = \begin{pmatrix} \partial_s^2 f_1 - \frac{2}{f_2} \partial_s f_1 \partial_s f_2 \\ \partial_s^2 f_2 + \frac{1}{f_2}((\partial_s f_1)^2 - (\partial_s f_2)^2) \end{pmatrix} = \begin{pmatrix} f_2^2 \partial_x^2 f_1 - f_2 \partial_x f_1 \partial_x f_2 \\ f_2^2 \partial_x^2 f_2 + f_2 (\partial_x f_1)^2 \end{pmatrix} \tag{4.1}$$

and

$$\begin{aligned} |\vec{\kappa}|_g^2 &= \frac{1}{f_2^2} \left[ (f_2^2 f_1'' - f_2 f_1' f_2')^2 + (f_2^2 f_2'' + f_2 (f_1')^2)^2 \right] \\ &= f_2^2 \Big[ (f_1'' - \frac{f_1'}{f_2} f_2')^2 + (f_2'' + \frac{(f_1')^2}{f_2})^2 \Big] \\ &= f_2^2 \Big[ (f_1'')^2 + \frac{(f_1')^2 (f_2')^2}{f_2^2} - 2 f_1'' \frac{f_1'}{f_2} f_2' + (f_2'')^2 + \frac{(f_1')^4}{(f_2)^2} + 2 f_2'' \frac{(f_1')^2}{f_2} \Big] \end{aligned}$$

which, using that $f$ satisfies $(f_1')^2 + (f_2')^2 \equiv 1$ and hence

$$f_1'(x) f_1''(x) + f_2'(x) f_2''(x) = 0 \,, \tag{4.2}$$

can be rewritten as

$$\begin{aligned} |\vec{\kappa}|_g^2 &= f_2^2 \Big[ (f_1'' f_2' - f_2'' f_1')^2 + \frac{(f_1')^2}{f_2^2} - 2 f_1'' \frac{f_1'}{f_2} f_2' + 2 f_2'' \frac{(f_1')^2}{f_2} \Big] \\ &= f_2^2 \Big[ (f_1'' f_2' - f_2'' f_1' + \frac{f_1'}{f_2})^2 - 4 f_1'' \frac{f_1'}{f_2} f_2' + 4 f_2'' \frac{(f_1')^2}{f_2} \Big] \\ &= f_2^2 \Big[ (f_1'' f_2' - f_2'' f_1' + \frac{f_1'}{f_2})^2 + \frac{4 f_2''}{f_2} \Big] . \end{aligned}$$

This formula gives directly the relation between the elastic energy in the hyperbolic plane and the Willmore energy of surfaces of revolution, indeed since the curve is closed

$$\begin{aligned} \mathcal{E}(f_{\mathbb{H}^2}) &= \int_{\mathbb{S}^1} |\vec{\kappa}|_g^2 \frac{1}{f_2} \, \mathrm{d}x = \int_{\mathbb{S}^1} \Big[ (f_1'' f_2' - f_2'' f_1' + \frac{f_1'}{f_2})^2 + \frac{4 f_2''}{f_2} \Big] f_2 \, \mathrm{d}x \\ &= \frac{2}{\pi} W(h_f) + 4 \int_{\mathbb{S}^1} f_2''(x) \, \mathrm{d}x = \frac{2}{\pi} W(h_f) . \end{aligned}$$

Notice that here $\mathcal{E}(f)$ denotes $\mathcal{E}_\lambda(f)$ with $\lambda = 0$.

**The Willmore flow starting from a surface of revolution** The Willmore flow of $h_f$ is given by

$$(\partial_t h_f)^\perp = -\nabla_{L^2} W(h_f) = -(\Delta H + 2H(H^2 - K)) \vec{N} \,, \tag{4.3}$$

with $\vec{N}$ the normal to the surface as fixed before and $\Delta$ the Laplace-Beltrami operator. Blatt in [1] proved that the Willmore flow, as long as it exists, keeps its rotational symmetry. Hence we can restrict the flow directly on the $\{(x,z) : z > 0\}$ plane (or take $\varphi = 0$ in the rotation) and obtain that the right hand side of (4.3) so restricted is given by

$$-(\Delta H + 2H(H^2 - K))(f_2', -f_1')^t \,.$$

Hence, by taking into account this rotational invariance, the Willmore flow corresponds to the following flow equation for the evolving curve in $\mathbb{R}^2$

$$(\partial_t f_{\mathbb{R}^2})^\perp = -(\Delta H + 2H(H^2 - K))\vec{n}_{\mathbb{R}^2}\,,$$

with $\vec{n}_{\mathbb{R}^2} = (f_2', -f_1')^t$.

We compute now $\Delta H + 2H(H^2 - K)$. We do the computations locally where $f_2' \neq 0$, so that using again (4.2) in the form of $f_2'' = -\frac{f_1' f_1''}{f_2'}$ we have

$$H = \frac{1}{2}\left(\frac{f_1''}{f_2'} + \frac{f_1'}{f_2}\right) \text{ and } K = \frac{f_1'' f_1'}{f_2 f_2'}\,.$$

Note that a similar formula holds where $f_1' \neq 0$, see [4]. By a direct computation (using (4.2) several times)

$$\begin{aligned}
2\Delta H &= 2\frac{1}{f_2}\partial_x(f_2\partial_x H)\\
&= \partial_x\left(\frac{f_1^{(3)}}{f_2'} - \frac{f_1'' f_2''}{(f_2')^2} + \frac{f_1''}{f_2} - \frac{f_1' f_2'}{f_2^2}\right) + \frac{f_2'}{f_2}\left(\frac{f_1^{(3)}}{f_2'} - \frac{f_1'' f_2''}{(f_2')^2} + \frac{f_1''}{f_2} - \frac{f_1' f_2'}{f_2^2}\right)\\
&= \partial_x\left(\frac{f_1^{(3)}}{f_2'} + \frac{(f_1'')^2 f_1'}{(f_2')^3} + \frac{f_1''}{f_2} - \frac{f_1' f_2'}{f_2^2}\right) + \frac{f_2'}{f_2}\left(\frac{f_1^{(3)}}{f_2'} + \frac{(f_1'')^2 f_1'}{(f_2')^3} + \frac{f_1''}{f_2} - \frac{f_1' f_2'}{f_2^2}\right)\\
&= \frac{f_1^{(4)}}{f_2'} - \frac{f_1^{(3)}}{(f_2')^2} f_2'' + 2\frac{f_1^{(3)} f_1'' f_1'}{(f_2')^3} + \frac{(f_1'')^3}{(f_2')^3} - 3\frac{(f_1'')^2 f_1' f_2''}{(f_2')^4}\\
&\quad + \frac{f_1^{(3)}}{f_2} - \frac{f_1'' f_2'}{f_2^2} - \frac{f_1'' f_2'}{f_2^2} - \frac{f_1' f_2''}{f_2^2} + 2\frac{f_1'(f_2')^2}{f_2^3} + \frac{f_1^{(3)}}{f_2} + \frac{(f_1'')^2 f_1'}{f_2 (f_2')^2} + \frac{f_1'' f_2'}{f_2^2} - \frac{f_1'(f_2')^2}{f_2^3}\\
&= \frac{f_1^{(4)}}{f_2'} + 3\frac{f_1^{(3)} f_1'' f_1'}{(f_2')^3} + 2\frac{f_1^{(3)}}{f_2}\\
&\quad + \frac{(f_1'')^3}{(f_2')^3} + 3\frac{(f_1'')^3 (f_1')^2}{(f_2')^5} - \frac{f_1'' f_2'}{f_2^2} + \frac{(f_1')^2 f_1''}{f_2' f_2^2} + \frac{f_1'(f_2')^2}{f_2^3} + \frac{(f_1'')^2 f_1'}{f_2 (f_2')^2}
\end{aligned}$$

whereas

$$\begin{aligned}
2H(H^2 - K) &= \Big(\frac{f_1''}{f_2'} + \frac{f_1'}{f_2}\Big)\Big(\frac{1}{4}\Big(\frac{f_1''}{f_2'} + \frac{f_1'}{f_2}\Big)^2 - \frac{f_1'' f_1'}{f_2 f_2'}\Big)\\
&= \frac{1}{4}\Big(\frac{f_1''}{f_2'} + \frac{f_1'}{f_2}\Big)\Big(\frac{f_1''}{f_2'} - \frac{f_1'}{f_2}\Big)^2 .
\end{aligned} \tag{4.4}$$

**The elastic flow** We consider now the gradient flow for the elastic energy in the hyperbolic half-plane

$$(\partial_t f_{\mathbb{H}^2})^\perp = -(\nabla_s^\perp)^2\vec{\kappa} - \frac{1}{2}|\vec{\kappa}|_g^2\vec{\kappa} + \vec{\kappa}\,.$$

Working again locally where $f_2' \neq 0$ using (4.2) we can rewrite the curvature (see (4.1)) as

$$\vec{\kappa} = \Big(-\frac{f_2 f_1''}{f_2'} + f_1'\Big)\vec{n}_{\mathbb{H}^2}\,,$$

with $\vec{n}_{\mathbb{H}^2}$ the normal vector in the hyperbolic half plane given by

$$\vec{n}_{\mathbb{H}^2} = f_2(-\partial_x f_2, \partial_x f_1) = -f_2\vec{n}_{\mathbb{R}^2}\,.$$

Hence it has the same direction as the normal vector to the curve $f_{\mathbb{R}^2}$ in the Euclidean plane but it differs by a factor of $-f_2$.

Since $\nabla_s \vec{n}_{\mathbb{H}^2}$ is tangential, we directly obtain that

$$\begin{aligned}\nabla_s^{\perp}\vec{\kappa} &= f_2\partial_x\Big(-\frac{f_2 f_1''}{f_2'}+f_1'\Big)\vec{n}_{\mathbb{H}^2}\\ &= f_2\Big(-f_1''-\frac{f_2 f_1^{(3)}}{f_2'}+\frac{f_2 f_1''}{(f_2')^2}f_2''+f_1''\Big)\vec{n}_{\mathbb{H}^2}\\ &= \Big(-\frac{f_2^2 f_1^{(3)}}{f_2'}-\frac{f_2^2 f_1'(f_1'')^2}{(f_2')^3}\Big)\vec{n}_{\mathbb{H}^2}\end{aligned}$$

and similarly

$$\begin{aligned}(\nabla_s^{\perp})^2\vec{\kappa} &= f_2\partial_x\Big(-\frac{f_2^2 f_1^{(3)}}{f_2'}-\frac{f_2^2 f_1'(f_1'')^2}{(f_2')^3}\Big)\vec{n}_{\mathbb{H}^2}\\ &= f_2\Big(-\frac{f_2^2 f_1^{(4)}}{f_2'}-2f_2 f_1^{(3)}+\frac{f_2^2 f_1^{(3)}}{(f_2')^2}f_2''-2\frac{f_2 f_1'(f_1'')^2}{(f_2')^2}-\frac{f_2^2(f_1'')^3}{(f_2')^3}\\ &\quad -2\frac{f_2^2 f_1' f_1'' f_1^{(3)}}{(f_2')^3}+3\frac{f_2^2 f_1'(f_1'')^2}{(f_2')^4}f_2''\Big)\vec{n}_{\mathbb{H}^2}\\ &= f_2\Big(-\frac{f_2^2 f_1^{(4)}}{f_2'}-2f_2 f_1^{(3)}-3\frac{f_2^2 f_1^{(3)}}{(f_2')^3}f_1''f_1'-2\frac{f_2 f_1'(f_1'')^2}{(f_2')^2}-\frac{f_2^2(f_1'')^3}{(f_2')^3}\\ &\quad -3\frac{f_2^2(f_1')^2(f_1'')^3}{(f_2')^5}\Big)\vec{n}_{\mathbb{H}^2}\end{aligned}$$

using (4.4), whereas

$$\begin{aligned}\frac{1}{2}|\vec{\kappa}|_g^2\vec{\kappa}-\vec{\kappa} &= \frac{1}{2}\Big(-\frac{f_2 f_1''}{f_2'}+f_1'\Big)^3\vec{n}_{\mathbb{H}^2}-\Big(-\frac{f_2 f_1''}{f_2'}+f_1'\Big)\vec{n}_{\mathbb{H}^2}\\ &= -\frac{1}{2}f_2^3\Big(\frac{f_1''}{f_2'}-\frac{f_1'}{f_2}\Big)^3\vec{n}_{\mathbb{H}^2}-\Big(-\frac{f_2 f_1''}{f_2'}+f_1'\Big)\vec{n}_{\mathbb{H}^2}\\ &= -4f_2^3 H(H^2-K)+f_2^3\Big(\frac{f_1''}{f_2'}-\frac{f_1'}{f_2}\Big)^2\frac{f_1'}{f_2}\vec{n}_{\mathbb{H}^2}+\Big(\frac{f_2 f_1''}{f_2'}-f_1'\Big)\vec{n}_{\mathbb{H}^2}\,.\end{aligned}$$

One sees that the first three terms in $(\nabla_s^{\perp})^2\vec{\kappa}$ differ only by a factor from the first three terms in $2\Delta H$.

**Relation between the $L^2$-gradients and the gradient flows** Comparing the expression we see that

$$\begin{aligned}-(\partial_t f_{\mathbb{H}^2})^{\perp} &= \nabla_{L^2}\mathcal{E}(f_{\mathbb{H}^2}) = (\nabla_s^{\perp})^2\vec{\kappa}+\frac{1}{2}|\vec{\kappa}|_g^2\vec{\kappa}-\vec{\kappa}\\ &= -f_2^3\Big(\frac{f_1^{(4)}}{f_2'}+2\frac{f_1^{(3)}}{f_2}+3\frac{f_1^{(3)}}{(f_2')^3}f_1''f_1'+2\frac{f_1'(f_1'')^2}{f_2(f_2')^2}+\frac{(f_1'')^3}{(f_2')^3}+3\frac{(f_1')^2(f_1'')^3}{(f_2')^5}\\ &\quad +4H(H^2-K)-\Big(\frac{f_1''}{f_2'}-\frac{f_1'}{f_2}\Big)^2\frac{f_1'}{f_2}-\frac{f_1''}{f_2'f_2^2}+\frac{f_1'}{f_2^3}\Big)\vec{n}_{\mathbb{H}^2}\\ &= -f_2^3\Big(2\Delta H-\frac{(f_1'')^3}{(f_2')^3}-3\frac{(f_1'')^3(f_1')^2}{(f_2')^5}+\frac{f_1''f_2}{f_2^2}-\frac{(f_1')^2 f_1''}{f_2' f_2^2}-\frac{f_1'(f_2')^2}{f_2^3}-\frac{(f_1'')^2 f_1'}{f_2(f_2')^2}\end{aligned}$$

$$
\begin{aligned}
&+2\frac{f_1'(f_1'')^2}{f_2(f_2')^2}+\frac{(f_1'')^3}{(f_2')^3}+3\frac{(f_1')^2(f_1'')^3}{(f_2')^5}\\
&+4H(H^2-K)-\Big(\frac{f_1''}{f_2'}-\frac{f_1'}{f_2}\Big)^2\frac{f_1'}{f_2}-\frac{f_1''}{f_2'(f_2)^2}+\frac{f_1'}{f_2^3}\Big)\vec{n}_{\mathbb{H}^2}\\
&=-f_2^3\Big(2\Delta H+\frac{f_1''f_2'}{f_2^2}-\frac{(f_1')^2f_1''}{f_2'f_2^2}-\frac{f_1'(f_2')^2}{f_2^3}+\frac{(f_1'')^2f_1'}{f_2(f_2')^2}\\
&+4H(H^2-K)-\Big(\frac{f_1''}{f_2'}-\frac{f_1'}{f_2}\Big)^2\frac{f_1'}{f_2}-\frac{f_1''}{f_2'(f_2)^2}+\frac{f_1'}{f_2^3}\Big)\vec{n}_{\mathbb{H}^2}\\
&=-f_2^3\Big(2\Delta H-2\frac{(f_1')^2f_1''}{f_2'f_2^2}+\frac{(f_1')^3}{f_2^3}+\frac{(f_1'')^2f_1'}{f_2(f_2')^2}\\
&+4H(H^2-K)-\Big(\frac{f_1''}{f_2'}-\frac{f_1'}{f_2}\Big)^2\frac{f_1'}{f_2}\Big)\vec{n}_{\mathbb{H}^2}\\
&=-2f_2^3\Big(\Delta H+2H(H^2-K)\Big)\vec{n}_{\mathbb{H}^2}\\
&=2f_2^4\Big(\Delta H+2H(H^2-K)\Big)\vec{n}_{\mathbb{R}^2}=-2f_2^4(\partial_t f_{\mathbb{R}^2})^\perp\,.
\end{aligned}
$$

Hence the two flows differ by the factor $2f_2^4$. □

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# A Analysis of the ordinary differential equation

Here, we shortly study the ODE for the curvature $\rho = \frac{a}{r}$. We remind the reader that we define $\mu_\lambda := \sqrt{2(\lambda+1)}$ for $\lambda > -\frac{1}{2}$, see Proposition 3.1.

**Lemma A.1.** *Let $\lambda \in \mathbb{R}$ and $\rho_0 > 1$. There exists a unique global smooth solution $\rho\colon [0,\infty) \to (1,\infty)$ to (3.4), i.e. to*

$$
\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) &= H_\lambda(\rho) = -\frac{1}{2}\rho(\rho^2-1)(\rho^2-2(\lambda+1))\\
\rho(0) &= \rho_0.
\end{cases}
$$

*Depending on $\lambda$, we have the following asymptotic behaviour of $\rho$:*

1. *If $\lambda > -\frac{1}{2}$, then the solution $\rho$ satisfies*
   *(i) if $\rho(0) > \mu_\lambda$, then $\rho(t) \searrow \mu_\lambda$ as $t \to \infty$,*
   *(ii) if $\rho(0) < \mu_\lambda$, then $\rho(t) \nearrow \mu_\lambda$ as $t \to \infty$.*
2. *If $\lambda \leq -\frac{1}{2}$, then $\rho(t) \searrow 1$ as $t \to \infty$.*

*Proof.* Existence and uniqueness of a global solution follow from smooothness of the right hand side and the existence and uniqueness theory of autonomous ODEs (see e.g. [15]). Concerning the asymptotic behavior, for $\lambda > -\frac{1}{2}$ we have $\mu_\lambda > 1$, and hence $H_\lambda(\rho) > 0$ for $1 < \rho < \mu_\lambda$ as well as $H_\lambda(\rho) < 0$ for $\rho > \mu_\lambda$ (see Figure 2). In the case of $\lambda \leq -\frac{1}{2}$ we have $H_\lambda(\rho) < 0$ for any $\rho > 1$, and since $H_\lambda(1) = 0$ we find $\rho > 1$ for all times $0 \leq t < \infty$. □

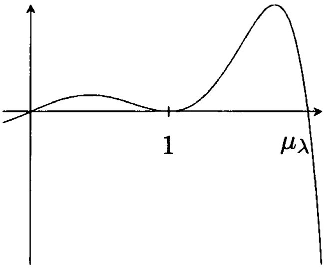


Figure 2: The graph of $H_\lambda$ for $\lambda > -\frac{1}{2}$.

## References


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