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On the bidomain equations as parabolic evolution equations

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1 Introduction

This paper is mainly based on the papers [9, 15]. We consider the bidomain equations that are commonly used as a model to represent the electrophysiological wave propagation in the heart. The bidomain system is as follows.

$$\begin{cases} \partial_t u + f(u,w) - \nabla\cdot(\sigma_i \nabla u_i) = s_i & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + f(u,w) + \nabla\cdot(\sigma_e \nabla u_e) = -s_e & \text{in } (0,\infty)\times\Omega, \\ \partial_t w + g(u,w) = 0 & \text{in } (0,\infty)\times\Omega, \\ u = u_i - u_e & \text{in } (0,\infty)\times\Omega, \\ \sigma_i\nabla u_i\cdot n = 0,\ \sigma_e\nabla u_e\cdot n = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0)=u_0,\ w(0)=w_0 & \text{in } \Omega. \end{cases} \tag{BDE}$$

Here $\Omega\subset\mathbb{R}^d$ denotes a domain describing the myocardium, whose outward unit normal vector to $\partial\Omega$ is denoted by n. The unknown functions u_i and u_e model the intra- and extracellular electric potentials, and u denotes the transmembrane potential. The variable w, the so-called gating variable, corresponding to the ionic transport through the cell membrane is also unknown function. On the other hand the conductivity matrices $\sigma_i(x)$ and $\sigma_e(x)$, and intra- and extracellular stimulation current $s_i(t,x)$ and $s_e(t,x)$ are given functions. The non-linear term f and g are given to be fixed later.

In [3], under physiological reasonable assumptions on $\sigma_{i,e}$ and $s_{i,e}$, they transformed the bidomain equations into an abstract form

$$\begin{cases} u' + Au + f(u,w) = s, & \text{in } (0,\infty), \\ w' + g(u,w) = 0, & \text{in } (0,\infty), \\ u(0)=u_0, w(0)=w_0 \end{cases} \tag{ABDE}$$

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by introducing a bidomain operator A and a modified source term s in L^2-setting. Functions $u_{i,e}$ can be recovered from u and $s_{i,e}$. Formally the bidomain operator is the harmonic mean of two elliptic operators, i.e. $(A_i^{-1} + A_e^{-1})^{-1}$ or $A_i(A_i + A_e)^{-1}A_e$, where $A_{i,e}$ is the elliptic operator $-\nabla \cdot (\sigma_{i,e}\nabla \ \cdot \ )$ with the homogeneous Neumann boundary condition, respectively. They proved that the bidomain operator is a non-negative self-adjoint operator by considering corresponding weak formulations. In the paper [3], they showed existence of a global weak solution and existence and uniqueness of a local strong solution from the theory of evolution equations in L^2 space. The uniqueness and regularity of the weak solutions have been considered in [18].

Our aim of this paper is to prove that the bidomain operator generates an analytic semigroup on $L^p(\Omega)$ for $1 < p < \infty$. To derive the analyticity, it is sufficient to derive resolvent estimates for the bidomain equations of the resolvent form. For L^p resolvent estimates a standard way is to use the Agmon's method (e.g. [19], [24]) and a localization method. The main idea of the first method is as follows. If we have a $W^{2,p}(\Omega \times \mathbb{R})$ a priori estimate for the operator $A - e^{i\theta}\partial_{tt}$, then A has an L^p resolvent estimate. Unfortunately, it seems difficult to derive such a $W^{2,p}$ a priori estimate and the localization method because of a nonlocal structure of the bidomain operator. Thus we argue in a different way. We first establish an L^∞ resolvent estimate for the bidomain equations by a contradiction argument including a blow-up argument. In section 2, we prove it. In section 3 we give an accurate definition of the bidomain operator in L^p spaces for $1 < p < \infty$ and derive an L^p resolvent estimate for $2 \leq p \leq \infty$ by interpolating L^2 and L^∞ results. The L^p-theory for $1 < p < 2$ is established by a duality argument. These are the main part of this paper.

Second part of this paper in section 4, we consider an application of a time-periodic solution. It is natural question whether the bidomain equations have a time-periodic solution since the bidomain equations is a model of the heart. We construct the linear theory of the time-periodic solution in a real interpolation spaces motivated by DaPrato-Grisvard's maximal regularity theory. For the bidomain equations which is non-linear equations with FitzHugh-Nagumo type f, g, we prove that the bidomain equations has a time-periodic solution for small periodic external forces s by Banach's fixed point theorem.

At last, recently, since the results mentioned here have been improved in many points, we give some bibliographical remarks on recent works for the bidomain equations.

2 L^∞-resolvent estimate for the bidomain equations

In this section we prove the L^∞ resolvent estimate. We need some assumptions on the functions $\sigma_{i,e}$. Let $a = a(x)$ denote unit tangent vector at the point $x \in \partial\Omega$. Set the longitudinal conductances $k_{i,e}^l : \partial\Omega \to \mathbb{R}$ and the transverse conductances $k_{i,e}^t : \partial\Omega \to \mathbb{R}$

along the fibers. Let the conductance tensors be of the form ([4])

$$\sigma_{i,e}(x) = k^t_{i,e}(x)I + (k^l_{i,e}(x) - k^t_{i,e}(x))a(x) \otimes a(x) \quad (x \in \partial\Omega).$$

By this form we have the normal n is the eigenvector of $\sigma_{i,e}$ whose eigenvalue is $k^t_{i,e}(x)$:

$$\sigma_{i,e}(x)n(x) = k^t_{i,e}(x)n(x) \ (x \in \partial\Omega).$$

Under these physiological reasonable assumptions of $\sigma_{i,e}$, we have the property of boundary conditions:

$$\sigma_{i,e}\nabla u \cdot n = 0 \Leftrightarrow \nabla u \cdot n = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Moreover we assume that there exist constants $0 < \underline{\sigma} < \overline{\sigma}$ such that

$$\underline{\sigma}|\xi|^2 \leq \langle \sigma_{i,e}(x)\xi, \xi\rangle \leq \overline{\sigma}|\xi|^2 \tag{2}$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^d$.

We consider the following resolvent equations

$$(*)\begin{cases} \lambda u - \nabla\cdot(\sigma_i \nabla u_i) = s & \text{in } \Omega, \\ \lambda u + \nabla\cdot(\sigma_e \nabla u_e) = s & \text{in } \Omega, \\ u = u_i - u_e & \text{in } \Omega, \\ \sigma_i\nabla u_i \cdot n = 0, \ \sigma_e \nabla u_e \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

which is the Laplace transformation of the linear part of the bidomain equations.

Let us state an L^∞ resolvent estimate. We set $\Sigma_{\theta,M} := \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| < \theta, M < |\lambda|\}$ and $N(u, u_i, u_e, \lambda)$ of the form

$$N(u,u_i,u_e,\lambda) := \sup_{x\in\Omega}\left(|\lambda||u(x)| + |\lambda|^{1/2}\left(|\nabla u(x)| + |\nabla u_i(x)| + |\nabla u_e(x)|\right)\right).$$

Theorem 2.1 (L^∞ resolvent estimate for bidomain equations). *Let $\Omega \subset \mathbb{R}^d$ be a uniformly C^2-domain and $\sigma_{i,e} \in C^1(\overline{\Omega}, \mathbb{S}^d)$ satisfy (1) and (2). Then for each $\varepsilon \in (0, \pi/2)$ there exist $C > 0$ and $M > 0$ such that*

$$N(u, u_i, u_e, \lambda) \leq C\|s\|_{L^\infty(\Omega)}$$

for all $\lambda \in \Sigma_{\pi-\varepsilon,M}$, $s \in L^\infty(\Omega)$ and strong solutions $u, u_{i,e} \in \bigcap_{d<p<\infty} W^{2,p}_{loc}(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ of $(*)$.

Remark 2.2. (i) *It is impossible to derive an estimate $|\lambda|\|u_{i,e}\|_\infty \leq C\|s\|_{L^\infty(\Omega)}$ because if (u, u_i, u_e) is a triplet of strong solutions then so is $(u, u_i + c, u_e + c)$ for all $c \in \mathbb{R}$.*
(ii) *By the Sobolev embedding theorem [1],*

$$\bigcap_{n<p<\infty} W^{2,p}_{loc}(\overline{\Omega}) \cap W^{1,\infty}(\Omega) \subset \bigcap_{0<\alpha<1} C^{1+\alpha}(\overline{\Omega}).$$

Hence (u, u_i, u_e) are C^1 functions and the left-hand side of the resolvent estimate makes sense.

Proof. We divide the proof into five steps. The first two steps are reformulation of equations and estimates. The last three steps (compactness, characterization of the limit and uniqueness) are crucial.

Step 1 (Normalization)

We argue by contradiction. Suppose that the statement were false. Then there would exist $\varepsilon \in (0, \pi/2)$, for any $k \in \mathbb{N}$ there would exist $\lambda_k = |\lambda_k| e^{i\theta_k} \in \Sigma_{\pi-\varepsilon,k}$, $s_k \in L^\infty(\Omega)$ and $u_k, u_{ik}, u_{ek} \in \bigcap_{d<p<\infty} W^{2,p}_{loc}(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ which are strong solutions of resolvent equations

$$\begin{cases} \lambda_k u_k - \nabla \cdot (\sigma_i \nabla u_{ik}) = s_k & \text{in } \Omega, \\ \lambda_k u_k + \nabla \cdot (\sigma_e \nabla u_{ek}) = s_k & \text{in } \Omega, \\ u_k = u_{ik} - u_{ek} & \text{in } \Omega, \\ \sigma_i \nabla u_{ik} \cdot n = 0, \ \sigma_e \nabla u_{ek} \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

with an L^∞ estimate $N(u_k, u_{ik}, u_{ek}, \lambda_k) > k \|s_k\|_{L^\infty(\Omega)}$.

We set

$$\begin{pmatrix} v_k \\ v_{ik} \\ v_{ek} \\ \tilde{s}_k \end{pmatrix} := \frac{1}{N(u_k, u_{ik}, u_{ek}, \lambda_k)} \begin{pmatrix} |\lambda_k| u_k \\ |\lambda_k| u_{ik} \\ |\lambda_k| u_{ek} \\ s_k \end{pmatrix}.$$

Then we get normalized resolvent equations of the form

$$\begin{cases} e^{i\theta_k} v_k - \frac{1}{|\lambda_k|} \nabla \cdot (\sigma_i \nabla v_{ik}) = \tilde{s}_k & \text{in } \Omega, \\ e^{i\theta_k} v_k + \frac{1}{|\lambda_k|} \nabla \cdot (\sigma_e \nabla v_{ek}) = \tilde{s}_k & \text{in } \Omega, \\ v_k = v_{ik} - v_{ek} & \text{in } \Omega, \\ \sigma_i \nabla v_{ik} \cdot n = 0, \ \sigma_e \nabla v_{ek} \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

with estimates $\frac{1}{k} > \|\tilde{s}_k\|_{L^\infty(\Omega)}$ and

$$\begin{aligned} &N\left(\frac{v_k}{|\lambda_k|}, \frac{v_{ik}}{|\lambda_k|}, \frac{v_{ek}}{|\lambda_k|}, \lambda_k\right) \\ =&\sup_{x\in\Omega} \left(|v_k(x)| + |\lambda_k|^{-1/2} \left(|\nabla v_k(x)| + |\nabla v_{ik}(x)| + |\nabla v_{ek}(x)|\right)\right) \\ =&1. \end{aligned}$$

Step 2 (Rescaling)

Secondly, we rescale variables near maximum points of normalized N. By definition of supremum there exists $\{x_k\}_{k=1}^{\infty} \subset \Omega$ such that

$$|v_k(x_k)| + |\lambda_k|^{-1/2} \left(|\nabla v_k(x_k)| + |\nabla v_{ik}(x_k)| + |\nabla v_{ek}(x_k)|\right) > \frac{1}{2}$$

for all $k \in \mathbb{N}$. We rescale functions $\{(w_k, w_{ik}, w_{ek})\}_{k=1}^{\infty}$, $\{t_k\}_{k=1}^{\infty}$, matrices $\{(\sigma_{ik}, \sigma_{ek})\}_{k=1}^{\infty}$ and domain Ω_k with respect to x_k. Namely, we set

$$\begin{aligned}
\begin{pmatrix} w_k \\ w_{ik} \\ w_{ek} \end{pmatrix}(x) &:= \begin{pmatrix} v_k \\ v_{ik} \\ v_{ek} \end{pmatrix}\left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right), \\
t_k(x) &:= \tilde{s}_k\left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right), \\
\sigma_{ik}(x) := \sigma_i\left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right), &\quad \sigma_{ek}(x) := \sigma_e\left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right), \\
\Omega_k &:= |\lambda_k|^{1/2}(\Omega - x_k).
\end{aligned}$$

By changing variables $\Omega \ni x \mapsto |\lambda_k|^{1/2}(x - x_k) \in \Omega_k$, we notice that our equations and our estimates can be rewritten of the form

$$\begin{cases}
e^{i\theta_k} w_k - \nabla \cdot (\sigma_{ik} \nabla w_{ik}) = t_k & \text{in } \Omega_k, \\
e^{i\theta_k} w_k + \nabla \cdot (\sigma_{ek} \nabla w_{ek}) = t_k & \text{in } \Omega_k, \\
w_k = w_{ik} - w_{ek} & \text{in } \Omega_k, \\
\sigma_{ik} \nabla w_{ik} \cdot n_k = 0, \ \sigma_{ek} \nabla w_{ek} \cdot n_k = 0 & \text{on } \partial\Omega_k,
\end{cases}$$

with estimates

$$\begin{aligned}
&\frac{1}{k} > \|t_k\|_{L^\infty(\Omega_k)}, \\
&|w_k(0)| + |\nabla w_k(0)| + |\nabla w_{ik}(0)| + |\nabla w_{ek}(0)| > \frac{1}{2}, \\
&\sup_{x \in \Omega_k} \left(|w_k(x)| + |\nabla w_k(x)| + |\nabla w_{ik}(x)| + |\nabla w_{ek}(x)|\right) = 1,
\end{aligned}$$

where n_k denotes the unit outer normal vector to Ω_k. Here, we remark that unknown functions w_{ik} and w_{ek} are defined up to an additive constant. So without loss of generality we may assume that $w_{ik}(0) := 0$.

Step 3 (Compactness)

In this step, we will show local uniform boundedness for $\{(w_k, w_{ik}, w_{ek})\}_{k=1}^{\infty}$. If these sequences are bounded, one can take subsequences $\{(w_{k_l}, w_{ik_l}, w_{ek_l})\}_{l=1}^{\infty}$ which uniformly convergences in the norm C^1 on each compact set. We need to divide two cases. One is the whole space case and the other is the half space case up to translation and rotation. We set $d_k = \mathrm{dist}(0, \partial\Omega_k) = |\lambda_k|^{1/2}\mathrm{dist}(x_k, \partial\Omega)$ and $D := \liminf_{k\to\infty} d_k$. The case of $D < \infty$, which is the case Ω_k goes to $\mathbb{R}^d_+$, is very hard in the points of a cut-off technique in this Step 3 and the dual problem in Step 5. Therefore we consider only the case of $D = \infty$ throughout the remained proof. More precisely see [9].

In the case $D = \infty$, the limit Ω_k is $\mathbb{R}^d$ in the sense that for any $R > 0$ there is k_0 such that $B(0, R) \subset \Omega_k$ for $k_0 \le k$. For the convergence of the domain, see [22]. Let cut-off function $\rho \in C_0^\infty(\mathbb{R}^d)$ be such that $\rho(x) \equiv 1$ for $|x| \le 1$ and $\rho(x) \equiv 0$ for $|x| \ge 3/2$. Here and hereafter by $C_0^m(E)$ we mean the space of all m-times continuously differentiable function with compact support in the set E. We localize functions w_k, w_{ik}, w_{ek} as follows

$$\begin{pmatrix} w_k^\rho \\ w_{ik}^\rho \\ w_{ek}^\rho \end{pmatrix} := \rho \begin{pmatrix} w_k \\ w_{ik} \\ w_{ek} \end{pmatrix} \quad \text{in } \Omega_k.$$

By multiplying rescaled resolvent equations by ρ, we consider the following localized equations

$$e^{i\theta_k} w_k^\rho - \nabla\cdot(\sigma_{ik}\nabla w_{ik}^\rho) = t_k\rho + I_{ik} \qquad \text{in } \Omega_k, \tag{3}$$
$$e^{i\theta_k} w_k^\rho + \nabla\cdot(\sigma_{ek}\nabla w_{ek}^\rho) = t_k\rho + I_{ek} \qquad \text{in } \Omega_k, \tag{4}$$
$$w_k^\rho = w_{ik}^\rho - w_{ek}^\rho \qquad \text{in } \Omega_k, \tag{5}$$
$$\sigma_{ik}\nabla w_{ik}^\rho\cdot n_k = 0,\ \sigma_{ek}\nabla w_{ek}^\rho\cdot n_k = 0 \qquad \text{on } \partial\Omega_k, \tag{6}$$

where

$$\begin{aligned} I_{ik} = -\sum_{1\le m,n\le d} \Big\{ & \big((\sigma_{ik})_{mn}\big)_{x_m}\rho_{x_n} w_{ik} + (\sigma_{ik})_{mn}\rho_{x_m x_n} w_{ik} \\ & + (\sigma_{ik})_{mn}\rho_{x_n}(w_{ik})_{x_m} + (\sigma_{ik})_{mn}\rho_{x_m}(w_{ik})_{x_n}\Big\}, \end{aligned}$$

$$\begin{aligned} I_{ek} = \sum_{1\le m,n\le d} \Big\{ & \big((\sigma_{ek})_{mn}\big)_{x_m}\rho_{x_n} w_{ek} + (\sigma_{ek})_{mn}\rho_{x_m x_n} w_{ek} \\ & + (\sigma_{ek})_{mn}\rho_{x_n}(w_{ek})_{x_m} + (\sigma_{ek})_{mn}\rho_{x_m}(w_{ek})_{x_n}\Big\} \end{aligned}$$

are lower order terms of w_{ik} and w_{ek}. Here, we take sufficiently large k such that $B(0, 2) \subset \Omega_k$.

Take some $p > d$ and apply $W^{2,p}(\Omega_k)$ a priori estimate for second order elliptic operators $-\nabla\cdot(\sigma_{ik}\nabla\cdot)$, which have the Neumann boundary (6). By (3) there exists $C > 0$

independent of $k \in \mathbb{N}$ such that

$$\begin{aligned}
&\|w_{ik}^{\rho}\|_{W^{2,p}(\Omega_k)}\\
\leq &C\left(\|w_{ik}^{\rho}\|_{L^p(\Omega_k)} + \|w_k^{\rho}\|_{L^p(\Omega_k)} + \|t_k\rho\|_{L^p(\Omega_k)} + \|I_{ik}\|_{L^p(\Omega_k)}\right)\\
\leq &C|B(0,2)|^{1/p}\left(\|w_{ik}^{\rho}\|_{L^\infty(\Omega_k)} + \|w_k^{\rho}\|_{L^\infty(\Omega_k)} + \|t_k\rho\|_{L^\infty(\Omega_k)} + \|I_{ik}\|_{L^\infty(\Omega_k)}\right)\\
=:&C|B(0,2)|^{1/p}\left(I + II + III + IV\right),
\end{aligned}$$

where we use Hölder inequality in the second inequality. The first term I is uniformly bounded in k since $w_{ik}(0) = 0$ and $\|\nabla w_{ik}\|_{L^\infty(\Omega_k)} \leq 1$. The second term II and the third term III are also uniformly bounded in k since $\|w_k\|_{L^\infty(\Omega_k)} \leq 1$, $\|\rho\|_{L^\infty(\Omega_k)} \leq 1$ and $\|t_k\|_{L^\infty(\Omega_k)} < 1/k$. Finally the forth term IV is also uniformly bounded in k since

$$\begin{aligned}
IV &\leq C(d, \sup_k \|\sigma_{ik}\|_{W^{1,\infty}(\Omega_k)})\|w_{ik}\|_{W^{1,\infty}(\Omega_k)}\\
&\leq C.
\end{aligned}$$

Here, the constant C may differ from line to line. Therefore the sequence $\{w_{ik}^{\rho}\}_{k=1}^{\infty}$ is uniformly bounded in $W^{2,p}(\Omega_k)$. Functions $\{w_{ek}^{\rho}\}_{k=1}^{\infty}$ and $\{w_k^{\rho}\}_{k=1}^{\infty}$ are also uniformly bounded in $W^{2,p}(\Omega_k)$ since the same calculation as above and (5). Here, Ω_k depends on $k \in \mathbb{N}$. By zero extension from Ω_k to $\mathbb{R}^d$, we have $\{(w_k^{\rho}, w_{ik}^{\rho}, w_{ek}^{\rho})\}_{k=1}^{\infty}$ is uniform bounded in the norm $\left(W^{2,p}(\mathbb{R}^d)\right)^3$. Thus we are able to take subsequences $\{(w_{k_l}^{\rho}, w_{ik_l}^{\rho}, w_{ek_l}^{\rho})\}_{l=1}^{\infty}$ and $w, w_i, w_e \in W^{2,p}(\mathbb{R}^d)$ such that

$$\begin{pmatrix} w_{k_l}^{\rho} \\ w_{ik_l}^{\rho} \\ w_{ek_l}^{\rho} \end{pmatrix} \to \begin{pmatrix} w \\ w_i \\ w_e \end{pmatrix} \quad \text{in the norm } C^1(\mathbb{R}^d) \text{ as } l \to \infty,$$

by Rellich's compactness theorem [1]. Since

$$|w_{k_l}(0)| + |\nabla w_{k_l}(0)| + |\nabla w_{ik_l}(0)| + |\nabla w_{ek_l}(0)| > \frac{1}{2},$$

we get

$$|w(0)| + |\nabla w(0)| + |\nabla w_i(0)| + |\nabla w_e(0)| \geq \frac{1}{2}.$$

Step 4 (Characterization of the limit)

We continue to consider the case of $D = \infty$, i.e. $\Omega_\infty = \mathbb{R}^d$. We have $w, w_i, w_e \in \bigcap_{d<p<\infty} W_{loc}^{2,p}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ and

$$\begin{pmatrix} w_{k_l} \\ \nabla w_{ik_l} \\ \nabla w_{ek_l} \end{pmatrix} \to \begin{pmatrix} w \\ \nabla w_i \\ \nabla w_e \end{pmatrix} \quad \text{weak} * \text{ in } L^\infty(\mathbb{R}^d) \text{ as } l \to \infty$$

since $\sup_{x\in\Omega_k} (|w_k(x)| + |\nabla w_k(x)| + |\nabla w_{ik}(x)| + |\nabla w_{ek}(x)|) = 1$.

We can characterize the limit functions by the weak formulation in the following proposition. We do not prove in this paper. See [9].

Proposition 2.3. *The limit $w, w_i, w_e \in \bigcap_{d<p<\infty} W^{2,p}_{loc}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ satisfy that for any $\phi_{i,e} \in C_0^\infty(\mathbb{R}^d)$*

$$\begin{cases} e^{i\theta_\infty}(w, \phi_i)_{L^2(\mathbb{R}^d)} + (\sigma_{i\infty}\nabla w_i, \nabla\phi_i)_{L^2(\mathbb{R}^d)} = 0, \\ e^{i\theta_\infty}(w, \phi_e)_{L^2(\mathbb{R}^d)} - (\sigma_{e\infty}\nabla w_e, \nabla\phi_e)_{L^2(\mathbb{R}^d)} = 0, \\ w = w_i - w_e, \end{cases}$$

where $\theta_\infty = \lim_{k\to\infty}\theta_k$ and $\sigma_{i\infty}, \sigma_{e\infty}$ are constant coefficients matrices which satisfy uniform ellipticity condition. Here, $(\cdot,\cdot)_{L^2(\mathbb{R}^d)}$ denotes L^2-inner product.

Step 5 (Uniqueness)

In this last step we prove that limit functions are unique and it is only trivial solution. The method is to reduce existence of solution to dual problems and use the fundamental lemma of calculus of variation.

Lemma 2.4. *Let $w, w_i, w_e \in \bigcap_{d<p<\infty} W^{2,p}_{loc}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ satisfy*

$$\begin{cases} e^{i\theta_\infty}(w, \phi_i)_{L^2(\mathbb{R}^d)} + (\sigma_{i\infty}\nabla w_i, \nabla\phi_i)_{L^2(\mathbb{R}^d)} = 0, \\ e^{i\theta_\infty}(w, \phi_e)_{L^2(\mathbb{R}^d)} - (\sigma_{e\infty}\nabla w_e, \nabla\phi_e)_{L^2(\mathbb{R}^d)} = 0, \\ w = w_i - w_e, \end{cases} \tag{7}$$

for all $\phi_{i,e} \in C_0^\infty(\mathbb{R}^d)$, then $w = 0$ and $w_i = w_e =$constant.

Equations (7) implies the following equations

$$\begin{cases} \left(w_i, e^{i\theta_\infty}\phi_i - \nabla\cdot(\sigma_{i\infty}\nabla\phi_i)\right)_{L^2(\mathbb{R}^d)} - (w_e, e^{i\theta_\infty}\phi_i)_{L^2(\mathbb{R}^d)} = 0, \\ (w_i, e^{i\theta_\infty}\phi_e)_{L^2(\mathbb{R}^d)} - \left(w_e, e^{i\theta_\infty}\phi_e - \nabla\cdot(\sigma_{e\infty}\nabla\phi_e)\right)_{L^2(\mathbb{R}^d)} = 0, \end{cases}$$

$$\begin{aligned} \left(w_i, e^{i\theta_\infty}(\phi_i - \phi_e) - \nabla\cdot(\sigma_{i\infty}\nabla\phi_i)\right)_{L^2(\mathbb{R}^d)} \\ - \left(w_e, e^{i\theta_\infty}(\phi_i - \phi_e) + \nabla\cdot(\sigma_{e\infty}\nabla\phi_e)\right)_{L^2(\mathbb{R}^d)} = 0. \end{aligned}$$

We do not prove the solvability of the dual problem in this paper. The solvability of the dual problems is as follows. For all $\psi_{i,e} \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d}(\psi_i - \psi_e)dx = 0$, we can find solutions $(\phi_i, \phi_e) \in \mathcal{S}'(\mathbb{R}^d)$ satisfying $\phi_i - \phi_e, \nabla \cdot (\sigma_{i\infty} \nabla \phi_i), \nabla \cdot (\sigma_{e\infty} \nabla \phi_e) \in L^1(\mathbb{R}^d)$ and such that

$$\begin{aligned} e^{i\theta_\infty}(\phi_i - \phi_e) - \nabla \cdot (\sigma_{i\infty} \nabla \phi_i) &= \psi_i && \text{in } \mathbb{R}^d, \\ e^{i\theta_\infty}(\phi_i - \phi_e) + \nabla \cdot (\sigma_{e\infty} \nabla \phi_e) &= \psi_e && \text{in } \mathbb{R}^d. \end{aligned}$$

Under this solvability, we have that for all $\psi_{i,e} \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} (\psi_i - \psi_e) dx = 0$,

$$(w_i, \psi_i)_{L^2(\mathbb{R}^d)} - (w_e, \psi_e)_{L^2(\mathbb{R}^d)} = 0.$$

Let $\psi_i = \psi_e$ then $(w, \psi_i)_{L^2(\mathbb{R}^d)} = 0$ for all $\psi_i \in C_0^\infty(\mathbb{R}^d)$. By fundamental lemma of calculus of variations, we get $w \equiv 0$. Let $\psi_e \equiv 0$ then $(w_i, \psi_i)_{L^2(\mathbb{R}^d)} = 0$ for all $\psi_i \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \psi_i dx = 0$. This means $w_i \equiv$ constant. Obviously $w_e = w_i$ since $w = w_i - w_e$.

Results of Step 3 and Step 5 are contradictory, so the proof of Theorem 2.1 is now complete. □

3 Definitions and L^p resolvent estimates for $1 < p < \infty$

In this section we define the bidomain operators in L^p spaces for $1 < p < \infty$ under assuming suitable physiological reasonable assumptions on $\sigma_{i,e}$.

Let Ω be a bounded C^2 domain. Consider the averaging zero space $L^p_{av}(\Omega)$ and its projection P_{av}:

$$L^p_{av}(\Omega) := \left\{ u \in L^p(\Omega) \mid \int_\Omega u dx = 0 \right\}$$
$$P_{av} u := u - \frac{1}{|\Omega|} \int_\Omega u dx \quad \text{for } u \in L^p(\Omega).$$

Let A_i and A_e be the second order elliptic operators in $L^p_{av}(\Omega)$ defined by

$$\begin{aligned} A_{i,e} u &:= -\nabla \cdot (\sigma_{i,e} \nabla u) \\ D(A_{i,e}) &:= \left\{ u \in W^{2,p}(\Omega) \cap L^p_{av}(\Omega) \mid \sigma_{i,e} \nabla u \cdot n = 0 \text{ a.e. in } \partial\Omega \right\} \subset L^p_{av}(\Omega). \end{aligned}$$

Let $\sigma_{i,e} \in C^1(\overline{\Omega})$ satisfy the uniform elliptic condition (2). Then we have that the operator A_i is densely defined closed linear operator on $L^p_{av}(\Omega)$ and for any $f \in L^p_{av}(\Omega)$ there uniquely exists $u \in D(A_i)$ such that $A_i u = f$. The operator A_e also has the same property.

Since we assume the special boundary conditions (1), we see

$$D(A_i) = \left\{ u \in W^{2,p}(\Omega) \cap L^p_{av}(\Omega) \mid \nabla u \cdot n = 0 \text{ a.e. in } \partial\Omega \right\} = D(A_e).$$

So we are able to consider the sum of the operators, $A_i + A_e$, with the domain $D(A_i)(= D(A_e))$ and we observe that inverse operator $(A_i + A_e)^{-1}$ on L^p_{av} is a bounded linear operator.

We reformulate resolvent equations corresponding to the parabolic and elliptic system as are derived in [3]. The new system contains only u and u_e as unknown functions. Since $u_i = u + u_e$, the new system is of the form:

$$\lambda u - \nabla \cdot (\sigma_i \nabla u) - \nabla \cdot (\sigma_i \nabla u_e) = s \qquad \text{in } \Omega, \tag{8}$$
$$- \nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e) \nabla u_e) = 0 \qquad \text{in } \Omega, \tag{9}$$
$$\sigma_i \nabla u \cdot n + \sigma_i \nabla u_e \cdot n = 0 \qquad \text{on } \partial\Omega, \tag{10}$$
$$\sigma_i \nabla u \cdot n + (\sigma_i + \sigma_e) \nabla u_e \cdot n = 0 \qquad \text{on } \partial\Omega. \tag{11}$$

Under $\int_\Omega u_e dx = 0$, which is often used assumption to study bidomain equations, from the equation (9),

$$\begin{aligned} & A_i P_{av} u + (A_i + A_e) u_e = 0 \\ \Leftrightarrow & (A_i + A_e) u_e = -A_i P_{av} u \quad (\in L^p_{av}(\Omega)) \\ \Leftrightarrow & u_e = -(A_i + A_e)^{-1} A_i P_{av} u \; (\in D(A_i)). \end{aligned}$$

We substitute this into (8) to set

$$\begin{aligned} & \lambda u + A_i P_{av} u - A_i (A_i + A_e)^{-1} A_i P_{av} u = s \\ \Leftrightarrow & \lambda u + A_i (A_i + A_e)^{-1} A_e P_{av} u = s. \end{aligned}$$

We are ready to define bidomain operators A.

Definition 3.1. *For $1 < p < \infty$, we define the bidomain operator A in L^p space by*

$$\begin{aligned} D(A) &:= \{u \in W^{2,p}(\Omega) \mid \nabla u \cdot n = 0 \text{ a.e. in } \partial\Omega\} \subset L^p(\Omega) \to L^p(\Omega) \\ A &:= A_i (A_i + A_e)^{-1} A_e P_{av}. \end{aligned}$$

Under $\int_\Omega u_e dx = 0$, the resolvent bidomain equations (8)-(11) for the function u can be written in a single resolvent equation of the form

$$(\lambda + A) u = s \quad \text{in } \Omega. \tag{12}$$

Once we solve this equation, we are able to recover from $u_e = -(A_i + A_e)^{-1} A_i P_{av} u$.

By operating $(A_i + A_e) A_i^{-1} P_{av}$ to $(\lambda + A) u = s$ and using a resolvent estimate for the operator A_e, we have the following a priori estimate for the bidomain operators.

Theorem 3.2 (A priori estimate for bidomain operators). *Let $1 < p < \infty$. For each $\lambda \in \Sigma_{\pi,0}$ there exists $C_\lambda > 0$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \le C_\lambda \left(\|(\lambda + A)u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right)$$

for all $u \in D(A)$.

By this theorem we observe that the bidomain operator A in L^p spaces is a densely defined closed linear operator. Let A_p be the bidomain operator in L^p spaces. We can prove for all $p \in (1, \infty)$, $A_p^* = A_{p'}$ and $\rho(-A_p) \supset \Sigma_{\pi,0}$, where the superscript $*$ means the adjoint operator and p' is the Hölder conjugate exponent of p.

Let us consider the L^p resolvent estimate for the bidomain operators. Note that the paper [3] showed that the bidomain operator A is a non-negative self-adjoint operator in $L^2(\Omega)$ so that it has a L^2 resolvent estimate. Namely, $\rho(-A_2) \supset \Sigma_{\pi,0}$ and for each $\varepsilon \in (0, \pi/2)$ there exists $C > 0$ such that

$$\sup_{\lambda \in \Sigma_{\pi-\varepsilon,0}} |\lambda| \|u\|_{L^2(\Omega)} \le C\|s\|_{L^2(\Omega)}$$

for all $s \in L^2(\Omega)$. We derived an L^∞ resolvent estimate (Theorem 2.1); for each $\varepsilon \in (0, \pi/2)$ there exist $C > 0$ and $M \ge 0$ such that $\rho(-A) \supset \Sigma_{\pi,M}$ and

$$\sup_{\lambda \in \Sigma_{\pi-\varepsilon,M}} |\lambda| \|u\|_{L^\infty(\Omega)} \le C\|s\|_{L^\infty(\Omega)}$$

and for all $s \in L^\infty(\Omega)$.

By using Riesz-Thorin interpolation theorem, we are able to derive an L^p resolvent estimate, i.e. for each $\varepsilon \in (0, \pi/2)$ and $2 \le p \le \infty$ there exist $C > 0$ and $M \ge 0$ such that $\rho(-A_p) \supset \Sigma_{\pi,M}$ and that

$$\sup_{\lambda \in \Sigma_{\pi-\varepsilon,M}} |\lambda| \|u\|_{L^p(\Omega)} \le C\|s\|_{L^p(\Omega)}$$

and for all $s \in L^p(\Omega)$.

For $2 \le p < \infty$ and its conjugate exponent $p'(\in (1, 2])$, we have

$$\|(\lambda + A_{p'})^{-1}\|_{\mathcal{L}(L^{p'}(\Omega))} = \|((\lambda + A_p)^{-1})^*\|_{\mathcal{L}(L^{p'}(\Omega))} = \|(\lambda + A_p)^{-1}\|_{\mathcal{L}(L^p(\Omega))} \le \frac{C}{|\lambda|}.$$

We derived the resolvent estimate for bidomain operators $-A_p$ in L^p spaces for the sufficiently large λ. However, in the next theorem, we estimate the resolvent for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$ and higher order derivatives $\|\nabla u\|_{L^p(\Omega)}$ and $\|\nabla^2 u\|_{L^p(\Omega)}$, which is similar to an elliptic operator in L^p spaces.

Theorem 3.3 (L^p resolvent estimates for bidomain operators). *Let $1 < p < \infty$. For each $\varepsilon \in (0, \pi/2)$ there exists $C > 0$ depending only on ε such that the unique solution $u \in D(A_p)$ of the resolvent equation $(\lambda + A_p)u = s$ satisfies*

$$|\lambda| \|u\|_{L^p(\Omega)} + |\lambda|^{1/2} \|\nabla u\|_{L^p(\Omega)} + \|\nabla^2 u\|_{L^p(\Omega)} \leq C \|s\|_{L^p(\Omega)}$$

for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$ and $s \in L^p(\Omega)$.

Proof. We divide the resolvent estimate $(\lambda + A_p)u = s$ into $(\lambda + A_p)u_1 = P_{av}s$ and $(\lambda + A_p)u_2 = s - P_{av}s$. Note that $u = u_1 + u_2$, $P_{av}s \in L^p_{av}(\Omega)$, $s - P_{av}s$ is a constant and the origin 0 belongs to $\rho(-A_p|_{L^p_{av}(\Omega)})$. For each $\varepsilon \in (0, \pi/2)$ we fix $M \geq 0$ which is the constant in the above explanation. Since $(\lambda + A_p)^{-1} P_{av}s = (\lambda + A_p|_{L^p_{av}(\Omega)})^{-1} P_{av}s$ and the resolvent operator $(\lambda + A_p|_{L^p_{av}(\Omega)})^{-1}$ is uniform bounded in a compact subset $\overline{\Sigma_{\pi-\varepsilon,0} \cap B(0, 2M)}$, we have there exists $C > 0$ depending on ε such that

$$\begin{aligned} \|u_1\|_{L^p(\Omega)} &= \|(\lambda + A_p)^{-1} P_{av}s\|_{L^p(\Omega)} \\ &= \|(\lambda + A_p|_{L^p_{av}(\Omega)})^{-1} P_{av}s\|_{L^p(\Omega)} \\ &\leq \frac{C}{|\lambda| + 1} \|P_{av}s\|_{L^p(\Omega)} \\ &\leq \frac{C}{|\lambda| + 1} \|s\|_{L^p(\Omega)} \end{aligned}$$

for all $\lambda \in \Sigma_{\pi-\varepsilon,0} \cap B(0, 2M)$. On the other hand we have $u_2 = \frac{1}{\lambda}(s - P_{av}s)$, so there exists $C > 0$ such that

$$\begin{aligned} \|u_2\|_{L^p(\Omega)} &= \|\frac{1}{\lambda}(s - P_{av}s)\|_{L^p(\Omega)} \\ &\leq \frac{C}{|\lambda|} \|s - P_{av}s\|_{L^p(\Omega)} \\ &\leq \frac{C}{|\lambda|} \|s\|_{L^p(\Omega)} \end{aligned}$$

for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$. We use the operator P_{av} is a bounded linear operator and combine two estimates. We have that there exists $C > 0$ such that $\|u\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|} \|s\|_{L^p(\Omega)}$ for all $\lambda \in \Sigma_{\pi-\varepsilon,0} \cap B(0, 2M)$. Since we have already proved the resolvent estimate for $|\lambda| > M$, the resolvent estimate holds for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$. We skip estimates for higher order derivatives. □

Therefore we can conclude that the bidomain operator generates an analytic semigroup on $L^p(\Omega)$ for $p \in (1, \infty)$.

4 Time-periodic solutions for the bidomain equations in real interpolation spaces

In this section we consider the time-periodic problem for the bidomain equations. We use the abstract form (ABDE) not (BDE). The source term s is a time periodic function when $s_{i,e}$ is the time periodic functions with the same period. The problem we would like to consider is whether the solution u is a time periodic when the source term s is a periodic function. We answer the problem in real interpolation spaces.

The outline of this section is as follows. We construct the linear theory of the time-periodic solutions in real interpolation spaces in the next subsection 4.1. The theorem is motivated by the theorem of DaPrato-Grisvard's maximal regularity, which implies that the analyticity of the semigroup derives maximal regularity in real interpolation spaces. Our linear theory is a time-periodic version of their theorem. In the subsection 4.2, we show that the non-linear bidomain equations with FitzHugh-Nagumo type non-linearities has a unique time periodic solution near the stable solution $(u, w) = (0, 0)$ if the time-periodic force s is sufficiently small. The proof is standard Banach's fixed point theorem combined with the maximal regularity estimate and non-linear estimate.

We prepare some function spaces and notations. Let X be a Banach space and $-A$ be the generator of a bounded analytic semigroup e^{-tA} on X with domain $D(A)$. For $\theta \in (0,1)$ and $1 \le p < \infty$, we denote by $D_A(\theta, p)$ space defined as

$$D_A(\theta,p) := \left\{ x \in X : [x]_{\theta,p} := \left(\int_0^\infty \|t^{1-\theta} A e^{-tA} x\|_X^p \frac{dt}{t} \right)^{1/p} < \infty \right\}. \tag{13}$$

When equipped with the norm $\|x\|_{\theta,p} := \|x\| + [x]_{\theta,p}$, the space $D_A(\theta,p)$ becomes a Banach space. It is well-known that $D_A(\theta,p)$ coincides with the real interpolation space $(X, D(A))_{\theta,p}$ and that the respective norms are equivalent. For details and more on interpolation spaces we refer, e.g., to [19, 20]. If $0 \in \rho(A)$, then the real interpolation space norm is equivalent to the homogeneous norm $[\cdot]_{\theta,p}$, see [11, Corollary 6.5.5]. Consider in particular the bidomain operator $A = A_q$ in $X = L^q(\Omega)$ for $1 < q < \infty$. Then, following Amann [2, Theorem 5.2], the space $(X, D(A))_{\theta,p}$ can be characterized as

$$(L^q(\Omega), D(A))_{\theta,p} = B^{2\theta}_{q,p}(\Omega), \quad 1 \le p < \infty, \tag{14}$$

provided $2\theta \in (0, 1 + 1/q)$. Here $B^s_{q,p}(\Omega)$ denotes, as usual, the Besov space of order $s \ge 0$.

For $0 < T < \infty$, we define the solution space $\mathbb{E}_A^{\mathrm{per}}$ as

$$\mathbb{E}_A^{\mathrm{per}} := \{ u \in W^{1,p}(0,T; D_A(\theta,p)) \mid Au \in L^p(0,T; D_A(\theta,p)) \text{ and } u(0) = u(T) \}$$

with norm

$$\|u\|_{\mathbb{E}_A^{\mathrm{per}}} := \|u\|_{W^{1,p}(0,T;D_A(\theta,p))} + \|Au\|_{L^p(0,T;D_A(\theta,p))},$$

which corresponds to the data space

$$\mathbb{F}_A := L^p(0,T;D_A(\theta,p)).$$

On the other hand, the solution space for the gating variable w is defined as

$$\mathbb{E}_w^{\mathrm{per}} := \{w \in W^{1,p}(0,T;D_A(\theta,p)) \mid w(0) = w(T)\}.$$

Then, the solution space for the periodic bidomain system is defined as the product space

$$\mathbb{E} := \mathbb{E}_A^{\mathrm{per}} \times \mathbb{E}_w^{\mathrm{per}}.$$

Finally, for a Banach space X we denote by $\mathbb{B}^X(u^*, R)$ the closed ball in X with center $u^* \in X$ and radius $R > 0$, i.e.,

$$\mathbb{B}^X(u^*, R) := \{u \in X \mid \|u - u^*\|_X \leq R\}.$$

4.1 The linear theory

Before we state the linear theory for the time-periodic solutions, we write down the theorem by DaPrato-Grisvard for the initial value problem. Let X be a Banach space and $-\mathcal{A}$ be the generator of a bounded analytic semigroup $e^{-t\mathcal{A}}$ on X. Assume that $\theta \in (0,1)$, $1 \leq p < \infty$, and $0 < T < \infty$. Then, for $f \in L^p(0,T;D_{\mathcal{A}}(\theta,p))$ we consider

$$u(t) := \int_0^t e^{-(t-s)\mathcal{A}} f(s)ds, \qquad 0 < t < T. \tag{15}$$

Then, u is the unique mild solution to the abstract Cauchy problem

$$\begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & 0 < t < T \\ u(0) = 0. \end{cases} \tag{ACP}$$

The classical DaPrato and Grisvard theorem in [7] is the following maximal regularity estimate.

Proposition 4.1 ([7, DaPrato-Grisvard]). *Let $\theta \in (0,1)$, $1 \leq p < \infty$, and $0 < T < \infty$. Then there exists a constant $C > 0$ such that for all $f \in L^p(0,T;D_{\mathcal{A}}(\theta,p))$, the function u given by* (15) *satisfies $u(t) \in D(\mathcal{A})$ and the equation* (ACP) *for almost every $0 < t < T$ and*

$$\|\mathcal{A}u\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))} \leq C\|f\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

We next consider the time periodic version of this theorem. Let $f : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$ be a periodic function of period T. Then the periodic problem of (ACP) reads as

$$\begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & t \in \mathbb{R}, \\ u(t) = u(t+T), & t \in \mathbb{R}. \end{cases} \tag{PACP}$$

Formally, a candidate for a solution u of (PACP) is given by

$$u(t) := \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} f(s)ds. \tag{16}$$

The periodicity of u can be seen the calculation.

$$u(t+T) = \int_{-\infty}^{t+T} e^{-(t+T-s)\mathcal{A}} f(s)ds = \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} f(s+T)ds = u(t)$$

since $f(s+T) = f(s)$ for all s. We prepare the following lemma that, under certain assumptions on $\mathcal{A}$ and f, u is indeed well-defined, continuous and periodic.

Lemma 4.2. *Let $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ be a T-periodic function satisfying*

$$f_{|(0,T)} \in L^p(0,T; D_{\mathcal{A}}(\theta, p))$$

and assume that $0 \in \rho(\mathcal{A})$. Then, the function u defined by (16) is well-defined, satisfies $u \in C(\mathbb{R}; D_{\mathcal{A}}(\theta, p))$, and is T-periodic.

Since we have the representation formula of the solution, the proof is direct calculation from Fubini's theorem and the characterization of the real interpolation spaces. To show the boundedness of the integral of the equation (16), we had to assume $0 \in \rho(\mathcal{A})$, which implies the exponential decay of the semigroup $e^{-t\mathcal{A}}$.

The linear theory of the time periodic version of the DaPrato-Grisvard theorem is as follows.

Theorem 4.3. *Let X be a Banach space and $-\mathcal{A}$ be the generator of a bounded analytic semigroup on X with $0 \in \rho(\mathcal{A})$. Let $\theta \in (0,1)$, $1 \leq p < \infty$, and $0 < T < \infty$.*

Then there exists a constant $C > 0$ such that for all periodic functions $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ with $f_{|(0,T)} \in L^p(0,T; D_{\mathcal{A}}(\theta, p))$ the function u defined by (16) is the unique strong solution $u \in \mathbb{E}_{\mathcal{A}}^{\mathrm{per}}$ of (PACP) and it satisfies

$$\|u\|_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}} \leq C\|f\|_{L^p(0,T;D_{\mathcal{A}}(\theta,p))} \tag{17}$$

for some $C > 0$.

4.2 The non-linear theory

At first we state the general setting of the time periodic solutions for a semi-linear parabolic equations. After that we apply the general statement to the bidomain equations.

Let $-\mathcal{A}$ be the generator of a bounded analytic semigroup $e^{-t\mathcal{A}}$ on a Banach space X with the domain $D(\mathcal{A})$ and $0 \in \rho(\mathcal{A})$. For $T > 0$, $\theta \in (0,1)$, and $1 \leq p < \infty$ let

$f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ be periodic of period T with $f_{|(0,T)} \in L^p(0, T; D_{\mathcal{A}}(\theta, p))$. We are aiming for the strong solvability of

$$\begin{cases} u'(t) + \mathcal{A}u(t) = F[u](t) + f(t) & (t \in \mathbb{R}) \\ \quad u(t) = u(t+T) & (t \in \mathbb{R}) \end{cases} \tag{NACP}$$

under some smallness assumptions on f. The solution u will be constructed in the space of maximal regularity $\mathbb{E}_{\mathcal{A}}^{\text{per}}$. Let $\mathbb{B}_\rho := \mathbb{B}^{\mathbb{E}_{\mathcal{A}}^{\text{per}}}(0, \rho)$ for some $\rho > 0$. For the nonlinear term F, we make the following standard assumption.

Assumption N There exists $R > 0$ such that the nonlinear term F is a mapping from $\mathbb{B}_R$ into $\mathbb{F}_{\mathcal{A}}$ and satisfies

$$F \in C^1(\mathbb{B}_R; \mathbb{F}_{\mathcal{A}}), \quad F(0) = 0, \quad \text{and} \quad DF(0) = 0,$$

where $DF : \mathbb{B}_R \to \mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\text{per}}, \mathbb{F}_{\mathcal{A}})$ denotes the Fréchet derivative.

The following theorem proves existence and uniqueness of solutions to (NACP) in the class $\mathbb{E}_{\mathcal{A}}^{\text{per}}$ for small forcings f.

Theorem 4.4. *Let $T > 0$, $0 < \theta < 1$, $1 \le p < \infty$, and F and $R > 0$ subject to Assumption* (N). *Then there is a constant $r \le R$ and $c = c(T, \theta, p, r) > 0$ such that if $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ is T-periodic with $\|f\|_{\mathbb{F}_{\mathcal{A}}} \le c$, then there exists a unique solution $u : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ of* (NACP) *with the same period T and $u_{|(0,T)} \in \mathbb{B}_r$.*

Proof. Let $S : \mathbb{B}_R \to \mathbb{E}_{\mathcal{A}}^{\text{per}}, v \mapsto u_v$ be the solution operator of the linear equation

$$u_v'(t) + \mathcal{A}u_v(t) = F[v(t)] + f(t) \quad \text{in } (0, T)$$

with $u_v(0) = u_v(T)$. This is well-defined since $F[v] \in \mathbb{F}_{\mathcal{A}}$ by Assumption (N), so that, by the theorem 4.3, u_v uniquely exists and lies in $\mathbb{E}_{\mathcal{A}}^{\text{per}}$.

We prove that this solution operator is a contraction on $\mathbb{B}_r$ for some $r \le R$. Let $M > 0$ denote the infimum of all constants C satisfying (17). Choose $r > 0$ small enough such that

$$\sup_{w \in \mathbb{B}_r} \|DF[w]\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\text{per}}, \mathbb{F}_{\mathcal{A}})} \le \frac{1}{2M},$$

which is possible by Assumption (N). By virtue of (17) as well as the mean value theorem, estimate for any $v \in \mathbb{B}_r$ and f satisfying $\|f\|_{\mathbb{F}_{\mathcal{A}}} \le r/(2M) =: c$,

$$\|S(v)\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}} \le M(\|F[v]\|_{\mathbb{F}_{\mathcal{A}}} + \|f\|_{\mathbb{F}_{\mathcal{A}}}) \le M(\sup_{w \in \mathbb{B}_r} \|DF[w]\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\text{per}}, \mathbb{F}_{\mathcal{A}})} \|v\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}} + \|f\|_{\mathbb{F}_{\mathcal{A}}}) \le r.$$

So $S(\mathbb{B}_r) \subset \mathbb{B}_r$. Similarly, for any $v_1, v_2 \in \mathbb{B}_r$,

$$\|S(v_1) - S(v_2)\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}} \le M \sup_{w \in \mathbb{B}_r} \|DF[w]\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\text{per}}, \mathbb{F}_{\mathcal{A}})} \|v_1 - v_2\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}} \le \frac{1}{2} \|v_1 - v_2\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}}.$$

Consequently, the solution operator S is a contraction on $\mathbb{B}_r$ and the contraction mapping theorem is applicable. The solution to (NACP) is defined as follows. Let u be the unique fixed point of S. Since $Su = u$, u satisfies $u(0) = u(T)$ and thus can be extended periodically to the whole real line. This function solves (NACP). $\square$

We consider the bidomain equations (ABDE) with FItzHugh-Nagumo type non-linearities:

$$\begin{aligned} f(u,w) &= u(u-a)(u-1) + w = u^3 - (a+1)u^2 + au + w, \\ g(u,w) &= bw - cu, \end{aligned}$$

where $0 < a < 1$ and b, $c > 0$. We can rewrite the matrix form:

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} A+a & 1 \\ -c & b \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} s - u^3 - (a+1)u^2 \\ 0 \end{pmatrix}. \tag{18}$$

This is just semi-linear equation. Therefore it is enough to prove that the matrix operator generates an analytic semigroup on a Banach space and that the non-linear term satisfies the assumption (N). We set the base space $X := L^q(\Omega) \times B^{2\theta}_{q,p}(\Omega)$.

Lemma 4.5. *Let $1 \leq p < \infty$, $1 < q < \infty$ and $\theta \in (0,1)$. Let $-A$ be the bidomain operator in $L^q(\Omega)$. Then the matrix operator*

$$\mathcal{A} := \begin{pmatrix} A+a & 1 \\ -c & b \end{pmatrix}.$$

generates a bounded analytic semigroup $e^{-t\mathcal{A}}$ on X and $0 \in \rho(\mathcal{A})$.

In the following proposition we elaborate the conditions on p, q, and θ that ensure that the right hand side of (18) maps $\mathbb{E}_A$ into $\mathbb{F}_A$.

Proposition 4.6. *Let $1 \leq p < \infty$, $d < q < \infty$ satisfy $1/p + d/(2q) \leq 3/4$ and $\theta \in (0, 1/2)$ there exists a constant $C > 0$ such that*

$$\|u^j\|_{\mathbb{F}_A} \leq C\|u\|^j_{\mathbb{E}_A}$$

for all $u \in \mathbb{E}_A$ and $j = 2, 3$.

Proof. We recall the mixed derivative theorem

$$W^{1,p}(0,T; L^q(\Omega)) \cap L^p(0,T; W^{2,q}(\Omega)) \subset \bigcap_{0 \leq \sigma \leq 1} W^{\sigma,p}(0,T; W^{2(1-\sigma),q}(\Omega)).$$

By the continuous embedding $W^{1,q}(\Omega) \subset B^{2\theta}_{q,p}(\Omega)$, Hölder's inequality and the mixed derivative theorem, we obtain for $j \in \{2,3\}$

$$\|u^j\|_{L^p(0,T;D_A(\theta,p))} \leq C\|u\|^j_{L^{jp}(0,T;W^{1,jq}(\Omega))} \leq C\|u\|^j_{W^{\sigma,p}(0,T;W^{2(1-\sigma),q}(\Omega))}.$$

provided $\sigma \in [0,1]$ satisfies

$$\sigma - 1/p \geq -1/(jp), \quad \text{and} \quad 2(1-\sigma) - d/q \geq 1 - d/(jq).$$

The condition $1/p + d/(2q) \leq 3/4$ guarantees the existence of σ for $j \in \{2,3\}$. □

For the bidomain equations with FitzHugh-Nagumo model, we can apply the theorem 4.4 since we have already checked the assumptions of the theorem and (N). Therefore we have the following theorem.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded C^2-domain and suppose that assumptions (1) and (2) hold true. Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|s\|_{\mathbb{F}_A} < C(R)$, the equation (ABDE) with*

$$\begin{aligned} f(u,w) &= u(u-a)(u-1) + w = u^3 - (a+1)u^2 + au + w, \\ g(u,w) &= bw - cu, \end{aligned}$$

for some $0 < a < 1$ and b, $c > 0$, admits a unique T-periodic strong solution (u,w) with $(u,w)_{|(0,T)} \in \mathbb{B}^{\mathbb{E}}((0,0),R)$.

5 Bibliographical remarks

In the last section, we give some bibliographical remarks. In this paper we prove that the bidomain operator generates an analytic semigroup on L^p spaces for $1 < p < \infty$. Recently, Hieber-Prüss in [14] proved that the the bidomain operator A_q in $L_q(\Omega)$ for $1 < q < \infty$ admits a bounded $\mathcal{H}^\infty$ calculus with $\mathcal{H}^\infty$ angle 0. This property is stronger property than the maximal L^p-L^q regularity. Moreover the same authors in [13] proved the global existence and uniqueness of the L_p-L_q strong solution for the bidomain equations with FitzHugh-Nagumo nonlinearities when $s = 0$. For the inhomogeneous case of $s \neq 0$ will be considered in [16].

To get the time-periodic solutions for the bidomain equations, we had to assume the smallness conditions since we use Banach's fixed point theorem. By using Galerkin method, Brower's fixed point theorem and weak-strong uniqueness method based on the result [16], existence of the time-periodic solutions for arbitrary large forces s will be considered in [10].

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