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# The Cahn–Hilliard Equation with a New Class of Dynamic Boundary Conditions

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**Abstract**

In this note, we report a new class of dynamic boundary conditions for the Cahn–Hilliard equation introduced by Liu and Wu [29]. We sketch the derivation of the model and present some results on the well-posedness and long-time behavior.

**Keywords**: Cahn–Hilliard equation, dynamic boundary condition, energetic variational approach, well-posedness, long-time behavior.

**AMS Subject Classification**: 35K25, 35K60, 35B40, 35A15, 37E35, 49S05, 80A22

## 1 Introduction

The Cahn–Hilliard equation describes important qualitative features of binary mixtures related to phase separation processes [2,3,32]. In recent years, the Cahn–Hilliard equation and its variants have also been successfully applied in a wide variety of segregation-like phenomena in science, see for instance [24,27,28,33,35] and the references therein. Assume that $T \in (0, +\infty)$, $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$ and $\mathbf{n} = \mathbf{n}(x)$ is the unit outer normal vector on $\Gamma$. The classical Cahn–Hilliard equation can be written in the following form:

$$\begin{cases} \phi_t = \Delta\mu, & \text{in } \Omega \times (0,T), \\ \mu = -\Delta\phi + F'(\phi), & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

The phase function $\phi$ represents the the difference of local relative concentrations for the two components of the mixture and $\mu$ stands for the chemical potential that equals to the Fréchet derivative of certain bulk free energy

$$E^{\text{bulk}}(\phi) = \int_\Omega \frac{1}{2}|\nabla\phi|^2 + F(\phi)\,\mathrm{d}x. \tag{1.2}$$

$F'$ is the first derivative of the bulk potential $F$ that has a double-well structure. A typical thermodynamically relevant potential is the following logarithmic potential (see [3])

$$F(y) = \frac{\theta}{2}(1+y)\ln(1+y) + (1-y)\ln(1-y) - \frac{\theta_c}{2}y^2, \quad y \in (-1,1),$$

for some constants $0 < \theta < \theta_c$, where $\theta$ is the temperature of the system and $\theta_c$ is the critical temperature of phase separation. Besides, the above singular potential is very often approximated by regular ones, typically $F(y) = \frac{1}{4}(y^2-1)^2$, $y \in \mathbb{R}$.

Suitable boundary conditions should be taken into account for system (1.1) and the classical choices are homogeneous Neumann boundary conditions for both $\mu$ and $\phi$:

$$\partial_{\mathbf{n}}\mu = 0, \quad \text{on } \Gamma \times (0,T), \tag{1.3}$$

$$\partial_{\mathbf{n}}\phi = 0, \quad \text{on } \Gamma \times (0,T), \tag{1.4}$$

where $\partial_{\mathbf{n}}$ denotes the outward normal derivative on $\Gamma$. The no-flux boundary condition (1.3) guarantees the conservation of mass in the bulk

$$\int_\Omega \phi(t)\,\mathrm{d}x = \int_\Omega \phi(0)\,\mathrm{d}x, \quad \forall\, t \in [0,T],$$

while the second boundary condition (1.4) means that the diffused interface, which separates the two phases of the material, intersects the solid wall (i.e., the boundary $\Gamma$) at a perfect static contact angle of $\frac{\pi}{2}$. Another important consequence of (1.3) and (1.4) is that the bulk free energy $E^{\mathrm{bulk}}(\phi)$ is decreasing in time, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{\mathrm{bulk}}(\phi(t)) + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x = 0, \quad \forall\, t \in (0,T).$$

Recently, in order to describe the effective short-range interactions between the wall and both mixture components, suitable surface free energy functional was introduced into the system by physicists (see e.g., [13,14,26])

$$E^{\mathrm{surf}}(\phi) = \int_\Gamma \frac{\kappa}{2}|\nabla_\Gamma\phi|^2 + G(\phi)\,\mathrm{d}S, \tag{1.5}$$

where $\nabla_\Gamma$ stands for the tangential (surface) gradient operator defined on $\Gamma$ and $G$ is a certain surface potential function that characterizes the possible preferential attraction (or repulsion) of one of the components by the wall. The coefficient $\kappa \geq 0$ is related to the surface diffusion. When $\kappa = 0$, the model is closely related to the evolution of an interface in contact with the solid boundary, i.e., the well-known moving contact line problem [37]. To ensure that the Cahn–Hilliard system tends to minimize the excess surface energy, the so-called dynamic boundary conditions have been proposed in the literature. For instance, we refer to [26] for the following boundary condition

$$\phi_t - \kappa\Delta_\Gamma\phi + \partial_{\mathbf{n}}\phi + G'(\phi) = 0, \quad \text{on } \Gamma \times (0,T), \tag{1.6}$$

where $\Delta_\Gamma$ denotes the Laplace–Beltrami operator on $\Gamma$. The above dynamic boundary condition can be viewed as a relaxation dynamics ($L^2$-gradient flow) of the surface free energy $E^{\text{surf}}$ on $\Gamma$ (together with the bulk interaction given by $\partial_{\mathbf{n}}\phi$), and it guarantees the energy dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[E^{\text{bulk}}(\phi(t)) + E^{\text{surf}}(\phi(t))\Big] + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x + \int_\Gamma |\phi_t|^2\,\mathrm{d}S = 0, \quad \forall\, t \in (0,T). \tag{1.7}$$

In the case of nonpermeable walls, it would seem more reasonable to write the conservation of mass both in the bulk and on the boundary such that

$$\int_\Omega \phi(t)\,\mathrm{d}x + \int_\Gamma \phi(t)\,\mathrm{d}S = \int_\Omega \phi(0)\,\mathrm{d}x + \int_\Gamma \phi(0)\,\mathrm{d}S, \quad \forall\, t \in [0,T].$$

Then a different type of dynamic boundary condition was derived in [23] (see also [7]),

$$\phi_t - \sigma\Delta_\Gamma\mu + \partial_{\mathbf{n}}\mu = 0, \qquad \text{on } \Gamma\times(0,T), \tag{1.8}$$

$$\mu = -\kappa\Delta_\Gamma\phi + \partial_{\mathbf{n}}\phi + G'(\phi), \qquad \text{on } \Gamma\times(0,T), \tag{1.9}$$

for some $\sigma \geq 0$, $\kappa \geq 0$. When $\sigma = 0$, it reduces to the so-called Wentzell boundary conditions introduced in [15] for permeable walls. Besides, from (1.8) and (1.9), it follows that the total energy of the Cahn–Hilliard system (1.1) still decreases in time (compare the relation (1.7)):

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[E^{\text{bulk}}(\phi(t)) + E^{\text{surf}}(\phi(t))\Big] + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x + \sigma\int_\Gamma |\nabla_\Gamma\mu|^2\,\mathrm{d}S = 0, \quad \forall\, t \in (0,T). \tag{1.10}$$

We see that under both choices of boundary conditions ((1.3) with (1.6), or (1.8) with (1.9)), the Cahn–Hilliard system (1.1) satisfies two important physical constraints, namely, the *mass conservation* and *energy dissipation*. Among them, (1.3), or respectively (1.8) is proposed to keep suitable mass conservation property in the physical domain, while the so-called variational boundary conditions (i.e., (1.6) or (1.9)) are chosen in a phenomenological way so that the validity of some specific energy dissipation relation is guaranteed (see (1.7) or (1.10)). Thus, (1.6) or (1.9) can be viewed as sufficient conditions for the energy dissipation of the system. However, as we have seen, such choice may not be unique.

Concerning mathematical analysis of the Cahn–Hilliard equation (1.1) subject to the dynamic boundary conditions (1.3) with (1.6), or (1.8) with (1.9), we refer to the recent works [4–12, 15–23, 25, 30, 31, 34, 36, 38, 39] and the references cited therein.

In the rest of this note, we first report a new type of dynamic boundary condition recently derived in [29] based on an energetic variational approach that combines the least action principle and Onsager's principle of maximum energy dissipation:

$$\begin{cases} \partial_{\mathbf{n}}\mu = 0, & \text{on } \Gamma \times (0,T), \\ \phi_t = \Delta_\Gamma(-\kappa\Delta_\Gamma\phi + \partial_{\mathbf{n}}\phi + G'(\phi)), & \text{on } \Gamma \times (0,T). \end{cases} \tag{1.11}$$

It is worth mentioning that (1.11) enjoys an interesting feature such that it naturally fulfills three important physical properties from its derivation:

- Kinematics: conservation of mass both in the bulk $\Omega$ and on the boundary $\Gamma$;
- Energetics: dissipation of the total free energy;
- Force balance: both in the bulk $\Omega$ and on the boundary $\Gamma$.

Then we present some results on well-posedness and long-time behavior of the Cahn–Hilliard system (1.1) subject to the dynamic boundary condition (1.11) under suitable assumptions on the nonlinearities $F$, $G$.

## 2 Model Derivation

In this section, we sketch the derivation of dynamic boundary conditions (1.11) using an energetic variational approach. For details we refer to our recent paper [29].

In the bulk $\Omega$, $\phi$ is assumed to be a locally conserved quantity that satisfies the continuity equation

$$\phi_t + \nabla \cdot (\phi \mathbf{u}) = 0, \quad (x,t) \in \Omega \times (0,T), \tag{2.1}$$

where $\mathbf{u} : \Omega \to \mathbb{R}^d$ stands for the microscopic effective velocity (e.g., due to diffusion process etc). We assume that $\mathbf{u}$ satisfies the no-flux boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad (x,t) \in \Gamma \times (0,T). \tag{2.2}$$

Next, we assume that the boundary dynamics is characterized by a local mass conservation law analogous to (2.1) such that

$$\phi_t + \nabla_\Gamma \cdot (\phi \mathbf{v}) = 0, \quad (x,t) \in \Gamma \times (0,T), \tag{2.3}$$

where $\mathbf{v} : \Gamma \to \mathbb{R}^d$ denotes the microscopic effective tangential velocity field on the boundary $\Gamma$. We note that there is no need to impose any boundary condition on $\mathbf{v}$, since here the boundary $\Gamma$ is assumed to be a closed manifold.

For an isothermal closed system, evolution of the binary mixtures satisfy the following energy dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t} E^{\text{total}}(t) = -\mathcal{D}^{\text{total}}(t), \quad t > 0. \tag{2.4}$$

Here we take

$$E^{\text{total}}(t) = E^{\text{bulk}}(t) + E^{\text{surf}}(t), \tag{2.5}$$

where $E^{\text{bulk}}$ and $E^{\text{surf}}$ are given by (1.2) and (1.5), respectively. On the other hand, the rate of energy dissipation $\mathcal{D}^{\text{total}}$ is chosen as

$$\mathcal{D}^{\text{total}}(t) = \mathcal{D}^{\text{bulk}}(t) + \mathcal{D}^{\text{surf}}(t), \tag{2.6}$$

which also consists of contributions from the bulk and the surface. Here, we assume that

$$\mathcal{D}^{\text{bulk}}(t) = \int_\Omega M_{\text{b}}^{-1}\phi^2\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x, \quad \mathcal{D}^{\text{surf}}(t) = \int_\Gamma M_{\text{s}}^{-1}\phi^2\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}S, \tag{2.7}$$

where $M_{\text{b}}$, $M_{\text{s}}$ are positive constants (mobility coefficients).

In order to derive a closed system of partial differential equations, it remains to determine the microscopic velocities $\mathbf{u}$, $\mathbf{v}$ in equations (2.1) and (2.3). Using the principle of least action (LAP) and Onsager's maximum dissipation principle (MDP), we are able to derive the conservative and dissipative forces according to the free energy (2.5) and the dissipation functional (2.6) (see [29] for details). Then by the force balance relation (Newton's second law), we obtain

$$\begin{cases} \phi\nabla\mu_{\text{b}} + M_{\text{b}}^{-1}\phi^2\mathbf{u} = 0, & \text{in } \Omega\times(0,T), \\ \phi\nabla_\Gamma(\mu_{\text{s}} + \partial_{\mathbf{n}}\phi) + M_{\text{s}}^{-1}\phi^2\mathbf{v} = 0, & \text{on } \Gamma\times(0,T), \end{cases} \tag{2.8}$$

where

$$\mu_{\text{b}} = -\Delta\phi + F'(\phi), \quad \mu_{\text{s}} = -\kappa\Delta_\Gamma\phi + \partial_{\mathbf{n}}\phi + G'(\phi).$$

At last, solving $\mathbf{u}$, $\mathbf{v}$ from (2.8) and inserting them back into (2.1), (2.3), we arrive at the Cahn–Hilliard system subject to a new class of dynamic boundary condition:

$$\begin{cases} \phi_t = \Delta(-\Delta\phi + F'(\phi)), & \text{in } \Omega\times(0,T), \\ \partial_{\mathbf{n}}\mu = 0, & \text{on } \Gamma\times(0,T), \\ \phi_t = \Delta_\Gamma(-\kappa\Delta_\Gamma\phi + \partial_{\mathbf{n}}\phi + G'(\phi)), & \text{on } \Gamma\times(0,T), \\ \phi|_{t=0} = \phi_0(x), & \text{in } \Omega. \end{cases} \tag{2.9}$$

**Remark 2.1.** *More general system can be derived via the same approach. For instance, we consider*

$$E^{bulk}(t) = \int_\Omega W_b(\phi,\nabla\phi)\,\mathrm{d}x, \quad E^{surf}(t) = \int_\Gamma W_s(\phi,\nabla_\Gamma\phi)\,\mathrm{d}S,$$

*where the energy density functions $W_b$ and $W_s$ are scalar functions that can take different forms under various physical considerations. Besides, the constant mobilities $M_b$, $M_s$ can*

*be replaced by $d \times d$ matrices $\mathbb{M}_b$, $\mathbb{M}_s$ that are symmetric and positive definite, i.e., we consider dissipation functionals as*

$$\mathcal{D}^{bulk}(t) = \int_\Omega \phi^2(\mathbb{M}_b^{-1}\mathbf{u}) \cdot \mathbf{u}\,\mathrm{d}x, \quad \mathcal{D}^{surf}(t) = \int_\Gamma \phi^2(\mathbb{M}_s^{-1}\mathbf{v}) \cdot \mathbf{v}\,\mathrm{d}S.$$

*Under the above choices, we can derive the following general system with dynamic boundary condition (see [29] for details):*

$$\begin{cases} \phi_t = \nabla \cdot (\mathbb{M}_b \nabla \mu_b), & \text{in } \Omega \times (0,T), \\ \mu_b = -\nabla \cdot \dfrac{\partial W_b}{\partial \nabla \phi} + \dfrac{\partial W_b}{\partial \phi}, & \text{in } \Omega \times (0,T), \\ \partial_{\mathbf{n}} \mu_b = 0, & \text{on } \Gamma \times (0,T), \\ \phi_t = \nabla_\Gamma \cdot \left[ \mathbb{M}_s \nabla_\Gamma \left( \mu_s + \dfrac{\partial W_b}{\partial \nabla \phi} \cdot \mathbf{n} \right) \right], & \text{on } \Gamma \times (0,T), \\ \mu_s = -\nabla_\Gamma \cdot \dfrac{\partial W_s}{\partial \nabla_\Gamma \phi} + \dfrac{\partial W_s}{\partial \phi}, & \text{on } \Gamma \times (0,T), \\ \phi|_{t=0} = \phi_0(x), & \text{in } \Omega, \end{cases} \tag{2.10}$$

*where $T \in (0, +\infty)$.*

# 3 Mathematical Analysis

Inspired by [30], it will be more convenient to view the trace of the order parameter $\phi$ as an unknown function on the boundary $\Gamma$. After introducing the new variable

$$\psi := \phi|_\Gamma,$$

the initial boundary value problem of the Cahn–Hilliard system (2.9) can be written in the following form:

$$\begin{cases} \phi_t = \Delta \mu, \quad \text{with } \mu = -\Delta \phi + F'(\phi), & \text{in } \Omega \times (0,T), \\ \partial_{\mathbf{n}} \mu = 0, & \text{on } \Gamma \times (0,T), \\ \phi|_\Gamma = \psi, & \text{on } \Gamma \times (0,T), \\ \psi_t = \Delta_\Gamma \mu_\Gamma, & \\ \quad \text{with } \mu_\Gamma = -\kappa \Delta_\Gamma \psi + \psi + \partial_{\mathbf{n}} \phi + G'(\psi), & \text{on } \Gamma \times (0,T), \\ \phi|_{t=0} = \phi_0(x), & \text{in } \Omega, \\ \psi|_{t=0} = \psi_0(x) := \phi_0(x)|_\Gamma, & \text{on } \Gamma. \end{cases} \tag{3.1}$$

In the remaining part of this section, we assume that the nonlinearities $F$, $G$ satisfy the following basic assumptions.

**(A1)** $F, G \in C^4(\mathbb{R})$.

(**A2**) There exist nonnegative constants $C_F$, $\widetilde{C}_F$, $C_G$, $\widetilde{C}_G \geq 0$ independent of $y \in \mathbb{R}$ such that

$$F(y) \geq -C_F, \quad F''(y) \geq -\widetilde{C}_F, \quad G(y) \geq -C_G, \quad G''(y) \geq -\widetilde{C}_G.$$

**Remark 3.1.** *Assumption (**A2**) can be regarded as a dissipative condition that guarantees the existence of global weak/strong solutions. It implies that both $F$ and $G$ are quadratic perturbations of some strictly convex functions $\widetilde{F}$, $\widetilde{G}$, for instance, we can put*

$$\widetilde{F}(y) = F(y) + \frac{\widetilde{C}_F + 1}{2} y^2 - F'(0)y - F(0), \quad \forall\, y \in \mathbb{R},$$

$$\widetilde{G}(y) = G(y) + \frac{\widetilde{C}_G + 1}{2} y^2 - G'(0)y - G(0), \quad \forall\, y \in \mathbb{R}.$$

When global weak solutions are concerned, we impose the following (subcritical) growth condition:

(**A3**) there exist positive constants $\widehat{C}_F$, $\widehat{C}_G > 0$ independent of $y \in \mathbb{R}$ such that

$$|F''(y)| \leq \widehat{C}_F(1 + |y|^p), \quad |G''(y)| \leq \widehat{C}_G(1 + |y|^q), \quad \forall\, y \in \mathbb{R},$$

where the exponents $p$, $q \in [0, +\infty)$ are fixed numbers such that
(i) when $\kappa > 0$, $p, q$ are arbitrary if $d = 2$; $p = 2$, $q$ is arbitrary if $d = 3$;
(ii) when $\kappa = 0$, $p$ is arbitrary if $d = 2$ and $p = 2$ if $d = 3$; $q = 0$ for $d = 2, 3$.

When global strong solutions for the case with surface diffusion (i.e., $\kappa > 0$) is concerned, the growth condition (**A3**) can be replaced by an alternative assumption:

(**A4**) there exist some positive constants $\rho_1$, $\rho_2$ such that

$$|\widetilde{F}'(y)| \leq \rho_1 |\widetilde{G}'(y)| + \rho_2, \quad \forall\, y \in \mathbb{R},$$

where $\widetilde{F}$ and $\widetilde{G}$ are defined in Remark 3.1.

**Remark 3.2.** *(1) It is easy to verify that the classical double well potential $F(y) = \frac{1}{4}(y^2 - 1)^2$ satisfies (**A1**)-(**A3**).*

*(2) According to (**A1**)-(**A3**), the surface potential function $G$ is allowed to be a polynomial of even degree with a positive leading coefficient provided that $\kappa > 0$. When $\kappa = 0$, it covers the typical form of the fluid-solid interfacial free energy in the moving contact line problem (see [35, Section 4]).*

*(3) Assumption (**A4**) removes the restriction on growth of potential functions $F$ and $G$. Instead it requires that the boundary potential $G$ plays a dominating role. This assumption can be viewed as a compatibility condition between $F$ and $G$, and it can be further extended to the case with physical relevant singular potentials (cf. [10, 11]). A similar assumption from the opposite side such that the bulk potential $F$ is dominative can be found in [21, (2.35)-(2.36)].*

## 3.1 Preliminaries

We introduce the product space

$$\mathcal{H} = L^2(\Omega) \times L^2(\Gamma),$$

which is a Hilbert space that can be viewed as the completion of $C^0(\overline{\Omega})$ with respect to the norm

$$\|(\phi, \psi)\|_{\mathcal{H}}^2 = \int_\Omega |\phi|^2 \, \mathrm{d}x + \int_\Gamma |\psi|^2 \, \mathrm{d}S, \quad \forall\, (\phi, \psi) \in \mathcal{H}. \tag{3.2}$$

We note that any element $h = (\phi, \psi) \in \mathcal{H}$ will be thought as a pair of functions belonging, respectively, to $L^2(\Omega)$ and to $L^2(\Gamma)$. If no additional regularity is imposed, the second component of $h$ (i.e., $\psi$) is not necessary to be the trace of the first one (i.e., $\phi$). We recall that the Dirichlet trace map $\gamma : \{\phi|_{\overline{\Omega}} : \ \phi \in C^\infty(\mathbb{R}^d)\} \to C^\infty(\Gamma)$, defined by $\gamma\phi = \phi|_\Gamma$, extends to a linear continuous operator $\gamma : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$, for all $s > \frac{1}{2}$. In the following text, we shall always use the notion $\phi|_\Gamma$ to indicate the trace operator defined in a suitable sense. Thus, when we consider a function $\phi \in H^s(\Omega)$ (with $s > \frac{1}{2}$), the symbol $\phi$ will be intended, as a pair $(\phi, \psi)$ formed by the function $\phi$ in $\Omega$ and its trace $\psi := \phi|_\Gamma$ on $\Gamma$. In this context, we introduce the notions

$$V^s = \big\{(\phi, \psi) \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\Gamma) : \ \psi = \phi|_\Gamma\big\}, \quad \forall\, s > \frac{1}{2},$$

with the equivalent norm given by $\|\phi\|_{H^s(\Omega)}$. Besides, we denote

$$\mathcal{V}^s = \big\{(\phi, \psi) \in H^s(\Omega) \times H^s(\Gamma) : \ \psi = \phi|_\Gamma\big\}, \quad \forall\, s > \frac{1}{2},$$

with the induced graph norm given by $\|(\phi, \psi)\|_{\mathcal{V}^s}^2 = \|\phi\|_{H^s(\Omega)}^2 + \|\psi\|_{H^s(\Gamma)}^2$. We note that $\mathcal{V}^s$ can be identified with a closed subspace of the product space $H^s(\Omega) \times H^s(\Gamma)$ and for any $s_1 > s_2 > \frac{1}{2}$, the dense and compact embeddings $\mathcal{V}^{s_1} \hookrightarrow \mathcal{V}^{s_2}$ hold. For later convenience, we set, for a parameter $\kappa \geq 0$

$$\mathbb{V}_\kappa^s := \mathcal{V}^s \ \text{ if } \kappa > 0, \qquad \mathbb{V}_\kappa^s := V^s \ \text{ if } \kappa = 0.$$

We see that for any fixed $\kappa \geq 0$, $\mathbb{V}_\kappa^1$ is a Hilbert space, which can be viewed as the completion of $C^1(\overline{\Omega})$ with respect to the following equivalent norm

$$\|(\phi, \psi)\|_{\mathbb{V}_\kappa^1}^2 = \int_\Omega |\nabla\phi|^2 \, \mathrm{d}x + \int_\Gamma (\kappa|\nabla_\Gamma\psi|^2 + |\psi|^2) \, \mathrm{d}S, \quad \forall\, (\phi, \psi) \in \mathbb{V}_\kappa^1. \tag{3.3}$$

For every $g \in (H^1(\Omega))^*$ (resp. $g \in (H^1(\Gamma))^*$), we denote by $\langle g \rangle_\Omega$ (resp. $\langle g \rangle_\Gamma$) the generalized average of $g$ over $\Omega$ (resp. $\Gamma$) such that

$$\langle g \rangle_\Omega = \frac{1}{|\Omega|} \langle g, 1 \rangle_{(H^1(\Omega))^*, H^1(\Omega)}, \quad \langle g \rangle_\Gamma = \frac{1}{|\Gamma|} \langle g, 1 \rangle_{(H^1(\Gamma))^*, H^1(\Gamma)}.$$

If $g \in L^2(\Omega)$ (resp. $g \in L^2(\Gamma)$), the above mean values simply reduce to

$$\langle g \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega g \, \mathrm{d}x, \quad \langle g \rangle_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma g \, \mathrm{d}S.$$

Next, we introduce the notion of (global) weak/strong solutions to problem (3.1):

**Definition 3.1.** *Let $\kappa \geq 0$, $T \in (0, +\infty)$. For any initial data $(\phi_0, \psi_0) \in \mathcal{V}^1$, a pair $(\phi, \psi)$ is called a weak solution to problem (3.1) on $[0, T]$, if it satisfies*

$$\begin{aligned}
&(\phi, \psi) \in L^\infty(0, T; \mathbb{V}_\kappa^1) \cap L^2(0, T; \mathbb{V}_\kappa^r), \\
&(\mu, \mu_\Gamma) \in L^2(0, T; H^1(\Omega) \times H^1(\Gamma)), \\
&(\phi_t, \psi_t) \in L^2(0, T; (H^1(\Omega))^* \times (H^1(\Gamma))^*),
\end{aligned}$$

*with $r = 3$ if $\kappa > 0$ and $r = \frac{5}{2}$ if $\kappa = 0$. The following weak formulations are satisfied*

$$\langle \phi_t(t), \zeta \rangle_{(H^1(\Omega))^*, H^1(\Omega)} + \int_\Omega \nabla \mu(t) \cdot \nabla \zeta \, \mathrm{d}x = 0, \tag{3.4}$$

$$\langle \psi_t(t), \eta \rangle_{(H^1(\Gamma))^*, H^1(\Gamma)} + \int_\Gamma \nabla_\Gamma \mu_\Gamma(t) \cdot \nabla_\Gamma \eta \, \mathrm{d}S = 0, \tag{3.5}$$

*for every $\zeta \in H^1(\Omega)$ and $\eta \in H^1(\Gamma)$ and almost every $t \in (0, T)$, with*

$$\mu = -\Delta \phi + F'(\phi), \qquad \text{a.e. in } \Omega \times (0, T), \tag{3.6}$$

$$\mu_\Gamma = -\kappa \Delta_\Gamma \psi + \psi + \partial_{\mathbf{n}} \phi + G'(\psi), \quad \text{a.e. on } \Gamma \times (0, T). \tag{3.7}$$

*Besides, the initial conditions are fulfilled*

$$\phi|_{t=0} = \phi_0(x) \quad \text{in } \Omega, \quad \text{and} \quad \psi|_{t=0} = \psi_0(x) \quad \text{on } \Gamma, \tag{3.8}$$

*and the bulk/surface mass conservation properties hold*

$$\langle \phi(t) \rangle_\Omega = \langle \phi_0 \rangle_\Omega, \quad \langle \psi(t) \rangle_\Gamma = \langle \psi_0 \rangle_\Gamma, \quad \forall\, t \in [0, T]. \tag{3.9}$$

*Moreover, for $t \in [0, T]$, $(\phi, \psi)$ satisfies the energy inequality*

$$E(\phi(t), \psi(t)) + \int_0^t \left( \|\nabla \mu(\tau)\|_{L^2(\Omega)}^2 + \|\nabla_\Gamma \mu_\Gamma(\tau)\|_{L^2(\Gamma)}^2 \right) \mathrm{d}\tau \leq E(\phi_0, \psi_0),$$

*where $E$ is defined by*

$$\begin{aligned}
E(\phi, \psi) = &\frac{1}{2} \int_\Omega |\nabla \phi|^2 \, \mathrm{d}x + \frac{1}{2} \int_\Gamma (\kappa |\nabla_\Gamma \psi|^2 + |\psi|^2) \, \mathrm{d}S \\
&+ \int_\Omega F(\phi) \, \mathrm{d}x + \int_\Gamma G(\psi) \, \mathrm{d}S.
\end{aligned} \tag{3.10}$$

**Definition 3.2.** *Let $\kappa \geq 0$, $T \in (0, +\infty)$. For any initial datum $(\phi_0, \psi_0) \in \mathcal{V}^3$, a pair $(\phi, \psi)$ is called a strong solution to problem (3.1) on $[0, T]$, if it satisfies the additional regularity*

$$
\begin{aligned}
&(\phi, \psi) \in L^\infty(0, T; \mathbb{V}_\kappa^{r_1}) \cap L^2(0, T; \mathbb{V}_\kappa^{r_2}),\\
&\mu \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)),\\
&\mu_\Gamma \in L^\infty(0, T; H^1(\Gamma)) \cap L^2(0, T; H^{r_3}(\Gamma)),\\
&(\phi_t, \psi_t) \in L^\infty(0, T; (H^1(\Omega))^* \times (H^1(\Gamma))^*) \cap L^2(0, T; \mathbb{V}_\kappa^1),
\end{aligned}
$$

*with $r_1 = 3$, $r_2 = 5$, $r_3 = 3$ if $\kappa > 0$ and $r_1 = \frac{5}{2}$, $r_2 = \frac{7}{2}$, $r_3 = 2$ if $\kappa = 0$. The equations and boundary conditions in problem (3.1) are satisfied a.e. in $\Omega \times (0, T)$ and on $\Gamma \times (0, T)$, respectively. The initial conditions and the mass conservation (3.9) hold as well. Besides, the following energy identity is satisfied*

$$
\frac{\mathrm{d}}{\mathrm{d}t} E(\phi(t), \psi(t)) + \|\nabla \mu\|_{L^2(\Omega)}^2 + \|\nabla_\Gamma \mu_\Gamma\|_{L^2(\Gamma)}^2 = 0, \quad \text{for a.e. } t \in (0, T).
$$

## 3.2 Main results

We are in a position to state the results on well-posedness and long-time behavior for problem (3.1) that have been proven in [29].

**Theorem 3.1** (Global weak/strong solutions for $\kappa > 0$). *Let $\kappa > 0$, $T \in (0, +\infty)$, $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with smooth boundary $\Gamma$.*

*(1) If assumptions **(A1)**–**(A3)** are satisfied, then for any initial datum $(\phi_0, \psi_0) \in \mathcal{V}^1$, problem (3.1) admits a unique global weak solution $(\phi, \psi)$ on $[0, T]$.*

*(2) If assumptions **(A1)**, **(A2)** are satisfied, besides, either **(A3)** or **(A4)** is fulfilled, then for any initial datum $(\phi_0, \psi_0) \in \mathcal{V}^3$, problem (3.1) admits a unique global strong solution $(\phi, \psi)$ on $[0, T]$.*

**Theorem 3.2** (Global weak/strong solutions for $\kappa = 0$). *Let $\kappa = 0$, $T \in (0, +\infty)$ and assumptions **(A1)**–**(A3)** be satisfied. Suppose that $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain with smooth boundary $\Gamma$ satisfying the condition*

$$
c_{\mathcal{R}} |\Gamma|^{\frac{1}{2}} |\Omega|^{-1} < 1, \tag{3.11}
$$

*where $c_{\mathcal{R}} > 0$ is the constant given in the inverse trace theorem. For any initial datum $(\phi_0, \psi_0) \in \mathcal{V}^1$ (resp. $(\phi_0, \psi_0) \in \mathcal{V}^3$), problem (3.1) admits a unique global weak (resp. strong) solution $(\phi, \psi)$ on $[0, T]$.*

Concerning the long-time behavior of problem (3.1), we need the following additional assumption:

**(AN)** $F$ and $G$ are real analytic functions on $\mathbb{R}$.

Then we have

**Theorem 3.3** (Uniqueness of asymptotic limit as $t \to +\infty$). *Suppose that the assumptions of Theorem 3.1 or Theorem 3.2 are satisfied. In addition, we assume (**AN**). Then for any initial datum $(\phi_0, \psi_0) \in \mathcal{V}^1$ (or $\in \mathcal{V}^3$), the corresponding global weak (or strong) solution $(\phi, \psi)$ of problem (3.1) converges to a single equilibrium $(\phi_\infty, \psi_\infty)$ as time goes to infinity such that*

$$\lim_{t\to+\infty} \|(\phi(t), \psi(t)) - (\phi_\infty, \psi_\infty)\|_{\mathbb{V}_\kappa^{r-\epsilon}} = 0, \tag{3.12}$$

*for any $\epsilon \in (0,1)$, where $r = 3$ if $\kappa > 0$, $r = \frac{5}{2}$ if $\kappa = 0$, $(\phi_\infty, \psi_\infty) \in \mathbb{V}_\kappa^r$ satisfies the following nonlinear nonlocal elliptic problem*

$$\begin{cases} -\Delta\phi_\infty + F'(\phi_\infty) = \lambda_1, & \text{in } \Omega, \\ -\kappa\Delta_\Gamma\psi_\infty + \psi_\infty + \partial_{\mathbf{n}}\phi_\infty + G'(\psi_\infty) = \lambda_2, & \text{on } \Gamma, \\ \quad \text{with } \ \langle\phi_\infty\rangle_\Omega = \langle\phi_0\rangle_\Omega, \quad \langle\psi_\infty\rangle_\Gamma = \langle\psi_0\rangle_\Gamma, \end{cases} \tag{3.13}$$

*with the constants $\lambda_1, \lambda_2$ given by*

$$\begin{cases} \lambda_1 = -|\Omega|^{-1}|\Gamma|\langle\partial_{\mathbf{n}}\phi_\infty\rangle_\Gamma + \langle F'(\phi_\infty)\rangle_\Omega, \\ \lambda_2 = \langle\partial_{\mathbf{n}}\phi_\infty\rangle_\Gamma + \langle\psi_\infty\rangle_\Gamma + \langle G'(\psi_\infty)\rangle_\Gamma. \end{cases} \tag{3.14}$$

*Moreover, we have the following estimate on convergence rate*

$$\|(\phi(t), \psi(t)) - (\phi_\infty, \psi_\infty)\|_{\mathbb{V}_\kappa^1} \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \ge 0, \tag{3.15}$$

*where the constant $\theta \in (0, \frac{1}{2})$ depends on $(\phi_\infty, \psi_\infty)$ and $C$ is a positive constant depending on $\|(\phi_0, \psi_0)\|_{\mathcal{V}^1}$, $\|(\phi_\infty, \psi_\infty)\|_{\mathbb{V}_\kappa^1}$, $\Omega$, $\Gamma$ and $\kappa$.*

Finally, for any given $\kappa > 0$ we are able to characterize the stability for local energy minimizers $(\phi^*, \psi^*)$ of $E(\phi, \psi)$ over the set

$$\mathcal{K}_1 = \{(\phi, \psi) \in \mathcal{V}^1 : \ \langle\phi\rangle_\Omega = \langle\hat{\phi}\rangle_\Omega, \quad \langle\psi\rangle_\Gamma = \langle\hat{\psi}\rangle_\Gamma\}, \tag{3.16}$$

where the pair $(\hat{\phi}, \hat{\psi}) \in \mathcal{V}^1$ is arbitrary but fixed. Namely,

$$E(\phi^*, \psi^*) = \inf\{E(\phi, \psi) : \ (\phi, \psi) \in \mathcal{K}_1 \cap \mathbf{B}_{\mathcal{V}^1}((\phi^*, \psi^*); \sigma)\}, \quad \text{for some } \sigma > 0.$$

Here, $\mathbf{B}_{\mathcal{V}^1}((\phi, \psi); \sigma)$ denotes the open ball in $\mathcal{V}^1$ with radius $\sigma$ centered at $(\phi, \psi)$.

**Theorem 3.4** (Stability criterion). *Suppose that $\kappa > 0$, $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain with smooth boundary $\Gamma$, moreover, the assumptions (**A2**), (**A3**) and (**AN**) are satisfied.*

*(1) Let $(\phi^*, \psi^*) \in \mathcal{K}_1$ be a local energy minimizer of $E(\phi, \psi)$ over the set $\mathcal{K}_1$. Then $(\phi^*, \psi^*)$ is Lyapunov stable. Namely, for any $\epsilon > 0$, there exists $\delta \in (0, \sigma)$ such that if*

*the initial datum* $(\phi_0, \psi_0) \in \mathcal{K}_1$ *satisfies* $\|(\phi_0, \psi_0) - (\phi^*, \psi^*)\|_{\mathcal{V}^1} < \delta$, *then the global weak solution* $(\phi, \psi)$ *to problem* (3.1) *(cf. Theorem 3.1) satisfies*

$$\|(\phi(t), \psi(t)) - (\phi^*, \psi^*)\|_{\mathcal{V}^1} < \epsilon, \quad \forall\, t \geq 0.$$

*(2) Let* $(\phi^*, \psi^*) \in \mathcal{K}_1$ *be a stationary point that is a weak solution of the stationary problem* (3.13)–(3.14) *(where the mass constraints are changed corresponding to the definition of the set* $\mathcal{K}_1$*). If* $(\phi^*, \psi^*)$ *does not attain any local minimum of* $E(\phi, \psi)$ *over* $\mathcal{K}_1$, *then* $(\phi^*, \psi^*)$ *is not Lyapunov stable.*

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