数理解析研究所講究録 第2094巻 2018年 44-54

Several numerical methods for the Navier-Stokes problem with Signorini's boundary condition

Guanyu Zhou[1], Takahito Kashiwabara[2], Norikazu Saito[2] and Yoshiki Sugitani[3]
[1]Department of Applied Mathematics, Tokyo University of Science
[2]The Graduate School of Mathematical Sciences, The University of Tokyo
[3]Tohoku University Advanced Institute for Materials Research(AIMR)

Abstract

We propose the outflow boundary condition of Signorini's type to ensure the energy inequality for the Navier-Stokes problem with an open outflow boundary. Since the Signorini's boundary condition involves the variational inequality, we propose several efficient numerical schemes to obtain the stable discrete solutions, including the penalty method and Lagrange multiplier approach. To evaluate the appropriateness of this artificial boundary condition, we carry out the numerical simulations using our schemes, and compare the simulation results with the data from physical experiments.

1 Introduction

The Navier-Stokes equation has been applied to investigate the blood flow in aorta (a pipe-shaped domain, see Figure 1), where the inflow velocity is specified and the non-slip boundary condition is imposed on the blood vessel wall. Let Ω be the computational domain with boundary $\Gamma = \Gamma_{in} \cup \Gamma_0 \cup \Gamma_{out}$, where Γ_{in}, Γ_0 and Γ_{out} denotes the inflow, wall and outflow boundaries, respectively. The Navier-Stokes problem is stated as follows.

$$
\begin{aligned}
&u_t + (u \cdot \nabla)u - \nabla \cdot \sigma(u,p) = f && \text{in } \Omega \times (0,T), && (1.1a)\\
&\nabla \cdot u = 0 && \text{in } \Omega \times (0,T), && (1.1b)\\
&u = g && \text{on } (\Gamma_{in} \cup \Gamma_0) \times (0,T), && (1.1c)\\
&u(x,0) = u_0 && \text{in } \Omega, && (1.1d)
\end{aligned}
$$

where (u,p) represent the velocity and pressure, $\sigma(u,p) := -pI + 2\nu D(u)$ denotes the traction tensor ($D(u) := (\nabla u + \nabla^\top u)/2$), and g is the inflow velocity satisfying

$$
g = 0 \quad \text{on } \Gamma_0 \times (0,T), \qquad \int_{\Gamma_{in}} g(t) \cdot n \, ds =: \beta(t) < 0 \text{ for } t \in (0,T]. \tag{1.2}
$$

Here, n is the unit outer normal vector to Γ. f and u_0 are the given force and initial velocity, respectively. In application, since the profile of velocity/pressure on the outflow boundary cannot be prescribed exactly, we need to put an appropriate artificial boundary condition on Γ_{out}, such that the model is mathematically well-posed and the simulation results agree with the experimental observation well.

As it requires no extra effort in implementation, the traction-free outflow boundary condition is popular in simulation:

$$
\sigma(u,p)n =: \tau(u,p) = 0 \quad \text{on } \Gamma_{out} \times (0,T). \tag{1.3}
$$

However, the mathematical well-posedness is questionable, and the numerical solution can easily become unstable when the Reynolds number is large, because the energy inequality may not hold. To verify the

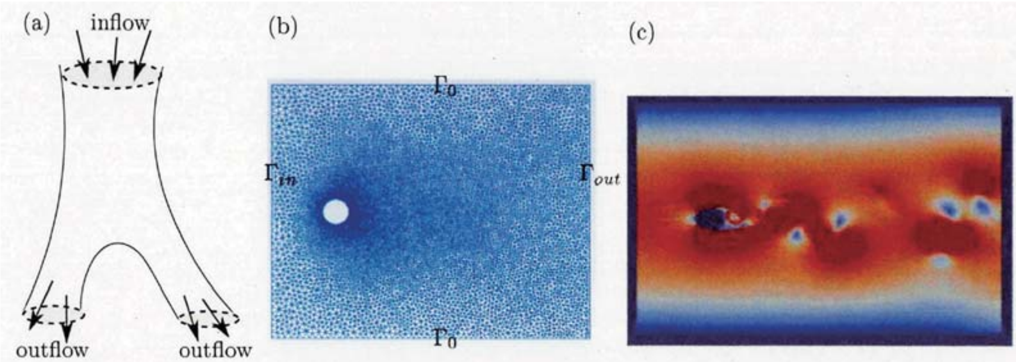

Figure 1: (a) A pipe-shaped domain. (b) The domain and mesh for numerical simulation. (c) A velocity profile of simulation.

energy inequality, we assume that Ω and g are sufficiently smooth so that there exists a smooth reference flow u_{ref} satisfying

$$u_{ref} = g \quad \text{on } (\Gamma_{in} \cup \Gamma_0) \times (0, T), \qquad u_{ref} \cdot n \geq 0 \quad \text{on } \Gamma_{out} \times (0, T), \tag{1.4a}$$

$$\nabla \cdot u_{ref} = 0 \quad \text{in } \Omega \times (0, T). \tag{1.4b}$$

Multiplying (1.1a) with $u - u_{ref}$ and using the integration by part (noting that $u - u_{ref} = 0$ on $\Gamma_{in} \cup \Gamma_0$),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u - u_{ref})(t)\|^2_{L^2(\Omega)} + 2\nu\|D(u - u_{ref})(t)\|^2_{L^2(\Omega)} + \int_\Omega (u \cdot \nabla)(u - u_{ref}) \cdot (u - u_{ref})\, dx \\
&= \int_\Omega (f - \partial_t u_{ref} - (u \cdot \nabla)u_{ref} - (u_{ref} \cdot \nabla)u) \cdot (u - u_{ref})\, dx \\
&\quad - \int_\Omega D(u_{ref}) : D(u - u_{ref})\, dx := RHS.
\end{aligned}$$

For 2d/3d case, one can bound the right-hand side of above equation as follows (see [10, 11])

$$RHS \leq C_{u_{ref}.f}(\|D(u - u_{ref})(t)\|_{L^2(\Omega)} + \|u - u_{ref}\|^2_{L^2(\Omega)}),$$

where $C_{u_{ref}.f}$ is a constant dependent on the norms of u_{ref} and f. If the non-negativity $\int_\Omega ((u - u_{ref}) \cdot \nabla)(u - u_{ref}) \cdot (u - u_{ref})\, dx \geq 0$ holds, then we can obtain the energy inequality

$$\|(u - u_{ref})(T)\|^2_{L^2(\Omega)} + \int_0^T \|(u - u_{ref})(t)\|^2_{H^1(\Omega)}\, dt \leq C_{u_{ref}.f.u_0.T}, \tag{1.5}$$

where $C_{u_{ref}.f.u_0.T}$ is a constant dependent on T and the norms of u_{ref}, f and u_0. (1.5) shows the boundedness of the velocity u in energy norm. Unfortunately, in view of

$$\int_\Omega (u \cdot \nabla)(u - u_{ref}) \cdot (u - u_{ref})\, dx = \frac{1}{2}\int_{\Gamma_{out}} u \cdot n|u - u_{ref}|^2\, ds, \tag{1.6}$$

since the sign of $u \cdot n$ on Γ_{out} is unknown, we cannot conclude the energy inequality (1.5), and the solution may blow up.

To ensure the energy inequality, a large number of works have been devoted to the open outflow boundary condition. [3, 4, Bruneau and Fabrie] proposed the following nonlinear type outflow boundary

$$\tau(u, p) = -\frac{[u \cdot n]_-}{2}(u - u_{ref}) + \nu D(u_{ref})n \quad \text{on } \Gamma_{out},$$

where $[s]_- := \max(0, -s)$ denotes the negative part of s. Since $[u \cdot n]_- > 0$ means the backward flow exists on Γ_{out}, the above boundary condition can be regarded as enforcing a traction vector $\tau(u,p)$ to control the backward flow, which has been applied to the simulation of blood flow in [1, Bazilevs, etc.]. The mathematical well-posedness of the non-homogeneous Navier-Stokes equations with this boundary condition is studied by [2, Boyer, etc.]. A class of more general energy-stable open boundary conditions has been proposed and investigated by [5, Dong] and [6, Dong and Shen], which is stated as follows:

$$\nu D_0 u_t - pn + \nu(n \cdot \nabla)u - \frac{1}{2}[|u|^2 n + (u \cdot n)]u\Theta_0(n,u) = f_b \quad \text{on } \Gamma_{out},$$

where D_0 and f_b are given parameter and function, and $\Theta_0(n,u)$ is a smooth approximation of $[u \cdot n]_-$. A comparison to the physical experiments has been discussed detailedly in [5]. In [9, 11], we proposed an outflow control boundary condition of Signorini's type and proved the mathematical well-posedness. This condition is an analogy to Signorini's condition in the theory of elasticity [7], which is to enforce the non-backward flow on Γ_{out}, i.e.,

$$u_n \geq 0, \quad \tau_n(u,p) \geq 0, \quad u_n\tau_n(u,p) = 0, \quad \tau_T(u) = 0 \quad \text{on } \Gamma_{out}, \tag{1.7}$$

where $u_n := u \cdot n$ denotes the normal component of velocity u, $\tau_n(u,p) := \tau(u,p) \cdot n$ and $\tau_T(u) := (I - n \otimes n)\tau(u,p)$ represent the normal and tangential parts of traction vector $\tau(u,p)$. Noting that $\tau(u,p) = \tau_n(u,p)n + \tau_T(u)$, we can regard (1.7) as an extension of the traction-free condition:

$$\tau(u,p) = 0 \qquad \text{on } \Gamma_{out.+}, \tag{1.8a}$$

$$u_n = 0, \quad \tau_T(u) = 0 \qquad \text{on } \Gamma_{out} \backslash \Gamma_{out.+}, \tag{1.8b}$$

where $\Gamma_{out.+} := \{x \in \Gamma_{out} \mid u_n > 0\}$. In view of $u_n \geq 0$ and (1.6), the energy inequality (1.5) holds true. We also proposed a penalty method and obtain the error estimates the P1b/P1 element for the stationary Stokes problem with the condition (1.7).

This paper is concerned with the numerical methods for the Navier-Stokes problem with Signorini's boundary condition (1.7). As a preliminary, we introduce the variational inequality for our model problem (1.1)(1.7) in Section 2. In Section 3, we apply the penalty method to approximate (1.7), and derive the energy-stability for the discrete solution. We consider the Lagrange multiplier approach in Section 4, where we consider the Uzawa method with projection and the active/inactive set method to implement (1.7). Finally, in Section 5, we study the convergence of our scheme. Moreover, we apply our schemes to the numerical simulation and comparing the results to the experimental data of [8], which indicates the suitability of Signorini's boundary condition in application.

2 The variational inequality

Assume that the model problem (1.1) (1.7) admits a unique strong solution (u,p) for $0 < t < T$ with regularity (cf. [11])

$$u \in L^\infty(0,T; H^1(\Omega)^d) \cap L^2(0,T; H^2(\Omega)^d), \quad u_t \in L^2(0,T; L^2(\Omega)^d), \quad p \in L^2(0,T; H^1(\Omega)).$$

Let us derive the variational inequality of (1.1) (1.7). For simplicity, from now on, we assume that g is independent of t (or else we can derive the variational form for $U = u - u_{ref}$). We define the function spaces:

$$V := \{v \in H^1(\Omega)^d \mid v = g \text{ on } \Gamma_{in} \cup \Gamma_0\}, \quad V_0 := \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_{in} \cup \Gamma_0\},$$
$$K := \{v \in V \mid v_n \geq 0 \text{ on } \Gamma_{out}\}, \quad Q := L^2(\Omega), \quad \mathring{Q} := L_0^2(\Omega).$$

For any $v \in V$, multiplying (1.1a) with $v - u$ and integrating with the integration by parts, we have (noting that $v - u = 0$ on $\Gamma_{in} \cap \Gamma_0$)

$$\int_\Omega u_t \cdot (v-u)\, dx + \int_\Omega (u \cdot \nabla)u \cdot (v-u)\, dx + 2\nu \int_\Omega D(u) : D(v-u)\, dx - \int_{\Gamma_{out}} \sigma(u,p)n \cdot (v-u)\, ds$$
$$- \int_\Omega p\nabla \cdot (v-u)\, dx = \int_\Omega f \cdot v\, dx.$$

Decomposing the traction vector $\sigma(u,p)n$ and $v-u$ into normal and tangential component yields

$$\int_{\Gamma_{out}} \sigma(u,p)n\cdot(v-u)\,ds = \int_{\Gamma_{out}} \tau_n(u,p)(v_n-u_n) + \underbrace{\tau_T(u)}_{=0}\cdot(v_T-u_T)\,ds,$$
$$= \int_{\Gamma_{out}} \underbrace{\tau_n(u,p)}_{\geq 0}\underbrace{v_n}_{\geq 0}\,ds - \int_{\Gamma_{out}} \underbrace{\tau_n(u,p)u_n}_{=0}\,ds \geq 0.$$

where $v_T := (I - n\otimes n)v$ denotes the tangential part of v, and we have applied the boundary condition of Signorini's type (1.7).

For $u, v, w \in H^1(\Omega)^d$ and $p \in L^2(\Omega)$, we introduce the notations

$$a(u,v) := 2\nu(D(u), D(v))_\Omega, \quad a_1(u,v,w) := ((u\cdot\nabla v), w)_\Omega,$$
$$b(v,p) := -(\nabla\cdot v, p)_\Omega,$$

where $(u,v)_\Omega := \int_\Omega u\cdot v\,dx$ is the inner produce of $L^2(\Omega)^d$ (or $L^2(\Omega)$). The variational form of (1.1)(1.7) is stated as follows.

$$(u_t, v-u)_\Omega + a_1(u,u,v-u) + a(u,v-u) + b(v-u,p) \geq (f, v-u)_\Omega \qquad \forall v\in K, \qquad (2.1a)$$
$$b(u,q) = 0 \qquad \forall q \in Q. \qquad (2.1b)$$

To overcome the difficulties of solving the variational inequality problem (2.1) numerically, we consider the penalty method and Lagrange multiplier approach respectively in the next sections.

3 The penalty method

The idea of the penalty technique is to approximate the Signorini boundary condition (1.7) by a Robin type boundary condition.

Introducing the penalty parameter ϵ $(0 < \epsilon \ll 1)$, we state the penalty problem:

$$u_{\epsilon,t} + (u_\epsilon\cdot\nabla)u_\epsilon - \nabla\cdot\sigma(u_\epsilon,p_\epsilon) = f \qquad \text{in } \Omega\times(0,T), \qquad (3.1a)$$
$$\nabla\cdot u_\epsilon = 0 \qquad \text{in } \Omega\times(0,T), \qquad (3.1b)$$
$$u_\epsilon = g \qquad \text{on } (\Gamma_{in}\cup\Gamma_0)\times(0,T), \qquad (3.1c)$$
$$\tau(u_\epsilon,p_\epsilon) = \epsilon^{-1}[u_\epsilon\cdot n]_- n \qquad \text{on } \Gamma_{out}\times(0,T), \qquad (3.1d)$$
$$u_\epsilon(x,0) = u_0 \qquad \text{in } \Omega. \qquad (3.1e)$$

As $[u_\epsilon\cdot n]_-$ denotes the backward flow on Γ_{out}, the Robin type boundary condition (3.1d) is to make the normal traction vector $\tau_n(u_\epsilon,p_\epsilon)$ sufficient large $(\epsilon^{-1} \gg 1)$ at the places where the backward flow occurs, so that the backward flow can be restrained. On the other hand, if the normal traction vector $\tau_n(u_\epsilon,p_\epsilon)$ is bounded, then (3.1d) indicates that

$$[u_\epsilon\cdot n]_- = \epsilon\tau_n(u_\epsilon,p_\epsilon) \to 0, \qquad \text{as } \epsilon\to 0,$$

which approximates to the boundary condition $[u\cdot n]_- = 0$ (i.e., $u\cdot n \geq 0$) on Γ_{out}. In [11], the existence of (u_ϵ,p_ϵ) has been proved, as well as the convergence $(u_\epsilon,p_\epsilon)\to(u,p)$ when passing to the limit $\epsilon\to 0$. Therefore, instead of solving the variational inequality (2.1a), we compute the solution of penalty problem (3.1) to approximate (u,p). In this section, we discuss the numerical schemes to (3.1) or its variational form presented as follows.

$$(u_{\epsilon,t}, v)_\Omega + a_1(u_\epsilon,u_\epsilon,v) + a(u_\epsilon,v) + b(v,p_\epsilon) - \frac{1}{\epsilon}\int_{\Gamma_{out}} [u_{\epsilon n}]_- v_n\,ds = (f,v)_\Omega \qquad \forall v\in V, \qquad (3.2a)$$
$$b(u_\epsilon,q) = 0 \qquad \forall q\in Q. \qquad (3.2b)$$

3.1 The spatial and time discretization

We consider the case that Ω is a polygon/polyhedron, and introduce a regular triangulation $\mathcal{T}_h$ to Ω. We apply the P1b/P1 element for velocity/pressure:

$$\begin{aligned}
V_h &:= \{v_h \in C(\Omega)^d \mid v_h|_T \in P_1(T)^d \oplus B(T)^d \ \forall T \in \mathcal{T},\ v_h = g_h \text{ on } \Gamma_{in} \cup \Gamma_0\},\\
V_{h0} &:= \{v_h \in C(\Omega)^d \mid v_h|_T \in P_1(T)^d \oplus B(T)^d \ \forall T \in \mathcal{T},\ v_h = 0 \text{ on } \Gamma_{in} \cup \Gamma_0\},\\
Q_h &:= \{a_h \in C(\Omega) \mid a_h|_T \in P_1(T) \ \forall T \in \mathcal{T}\},
\end{aligned}$$

where $P_1(T)$ and $B(T)$ stands for the sets of linear functions and bubble functions in T respectively, and g_h is an interpolation of g.

For time-discretization, we set the time-step-size $\Delta t := \frac{T}{M}$ for some integer M ($M \gg 1$), and divide the time interval $(0, T)$ into M segments $\{(t_{m-1}, t_m)\}_{m=1}^{M}$ with $t_m := m\Delta t$. Let $u_h^0 \in V_h$ be the approximated initial velocity (i.e., $u_h^0 \approx u_0$) satisfying

$$u_h^0 \cdot n \geq 0 \quad \text{on } \Gamma_{out}. \tag{3.3}$$

We shall use the backward Euler scheme for the time-differential approximation:

$$\bar{\partial} u^m := \frac{u^m - u^{m-1}}{\Delta t} \approx u_t(t_m) \qquad (u^m := u(t_m)).$$

3.2 The discrete penalty problem

With the above settings, we give the discretization of penalty problem (3.1). Find $\{(u_h^m, p_h^m)\}_{m=1}^{M} \subset V_h \times Q_h$ satisfying: for all $(v_h, q_h) \in V_{h0} \times Q_h$,

$$(\bar{\partial} u_h^m, v_h)_\Omega + a_1(u_h^{m-1}, u_h^m, v_h) + a(u_h^m, v_h) + b(v_h, p_h^m) - \frac{1}{\epsilon}\int_{\Gamma_{out}} [u_{hn}^m]_- v_{hn}\, ds = (f^m, v_h)_\Omega, \tag{3.4a}$$

$$b(u_h^m, q_h) = 0 \tag{3.4b}$$

where $f^m := \frac{1}{\delta t}\int_{t_{m-1}}^{t_m} f(t)\, dt$ and $u_{hn}^m := u_h^m \cdot n$.

To derive the discrete energy inequality for u_h^m, we introduce the approximated reference flow $\{u_{ref.h}^m\}_{m=1}^{M} \subset V_h$ which satisfies

$$u_{ref.h}^m = g_h \quad \text{on } \Gamma_{in} \cap \Gamma_0, \qquad u_{ref.h}^m \cdot n \geq 0 \quad \text{on } \Gamma_{out}, \tag{3.5a}$$

$$b(u_{ref.h}^m, q_h) = 0 \quad \text{for all } q_h \in Q_h. \tag{3.5b}$$

Theorem 3.1. *Given $\{f^m\}_{m=1}^{M}$ and u_h^0, for sufficiently small ϵ, there exists a unique solution $\{(u_h^m, p_h^m)\}_{m=1}^{M}$ to (3.4), and the discrete energy inequality holds:*

$$\|u_h^M\|_{L^2(\Omega)}^2 + \Delta t\sum_{m=1}^{M} \|u_h^m\|_{H^1(\Omega)}^2 + \frac{\Delta t}{\epsilon}\sum_{m=1}^{M} \|[u_{hn}^m]_-\|_{L^2(\Gamma)}^2 \leq C(\Delta t\sum_{m=1}^{M} \|f^m\|_{L^2(\Omega)}^2 + \|u_h^0\|_{L^2(\Omega)}^2).$$

3.3 The approximation of $[u_{hn}]_-$

Since the nonlinear term $[u_{hn}]_-$ is not C^1-differentiable, we introduce the regularization to $[u_{hn}]_-$ and consider the Newton iteration for solving the nonlinear problem. Let δ ($0 < \delta \ll 1$) be the regularization parameter. The regularization to $[s]_-$ is given by

$$\phi_\delta(s) := \frac{1}{2}(\sqrt{s^2 + \delta^2} - s). \tag{3.6}$$

We see that $\phi_\delta(s) \to [s]_-$ as $\delta \to 0$. Then, we replace $[u_{hn}]_-$ in (3.4) with $\phi_\delta(u_{hn})$, i.e., we solve the following regularization problem.

Find $\{(u_h^m, p_h^m)\}_{m=1}^M \subset V_h \times Q_h$ satisfying: for all $(v_h, q_h) \in V_{h0} \times Q_h$,

$$(\bar{\partial} u_h^m, v_h)_\Omega + a_1(u_h^{m-1}, u_h^m, v_h) + a(u_h^m, v_h) + b(v_h, p_h^m) - \frac{1}{\epsilon}\int_{\Gamma_{out}} \phi_\delta(u_{hn}) v_{hn}\, ds = (f^m, v_h)_\Omega. \quad (3.7a)$$

$$b(u_h^m, q_h) = 0. \quad (3.7b)$$

Since $\phi_\delta(s) \in C^2$, we can apply Newton's iteration to (3.7).

(Algorithm 1)

(Step 1). Set $j = 1$. We compute the initial value $(u_h^{m.(0)}, p_h^{m.(0)}) \in V_h \times Q_h$ for iteration, which satisfies: for any $(v_h, q_h) \in V_{h0} \times Q_h$,

$$\frac{1}{\Delta t}(u_h^{m.(0)} - u_h^{m-1}, v_h)_\Omega + a_1(u_h^{m-1}, u_h^{m.(0)}, v_h) + a(u_h^{m.(0)}, v_h) + b(v_h, p_h^{m.(0)}) = (f^m, v_h)_\Omega, \quad (3.8a)$$

$$b(u_h^{m.(0)}, q_h) = 0. \quad (3.8b)$$

(Step 2). Solve the problem: find $(du_h^{m.(j)}, dp_h^{m.(j)}) \in V_{h0} \times Q_h$ satisfying for all $(v_h, q_h) \in V_{h0} \times Q_h$,

$$\begin{aligned} &\frac{1}{\Delta t}(du_h^{m.(j)}, v_h)_\Omega + a_1(u_h^{m-1}, du_h^{m.(j)}, v_h) + a(du_h^{m.(j)}, v_h) \\ &\qquad - \frac{1}{\epsilon}\int_{\Gamma_{out}} \phi'_\delta(u_{hn}^{m.(j-1)}) du_{hn}^{m.(j)} v_{hn}\, ds + b(v_h, dp_h^{m.(j)}) = F^{m.j}(v_h), \end{aligned} \quad (3.9a)$$

$$b(du_h^{m.(j)}, q_h) = 0, \quad (3.9b)$$

where $\phi'_\delta(s) := \frac{1}{2}(\frac{s}{\sqrt{s^2+\delta^2}} - 1)$ denotes the derivative of $\phi_\delta(s)$, $u_{hn}^{m.(j-1)} := u_h^{m.(j-1)} \cdot n$, $du_{hn}^{m.(j-1)} := du_h^{m.(j-1)} \cdot n$, and $F^{m.j}(v_h)$ is defined by

$$\begin{aligned} F^{m.j}(v_h) :=& (f^m, v_h)_\Omega - \frac{1}{\Delta t}(u_h^{m.(j-1)} - u_h^{m-1}, v_h)_\Omega - a_1(u_h^{m-1}, u_h^{m.(j-1)}, v_h) - a(u_h^{m.(j-1)}, v_h) \\ &+ \frac{1}{\epsilon}\int_{\Gamma_{out}} \phi_\delta(u_{hn}^{m.(j-1)}) v_{hn}\, ds - b(v_h, p_h^{m.(j-1)}) \end{aligned}$$

Then we update the solution:

$$(u_h^{m.(j)}, p_h^{m.(j)}) := (u_h^{m.(j-1)} + du_h^{m.(j)}, p_h^{m.(j-1)} + dp_h^{m.(j)}) \quad (3.10)$$

(Step 3). Increase j by 1 and iterate (Step 2) until convergence.

4 The Lagrange multiplier approach

In this section, we consider the Lagrange multiplier approach to treat the Signorini's boundary condition in numerical computation. First, let us pay attention to the continuous problem (1.1)(1.7) and its variational inequality (2.1). Introducing the Lagrange multiplier

$$\lambda := -\tau_n(u, p) \leq 0,$$

we enforce the boundary condition $u_n \geq 0$ by the weak form

$$[u_n, \mu - \lambda] \leq 0 \qquad \text{for all } \mu \in \Lambda^*\},$$

where $[v_n, \mu] := \int_{\Gamma_{out}} v_n \mu \, ds$ is the dual product between $H_{00}^{\frac{1}{2}}(\Gamma_{out})$ and its dual space $H_{00}^{\frac{1}{2}}(\Gamma_{out})^*$, and

$$\Lambda^* := \{\mu \in H_{00}^{\frac{1}{2}}(\Gamma_{out})^* \mid [v_n, \mu] \le 0 \ \ \forall v \in K\}.$$

Then, with the help of λ, we write an equivalent form of (2.1).

$$\begin{aligned}
&(u_t, v)_\Omega + a_1(u, u, v) + a(u, v) + b(v, p) + [v_n, \lambda] = (f, v)_\Omega && \forall v \in V, && (4.1a)\\
&b(u, q) = 0 && \forall q \in Q, && (4.1b)\\
&[u_n, \mu - \lambda] \le 0 && \forall \mu \in \Lambda^*. && (4.1c)
\end{aligned}$$

The equivalence between the boundary condition of Signorini's type (1.7) and (4.1c) has been derived in [10]. Therefore, we can solve the Lagrange multiplier problem (4.1) instead of (2.1). To discretize (4.1), we need to introduce a finite element function space for λ. Let $\mathcal{E}_{out}$ be the mesh of the outflow boundary Γ_{out} inherited from the triangulation $\mathcal{T}$. We define the function space

$$\Lambda_h := \{\mu_h \in C(\Gamma_{out}) \mid \mu_h|_e \in P_1(e) \ \ \forall e \in \mathcal{E}_{out}\}, \quad \Lambda_{h,-} := \{\mu_h \in \Lambda_h \mid \mu_h \le 0\},$$

and the bilinear form

$$c(v_{hn}, \mu_h) := \sum_{e \in \mathcal{E}_{out}} \frac{|e|}{d} \sum_{i=1}^{d} \mu_h(N_e^i) v_{hn}(N_e^i) \qquad \forall v_h \in V_h, \ \mu_h \in \Lambda_h,$$

where $\{N_e^i\}_{i=1}^d$ denotes the vertices of the edge/face e.

The discretization of the problem (4.1) reads as: find $\{(u_h^m, p_h^m, \lambda_h^m)\}_{m=1}^M \subset V_h \times Q_h \times \Lambda_{h,-}$ such that for all $(v_h, q_h, \mu_h) \in V_{h0} \times Q_h \times \Lambda_{h,-}$,

$$\begin{aligned}
&(\bar{\partial} u_h^m, v_h)_\Omega + a_1(u_h^m, u_h^m, v_h) + a(u_h^m, v_h) + b(v_h, p_h^m) + c(v_{hn}, \lambda_h^m) = (f^m, v_h)_\Omega && \forall v_h \in V_h, && (4.2a)\\
&b(u_h^m, q_h) = 0 && \forall q_h \in Q_h, && (4.2b)\\
&c(u_{hn}^m, \mu_h - \lambda_h^m) \le 0 && \forall \mu_h \in \Lambda_{h,-}. && (4.2c)
\end{aligned}$$

The discrete problem (4.2) preserves the energy inequality.

Theorem 4.1. *Let $\{(u_h^m, p_h^m, \lambda_h^m)\}_{m=1}^M$ be a solution of (4.2). On the outflow boundary Γ_{out}, $u_{hn}^m \ge 0$ holds exactly. Moreover, we have the discrete energy inequality:*

$$\|u_h^M\|_{L^2(\Omega)}^2 + \Delta t \sum_{m=1}^{M} \|u_h^m\|_{H^1(\Omega)}^2 \le C(\Delta t \sum_{m=1}^{M} \|f^m\|_{L^2(\Omega)}^2 + \|u_h^0\|_{L^2(\Omega)}^2).$$

Next, we consider two methods to implement (4.2): the Uzawa method with projection and the active/inactivie set method.

4.1 The Uzawa method with projection

We define a projection operator:

$$\mathcal{P} : \Lambda_h \to \Lambda_{h,-}, \quad \mu_h \mapsto \mathcal{P}\mu_h,$$

$$\mathcal{P}\mu_h := \begin{cases} \mu_h(N_e^i) & \text{if } \mu_h(N_e^i) \le 0, \\ 0 & \text{if } \mu_h(N_e^i) > 0. \end{cases} \quad \text{for all } e \in \mathcal{E}_{out} \text{ and } i = 1, \dots, d,$$

The projection $\mathcal{P}\mu_h$ is a truncation of the positive part of μ_h. With the help of $\mathcal{P}$, we state the Uzawa method for (4.2).

(Algorithm 2).

(Step 1). Let $\lambda_h^{m.(0)} = 0$. Find $(u_h^{m.(0)}, p_h^{m.(0)}) \in V_h \times Q_h$ satisfying: for all $(v_h, q_h) \in V_{h0} \times Q_h$,

$$\begin{aligned}\frac{1}{\Delta t}(u_h^{m.(0)} - u_h^{m-1}, v_h)_\Omega + a_1(u_h^{m.(0)}, u_h^{m.(0)}, v_h) + a(u_h^{m.(0)}, v_h) + b(v_h, p_h^{m.(0)})& \\ = (f^m, v_h)_\Omega - c(v_{hn}, \lambda_h^{m.(0)}),&\end{aligned} \tag{4.3a}$$

$$b(u_h^{m.(0)}, q_h) = 0. \tag{4.3b}$$

We take the solution $(u_h^{m.(0)}, p_h^{m.(0)})$ as the initial value of the Uzawa method. Set $j = 1$.

(Step 2). We update the Lagrange multiplier using the obtained solution $u_h^{m.(j-1)}$ and the projection operator $\mathcal{P}$:

$$\lambda_h^{m.(j)} := \mathcal{P}(\lambda_h^{m.(j-1)} + \rho u_{hn}^{m.(j-1)}) \qquad (\rho > 0). \tag{4.4}$$

Then find $(u_h^{m.(j)}, p_h^{m.(j)}) \in V_h \times Q_h$ satisfying: for all $(v_h, q_h) \in V_{h0} \times Q_h$,

$$\begin{aligned}\frac{1}{\Delta t}(u_h^{m.(j)} - u_h^{m-1}, v_h)_\Omega + a_1(u_h^{m.(j)}, u_h^{m.(j)}, v_h) + a(u_h^{m.(j)}, v_h) + b(v_h, p_h^{m.(j)})& \\ = (f^m, v_h)_\Omega - c(v_{hn}, \lambda_h^{m.(j)}),&\end{aligned} \tag{4.5a}$$

$$b(u_h^{m.(0)}, q_h) = 0. \tag{4.5b}$$

(Step 3). Increase j by 1 and iterate (Step 2) until convergence.

4.2 The active/inactivie set method

Noting that the projection (4.4) in Uzawa's method only ensures the non-positivity of the Lagrange multiplier λ_h^m, the condition $u_{hn}^m \lambda_h^m = 0$ has not been treated explicitly, whereas the active/inactivie set method ensures the condition $u_{hn}^{m.(j)} \lambda_h^{m.(j)} = 0$ at every iteration.

The idea of the active/inactivie set method is to think of the Signorini's boundary condition as a combination of the traction-free boundary condition on the active set $\Gamma_{out,+}$ and the slip boundary condition $u_n = 0$ on the inactive set $\Gamma_{out} \backslash \Gamma_{out,+}$ (see (1.8)).

The algorithm is presented as follows.

(Algorithm 3).

(Step 1). Let $\lambda_h^{m.(0)} = 0$. Find $(u_h^{m.(0)}, p_h^{m.(0)}) \in V_h \times Q_h$ satisfying: for all $(v_h, q_h) \in V_{h0} \times Q_h$,

$$\frac{1}{\Delta t}(u_h^{m.(0)} - u_h^{m-1}, v_h)_\Omega + a_1(u_h^{m.(0)}, u_h^{m.(0)}, v_h) + a(u_h^{m.(0)}, v_h) + b(v_h, p_h^{m.(0)}) = (f^m, v_h)_\Omega, \tag{4.6a}$$

$$b(u_h^{m.(0)}, q_h) = 0. \tag{4.6b}$$

We take the solution $(u_h^{m.(0)}, p_h^{m.(0)})$ as the initial value for iteration. Set $j = 1$.

(Step 2). We define the active set $A^{m.(j)}$ and inactive set $I^{m.(j)}$ by :

$$A^{m.(j)} := \{x \in \Gamma_{out} \mid \lambda_h^{m.(j-1)} + \rho u_{hn}^{m.(j-1)} > 0\}, \qquad I^{m.(j)} := \Gamma_{out} \backslash A^{m.(j)}. \tag{4.7}$$

(Step 3). Find $(u_h^{m.(j)}, p_h^{m.(j)}) \in V_h \times Q_h$ satisfying $u_{hn}^{m.(j)} = 0$ on $I^{m.(j)}$ and for all $(v_h, q_h) \in \{v_h \in V_{h0} \mid u_{hn}^{m.(j)} = 0 \text{ on } I^{m.(j)}\} \times Q_h$,

$$\frac{1}{\Delta t}(u_h^{m.(j)} - u_h^{m-1}, v_h)_\Omega + a_1(u_h^{m.(j)}, u_h^{m.(j)}, v_h) + a(u_h^{m.(j)}, v_h) + b(v_h, p_h^{m.(j)}) = (f^m, v_h)_\Omega, \tag{4.8a}$$

$$b(u_h^{m.(0)}, q_h) = 0. \tag{4.8b}$$

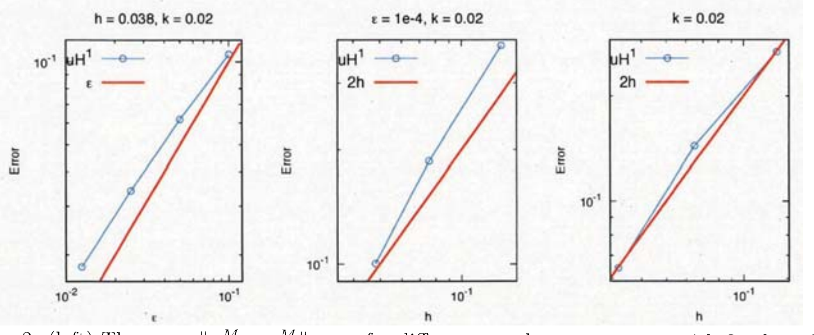

Figure 2: (left) The error $\|u_f^M - u_h^M\|_{H^1(\Omega)}$ for different penalty parameter ϵ with fixed mesh size $h = 0.038$ and $\Delta t = 0.02$. (middle) The error $\|u_f^M - u_h^M\|_{H^1(\Omega)}$ for different mesh size h with fixed penalty parameter $\epsilon = 10^{-4}$ and $\Delta t = 0.02$. (right) The error $\|u_f^M - u_h^M\|_{H^1(\Omega)}$ for different mesh size h with $\Delta t = 0.02$ for the Lagrange multiplier approach.

(Step 4). With obtained $(u_h^{m.(j)}, p_h^{m.(j)})$, we calculate $\lambda_h^{m.(j)}$, which satisfies: for all $v_h \in V_{h0}$,

$$\begin{aligned} c(v_{hn}, \lambda_h^{m.(j)}) =& (f^m, v_h)_\Omega - \frac{1}{\Delta t}(u_h^{m.(j)} - u_h^{m-1}, v_h)_\Omega - a_1(u_h^{m.(j)}, u_h^{m.(j)}, v_h) \\ & - a(u_h^{m.(j)}, v_h) - b(v_h, p_h^{m.(j)}), \end{aligned} \tag{4.9}$$

(Step 5). Increase j by 1 and iterate (Step 2)-(Step 4) until convergence.

The condition $u_{hn}^{m.(j)}\lambda_h^{m.(j)} = 0$ on Γ_{out} is satisfied at each iteration. In fact, at (Step 3), the problem (4.8) implies that $\lambda_h^{m.(j)} = 0$ on the active set $A^{m.(j)}$. On the other hand, $u_{hn}^{m.(j)} = 0$ on the inactive set $I^{m.(j)}$ is explicitly implemented. Hence, we have $u_{hn}^{m.(j)}\lambda_h^{m.(j)} = 0$ on $I^{m.(j)} \cup A^{m.(j)} = \Gamma_{out}$.

5 The numerical experiments

5.1 The convergence rate

5.1.1 The convergence rate of the penalty approach

We investigate the convergence rate of the penalty approach by numerical experiment. We simulate the flow in domain $\Omega := \{(x, y) \mid 0 < x < 3,\ -1 < y < -1,\ (x - 0.4)^2 + y^2 > 0.4^2\}$ with inflow boundary $\Gamma_{in} := \{(x, y) \mid x = 0\}$, the no-slip boundary $\Gamma_0 := \{(x, y) \mid y = \pm 1\}$, and the outflow boundary $\Gamma_{out} := \{(x, y) \mid x = 3\}$. We set the viscosity $\nu = 0.01$ and the inflow velocity $g = 2(1 + 0.3\sin(2\pi t))(1 - y^2)$ on Γ_{in}.

For a very fine mesh $\mathcal{T}_f$ and tiny penalty parameter ϵ_f and time-step size Δt, we compute the numerical solution at $T = 0.6$, which is denoted by (u_f^M, p_f^M). We regard (u_f^M, p_f^M) as the exact solution and calculate the error $\|u_f^M - u_h^M\|_{H^1(\Omega)}$, where u_h^M is the discrete solution with coarser mesh $\mathcal{T}$ or the penalty parameter ϵ ($\epsilon_f < \epsilon \ll 1$).

First, we fix the mesh $\mathcal{T}_f$, and plot the errors for different ϵ in Figure 5.1.1 (left), which indicates the convergence of order $O(\epsilon)$. Next, for fixed penalty parameter ϵ_f, we compute the error for different mesh size h (see Figure 5.1.1 (middle)), which shows the convergence of order $O(h)$.

For the Lagrange multiplier approach, we investigate the convergence order depending on mesh size h with a fixed time-step size Δt. The error is plotted in Figure 5.1.1 (right), which also shows the $O(h)$-convergence.

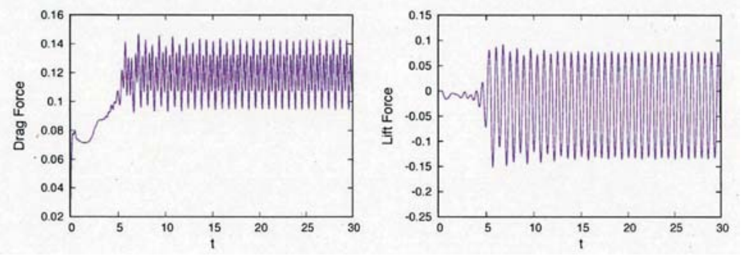

Figure 3: The drag and lift forces for $\nu = 9.76 \times 10^{-5}$.

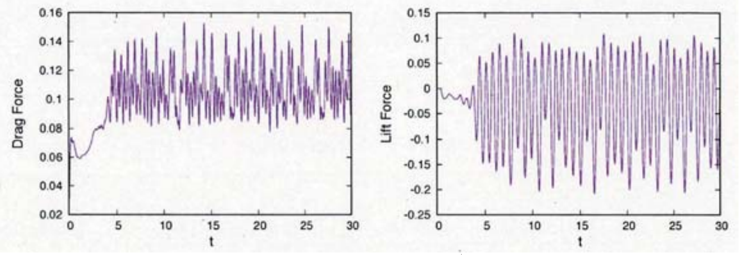

Figure 4: The drag and lift forces for $\nu = 4.88 \times 10^{-5}$.

5.2 The numerical simulations

To validate the suitability of applying the Signorini boundary condition for real-world fluid simulation, we compare the simulation results to the experimental data. We simulate the flow in domain $\Omega := \{(x, y) \mid 0 < x < 2.5L,\ -L < y < -L,\ (x-0.5)^2 + y^2 > r^2\}$ and time interval $(0, T)$, where $T = 30$, $r = 0.1$, $L = 1$, $\Gamma_{in} := \{(x, y) \mid x = 0\}$, the no-slip boundary $\Gamma_0 := \{(x, y) \mid y = \pm L\}$, and the outflow boundary $\Gamma_{out} := \{(x, y) \mid x = 2.5L\}$. The inflow velocity is given by $g = L^2 - y^2$ on Γ_{in}. We compute the drag force D_f and the lift force L_f on the circle boundary $\mathcal{C}_1 := \{(x, y) \mid (x - 0.5)^2 + y^2 = r^2\}$:

$$D_f := -\int_{\mathcal{C}_1} \sigma(u, p)n \cdot \boldsymbol{d}\, ds, \qquad L_f := \int_{\mathcal{C}_1} \sigma(u, p)n \cdot \boldsymbol{l}\, ds,$$

where $\boldsymbol{d} = (1, 0)^\top$ and $\boldsymbol{l} = (0, 1)^\top$. We plot two profiles of the drag and lift forces in Figure 5.2 and Figure 5.2 for $\nu = 9.76 \times 10^{-5}$ and $\nu = 4.88 \times 10^{-5}$ respectively. The average of drag and lift forces for various Reynolds number $\propto Re = \nu^{-1}$ are plotted in Figure 5.2, which somehow corresponds to the experimental data [8] for $\nu \geq 10^{-3}$.

References

[1] Y. Bazilevs, J. R. Hohean, T. J. R. Hughes, R. D. Moser, and Y. Zhang. Patient-specific isogeometric fluid–structure interaction analysis of thoracicaortic blood flow due to implantationof the Jarvik2000 left ventricular assist device. *Comput. Methods Appl. Mech. Eng.*, 198:3534–3550, 2009.

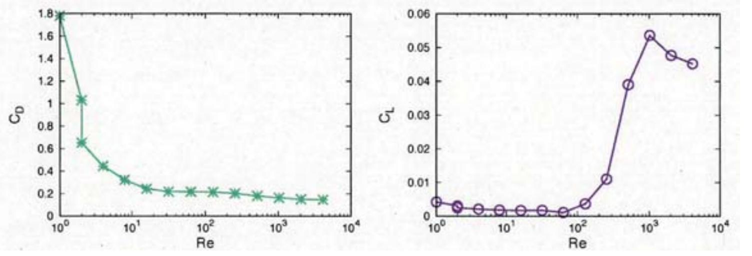

Figure 5: The average drag and lift forces for different $Re = \nu^{-1}$.

[2] F. Boyer and P. Fabrie. Outflow boundary conditions for the incompressible non-homogeneous Navier-Stokes equations. *Discrete and Continuous Dynamical Systems-Series B, American Institute of Mathematical Sciences*, 7:219–250, 2007.

[3] C.-H. Bruneau and P. Fabrie. Effective downstream boundary conditions for incompressible Navier-Stokes equations. *Int. J. Numer. Methods Fluids*, 19:693–705, 1994.

[4] C.-H. Bruneau and P. Fabrie. New efficient boundary conditions for incompressible Navier-Stokes equations: a well-posedness result. *Math. Model. Numer. Anal.*, 30:815–840, 1996.

[5] S. Dong. A convective-like energy-stable open boundary condition for simulations of incompressible flows. *J. Comput. Phys.*, 302:300–328, 2015.

[6] S. Dong and J. Shen. A pressure-correction scheme for generalized form of energy-stable open boundary conditions for incompressible flows. *J. Comput. Phys.*, 291:254–278, 2015.

[7] N. Kikuchi and J. T. Oden. *Contact Problems in Elasticity.* SIAM, 1988.

[8] C. Norberg. Fluctuating lift on a circular cylinder: review and new measurements. *Journal of Fluids and Structures*, 17:57–96, 2003.

[9] N. Saito, Y. Sugitani, and G. Zhou. *Energy inequalities and outflow boundary conditions for the Navier-Stokes equations*, chapter Advances in computational fluid-structure interaction and flow simulation: New methods and challenging computations, pages 307–317. Birkhauser, 2016.

[10] N. Saito, Y. Sugitani, and G. Zhou. Unilateral problem for the Stokes equations: the well-posedness and finite element approximation. *Appl. Numer. Math.*, 105:124–147, 2016.

[11] G. Zhou and N. Saito. The Navier-Stokes equations under a unilateral boundary condition of Signorini's type. *Journal of Mathematical Fluid Mechanics*, 18:481–510, 2016.