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COMMENTS ON MULTILINEAR STRONG MAXIMAL OPERATORS ON MIXED LEBESGUE SPACES

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0.1. **Maximal type operators.** Let $n \geq 1$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $\mathcal{M}$ be the well-known Hardy-Littlewood maximal operator defined on $\mathbb{R}^n$ as follows.

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy,$$

where $B_r(x)$ is the open ball in $\mathbb{R}^n$ centered at x with radius r and $|B_r(x)|$ denotes the volume of $B_r(x)$. The corresponding uncentered maximal function will be denoted by $\widetilde{\mathcal{M}}f$.

It is well known that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are of type (p, p) for $1 < p \leq \infty$ and weak type $(1, 1)$. In 1997, Kinnunen [18] first showed that $\mathcal{M}$ is bounded from the first order Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$. Later on, the $W^{1,p}$-bounds for the uncentered operator $\widetilde{\mathcal{M}}$ was obtained by Hajłasz and Onninenin [15]. Since then, many works have been done to extend the above results in a more general setting. We refer the reader to see [19], [20], [7, 28], [27], [31], [32]. However, the results for $p = 1$ are quite different. In 2004, Hajłasz and Onninen [15] surprisingly pointed out that the Hardy-Littlewood maximal operator is not bounded on $W^{1,1}$ space. Thus, it is quite natural to consider whether the operator $f \mapsto |\nabla \mathcal{M}f|$ is bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ or not. When $n = 1$, this problem was completely solved in [2, 24, 26, 39]. But for $n \geq 2$, only partial results were given by Hajłasz and Malý [14] and Luiro [33]. For more previous works or related topic we refer the readers to consult [3, 5, 6, 8, 24, 21, 22, 29], and the references therein.

In order to study the multilinear singular integral operators with multiple weights, in 2009, Lerner et al [25] introduce the multilinear version of Hardy-Littlewood maximal functions. In 2011, Grafakos, Liu, Pérez and Torres [13] introduced and studied the weighted strong and endpoint estimates for the following multilinear strong maximal function $\mathscr{M}_{\mathcal{R}}$.

Definition 0.1 (**Multilinear strong maximal function** [13]). Let $\vec{f} = (f_1, \cdots, f_m)$ be an m-dimensional vector of locally integrable functions. Define the multilinear strong maximal function $\mathscr{M}_{\mathcal{R}}$ by

$$\mathscr{M}_{\mathcal{R}}(\vec{f})(x) = \sup_{\substack{R \ni x \\ R \in \mathcal{R}}} \prod_{i=1}^{m} \frac{1}{|R|} \int_R |f_i(y_i)|dy_i, \tag{0.1}$$

where $x \in \mathbb{R}^n$ and $\mathcal{R}$ denotes the family of all rectangles in $\mathbb{R}^n$ with sides parallel to the axes.

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Whenever $m = 1$, we simply denote $\mathscr{M}_{\mathcal{R}}$ by $\mathcal{M}_{\mathcal{R}}$. Then $\mathcal{M}_{\mathcal{R}}$ coincides with the classical strong maximal operator studied by Jessen, Marcinkiewicz and Zygmund [17] in 1935. Unlike the Hardy-Littlewood maximal function, the strong maximal function $\mathcal{M}_{\mathcal{R}}$ is not of weak type $(1,1)$. Therefore, as an replacement, Jessen, Marcinkiewicz and Zygmund [17] showed that it is bounded from $L(\log^+ L)(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. In 1975, the $L(\log^+ L)(\mathbb{R}^d)$ type estimate was again proved by Córdoba and Fefferman [9] by using an alternative geometric method [9].

It is known that [13] $\mathscr{M}_{\mathcal{R}}$ is bounded from $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for all $1 < p_1, \dots, p_m, p \leq \infty$ and $1/p = \sum_{i=1}^m 1/p_i$. Moreover, for each $f_i \in L^{p_i}(\mathbb{R}^d)$, it holds that

$$\|\mathscr{M}_{\mathcal{R}}(\vec{f})\|_{L^p(\mathbb{R}^d)} \lesssim_{p_1,\dots,p_m} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^d)}. \tag{0.2}$$

0.2. **Mixed Lebesgue spaces.** We first introduce the definition of mixed Lebesgue spaces.

Definition 0.2 (**Mixed Lebesgue space** $L^{p,q}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, [23]). Let $1 \leq p,\, q < \infty$ and $n_i \geq 1\, (i = 1, 2)$, the mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$L^{p,q}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) := \{f : \mathbb{R}^{n_1+n_2} \to \mathbb{R};\ \ \|f\|_{L^{p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} < \infty\},$$

where

$$\|f\|_{L^{p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} := \Big(\int_{\mathbb{R}^{n_1}} \Big(\int_{\mathbb{R}^{n_2}} |f(x,y)|^q dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}.$$

Similarly, we can define mixed Lebesgue space with ℓ terms by $L^{p_1,p_2,\dots,p_\ell}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_\ell})$. It is easy to see that $L^{p,q}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = L^p(\mathbb{R}^{n_1+n_2})$ if $p = q$.

It is well known that this space still preserves the following properties: it is a Banach space and some classical theorems, such like monotone convergence theorem, Lebesgue's dominated convergence theorem still hold. The definition of mixed Lebesgue space can be traced back to the nice work of Benedek and Panzone [4], in 1961. Since then, achievements have been made in the study of some classical operators on mixed Lebesgue spaces. Among these achievements, there are nice works of Adams and Bagby [1], Lizorkin [34], and Milman [35].These works mainly focused on the translation-dilation invariant estimates for Riesz potentials, the multipliers of Fourier integrals and bounds of convolutions, and the interpolation problem of Banach spaces and Lorentz spaces with mixed norms. Later on, the maximal inequalities and Fourier multipliers for spaces with mixed quasi-norms were studied by Schmeisser [37] and the theory of vector-valued singular operators on mixed Lebesgue spaces was considered by Fernandez [11]. Recently, some weighted theory for maximal operators associated with some special rectangles constructed by the products of two cubes was developed by Kurtz [23]. Still more recently, Radial multipliers and restriction to surfaces of the Fourier transform in mixed-norm spaces were demonstrated by Córdoba and Latorre Crespo [10]. Moreover, the smoothing properties of bilinear operators and Leibniz-type rules in mixed Lebesgue spaces were presented very recently by Hart, Torres and Wu [16].

Based on the previous results for the Hardy-Littlewood maximal operators, the multilinear strong maximal functions and some other classical operators, it is therefore a quite natural question to ask whether the multilinear strong maximal operators are still bounded on the product of mixed Lesbegue spaces. and enjoy the regularity and continuity properties.

0.3. We will consider the boundedness, regularity and continuity properties of the multilinear strong maximal operators on the mixed Lebesgue spaces and mixed Lebesgue-Sobolev spaces. We will see that these results rely heavily on one dimensional results. To begin with, we introduce the definition of mixed Lebesgue-Sobolev spaces.

Definition 0.3 (**Mixed Lebesgue-Sobolev spaces** $W^{1,p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$). Let $1\le p\le\infty$, $1\le q\le\infty$ and $n_i\ge 1\,(i=1,2)$, the mixed Sobolev Lebesgue space $W^{1,p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is defined by

$$W^{1,p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}):=\{f:\mathbb{R}^{n_1+n_2}\to\mathbb{R};\quad \|f\|_{W^{1,p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}<\infty\},$$

where $\|f\|_{W^{1,p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n-2})}:=\|\nabla f\|_{L^{p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}+\|f\|_{L^{p,q}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$. Similarly, we can define mixed Lebesgue-Sobolev spaces with ℓ terms by $W^{1,p_1,p_2,\dots,p_\ell}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})$.

We obtain the following boundedness of $\mathscr{M}_{\mathcal{R}}$ on mixed Lebesgue spaces.

Theorem 0.1 (**Estimates on mixed Lebesgue spaces**). *Let $\ell\ge 1$ and $n=\sum_{i=1}^{\ell}n_i$ with $n_i\ge 1\,(i=1,\dots,\ell)$. Let $1<p_j,p_{1j},p_{2j},\dots,p_{mj}<\infty$ and $1/p_j=\sum_{i=1}^{m}1/p_{ij}\,(j=1,\dots,\ell)$. Then, the following inequality holds*

$$\|\mathscr{M}_{\mathcal{R}}(\vec f)\|_{L^{p_1,p_2,\dots,p_\ell}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})}\lesssim_{n,p_1,\dots,p_l}\prod_{j=1}^{m}\|f_j\|_{L^{p_{j1},p_{j2},\dots,p_{j\ell}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})}.$$

Now, we may consider the properties of $\mathscr{M}_{\mathcal{R}}$ on mixed Lebesgue-Sobolev spaces.

Theorem 0.2 (**Estimates on mixed Lebesgue-Sobolev spaces**). *Let $\ell\ge 1$ and $n=\sum_{i=1}^{\ell}n_i$ with $n_i\ge 1\,(i=1,\dots,\ell)$. Let $1<p_j,p_{1j},p_{2j},\dots,p_{mj}<\infty$ and $1/p_j=\sum_{i=1}^{m}1/p_{ij}\,(j=1,\dots,\ell)$. Then $\mathscr{M}_{\mathcal{R}}$ is bounded from products of Lebesgue-Sobolev spaces $W^{1,p_{11},p_{12},\dots,p_{1\ell}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})\times\cdots\times W^{1,p_{m1},p_{m2},\dots,p_{m\ell}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})$ to Lebesgue-Sobolev spaces $W^{1,p_1,p_2,\dots,p_\ell}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})$, and it holds that*

$$\|\mathscr{M}_{\mathcal{R}}(\vec f)\|_{W^{1,p_1,p_2,\dots,p_\ell}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})}\lesssim_{n,p_1,\dots,p_l}\prod_{j=1}^{m}\|f_i\|_{W^{1,p_{j1},p_{j2},\dots,p_{j\ell}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})}$$

Moreover, the following property holds

$$|D_l\mathscr{M}_{\mathcal{R}}(\vec f)(x)|\le\sum_{j=1}^{m}\mathscr{M}_{\mathcal{R}}(\vec F_j)(x),\quad \text{a.e. } x\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell},$$

where $\vec F_j=(f_1,\dots,f_{j-1},D_lf_j,f_{j+1},\dots,f_m)$.

Theorem 0.3 (**Continuity on mixed Lebesgue-Sobolev spaces**). *Let $\ell\ge 1$ and $n=\sum_{i=1}^{\ell}n_i$ with $n_i\ge 1\,(i=1,\dots,\ell)$. Let $1<p_j,p_{1j},p_{2j},\dots,p_{mj}<\infty$, $p_{1j}\ge p_{2j},\ge\cdots\ge p_{mj}>1$ and $1/p_j=\sum_{i=1}^{m}1/p_{ij}\,(j=1,\dots,\ell)$.Then $\mathscr{M}_{\mathcal{R}}$ is continuous from products of Lebesgue-Sobolev spaces $W^{1,p_{11},p_{12},\dots,p_{1\ell}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})\times\cdots\times W^{1,p_{m1},p_{m2},\dots,p_{m\ell}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})$ to Lebesgue-Sobolev spaces $W^{1,p_1,p_2,\dots,p_\ell}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})$.*

Remark 0.4. If $f_i,\nabla f_i\in L^r(\mathbb{R}^n)$ $(i=1,2,\dots,m)$ for some $1<r<\infty$, then the assumption $p_{1j}\ge p_{2j},\ge\cdots\ge p_{mj}>1$ can be removed in Theorem 0.3.

We state here some comments on fundamental facts on mixed L^p spaces, used in the proofs of the above results.

Lemma 0.4. *Let* $1 \le p_2 \le p_1 < \infty$ *and* $n_1, n_2 \in \mathbb{N}$. *Then*

$$\|f\|_{L^{p_2,p_1}(\mathbb{R}^{n_2}\times\mathbb{R}^{n_1})} \le \|f\|_{L^{p_1,p_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Proof. By Minkowski's inequality

$$\begin{aligned}
\|f\|_{L^{p_2,p_1}(\mathbb{R}^{n_2}\times\mathbb{R}^{n_1})} &= \left(\int_{\mathbb{R}^{n_1}}\left(\int_{\mathbb{R}^{n_2}} |f(x_1,x_2)|^{p_2}dx_2\right)^{p_1/p_2} dx_1\right)^{1/p_1} \\
&\le \left(\int_{\mathbb{R}^{n_2}}\left(\int_{\mathbb{R}^{n_1}} |f(x_1,x_2)|^{p_1}dx_1\right)^{p_2/p_1} dx_2\right)^{1/p_2} \\
&= \|f\|_{L^{p_1,p_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.
\end{aligned}$$

□

From this, it follows immediately the following:

Lemma 0.5. *Let* $1 \le p_1, p_2, \ldots, p_\ell < \infty$, $p_j \le p_{j-1}$ *and* $n_1, n_2, \ldots, n_\ell \in \mathbb{N}$. *Then,*

$$\|f\|_{L^{p_1,\ldots,p_{j-2},p_j,p_{j-1},p_{j+1},\ldots,p_\ell}(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_{j-2}}\times\mathbb{R}^{n_j}\times\mathbb{R}^{n_{j-1}}\times\mathbb{R}^{n_{j+2}}\times\cdots\times\mathbb{R}^{n_\ell})} \le \|f\|_{L^{p_1,p_2,\ldots,p_\ell}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})}.$$

Using this we get

Lemma 0.6. *Let* $1 \le p_\ell \le p_{\ell-1} \le \cdots \le p_1 < \infty$, *and* $n_1, n_2, \ldots, n_\ell \in \mathbb{N}$. *Then, for any* $2 \le j \le \ell$

$$\|f\|_{L^{p_j,p_1,\ldots,p_{j-1},p_{j+1},\ldots,p_\ell}(\mathbb{R}^{n_j}\times\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_{j-1}}\times\mathbb{R}^{n_{j+2}}\times\cdots\times\mathbb{R}^{n_\ell})} \le \|f\|_{L^{p_1,p_2,\ldots,p_\ell}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})}.$$

From this it follows

Remark 0.1. Under the assumption of Lemma 0.6, for $f \in L^{p_1,p_2,\ldots,p_\ell}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_\ell})$ we have

$$f(x_1,\ldots,x_{j-1},\cdot,x_{-j+1},\ldots,x_\ell) \in L^{p_j}(\mathbb{R}^{n_j})$$

for almost every $(x_1,\ldots,x_{j-1},x_{-j+1},\ldots,x_\ell) \in \mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_{j-1}}\times\mathbb{R}^{n_{j+2}}\times\cdots\times\mathbb{R}^{n_\ell}$.

Remark 0.2. Without the assumption of Lemma 0.6, the conclusion in the above remark does not hold. In fact, let $1 \le p < q < \infty$. Taking $1/q < a < 1/p$, set

$$f(x) = \chi_{-|x|<1,|y|<1}\frac{1}{|x-y|^a}.$$

Then $f \in L^{p,q}(\mathbb{R}\times\mathbb{R})$ but $f(x,y) \notin L^q(\mathbb{R}_y)$ for $|x|<1$, and hence $f \notin L^{q,p}(\mathbb{R}\times\mathbb{R})$.

Proof.

$$\begin{aligned}
\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x,y)|^p dx\right)^{q/p} dy\right)^{1/q} &= \left(\int_{|y|<1}\left(\int_{|x|<1}\frac{1}{|x-y|^{ap}}dx\right)^{q/p} dy\right)^{1/q} \\
&\le \left(\int_{|y|<1}\left(\int_0^1 \frac{2}{x^{ap}}dx\right)^{q/p} dy\right)^{1/q} \\
&= 2^{1/q}\left(\int_0^1 \frac{2}{x^{ap}}dx\right)^{1/p} < \infty.
\end{aligned}$$

But

$$\int_{\mathbb{R}}|f(x,y)|^q dy = \int_{|y|<1}\frac{dy}{|x-y|^{aq}} = +\infty \quad \text{for } |x|<1,$$

and hence $f \notin L^{q,p}(\mathbb{R}\times\mathbb{R})$. □

Example 0.5. Let $1 \leq p < q < \infty$ and $1/q < \alpha < 1/p$. Set

$$f(x,y) = \chi_{\{|x-y|\leq 1\}} \frac{1}{(|x-y|(1+|y|))^\alpha}.$$

Then $f \in L^{p,q}(\mathbb{R}\times\mathbb{R})$, but $f(x,\cdot) \notin L^q(\mathbb{R}_y)$ for any $x \in \mathbb{R}$. In particular, $f \notin L^{q,p}(\mathbb{R}\times\mathbb{R})$.

Furthermore, $f \notin L^r(\mathbb{R}^2)$ for any $1 \leq r < \infty$.

Proof.

$$\begin{aligned}
\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x,y)|^p dx\right)^{q/p} dy\right)^{1/q} &= \left(\int_{\mathbb{R}}\left(\int_{|x-y|\leq 1}\frac{1}{|x-y|^{p\alpha}}dx\right)^{q/p}\frac{1}{(1+|y|)^{q\alpha}}dy\right)^{1/q} \\
&\leq \left(\int_{\mathbb{R}}\left(\int_0^1 \frac{2}{x^{p\alpha}}dx\right)^{q/p}\frac{1}{(1+|y|)^{q\alpha}}dy\right)^{1/q} \\
&= \left(\frac{2}{1-p\alpha}\right)^{1/p}\left(\frac{2}{q\alpha-1}\right)^{1/q} < \infty.
\end{aligned}$$

On the other hand, for $|x-y|\leq 1$, we have $|y| \leq |x| + |y-x| \leq |x|+1$. So, we get

$$\left(\int_{\mathbb{R}}|f(x,y)|^q dy\right)^{1/q} = \left(\int_{|x-y|\leq 1}\frac{1}{(|x-y|(1+|y|)^{q\alpha}}dy\right)^{1/q} \geq \left(\int_{|y|\leq 1}\frac{dy}{|y|^{q\alpha}}\right)^{1/q}\cdot\frac{1}{(|x|+2)^\alpha} = +\infty.$$

which implies $f(x,\cdot) \notin L^q(\mathbb{R}_y)$ for any $x \in \mathbb{R}$.

Next, for $1/\alpha \leq r < \infty$ we have as above

$$\left(\int_{\mathbb{R}}|f(x,y)|^r dy\right)^{1/r} = \left(\int_{|x-y|\leq 1}\frac{1}{(|x-y|(1+|y|)^{r\alpha}}dy\right)^{1/r} \geq \left(\int_{|y|\leq 1}\frac{dy}{|y|^{r\alpha}}\right)^{1/r}\cdot\frac{1}{(|x|+2)^\alpha} = +\infty.$$

which implies $f(x,\cdot) \notin L^r(\mathbb{R}_y)$ for any $x \in \mathbb{R}$, and so $f(x,y) \notin L^r(\mathbb{R}^2)$.

If $1 \leq r < 1/\alpha$, we get

$$\begin{aligned}
\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x,y)|^r dx\right) dy\right)^{1/r} &= \left(\int_{\mathbb{R}}\left(\int_{|x-y|\leq 1}\frac{1}{|x-y|^{r\alpha}}dx\right)\frac{1}{(1+|y|)^{r\alpha}}dy\right)^{1/r} \\
&= \left(\int_{\mathbb{R}}\left(\int_0^1 \frac{2}{x^{r\alpha}}dx\right)\frac{1}{(1+|y|)^{r\alpha}}dy\right)^{1/r} \\
&= \left(\frac{2}{1-r\alpha}\right)^{1/r}\left(\int_{\mathbb{R}}\frac{1}{(1+|y|)^{r\alpha}}dy\right)^{1/r} = \infty,
\end{aligned}$$

i.e. $f \notin L^r(\mathbb{R}^2)$. □

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