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A construction of Lie algebras and (ε, δ)-Freudenthal Kantor triple systems associated with bilinear forms

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Abstract. In this work we discuss a characterization of (ϵ, δ)-Freudenthal Kantor triple systems defined by bilinear forms and give examples of such triple systems. From these results, we construct some Lie algebras or superalgebras.

1 Introduction

The concept discussed here first appeared with a class of nonassociative algebras, that is commutative Jordan algebras, which was the defining subspace g_{-1} in the Tits-Kantor-Koecher (for short TKK) construction of 3-graded Lie algebras $g = g_{-1} \oplus g_0 \oplus g_1$, such that $[g_i, g_j] \subseteq g_{i+j}$. Nonassociative algebras are rich in algebraic structures, and they provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry. Specially, the concept of nonassociative algebras such as Jordan and Lie (super)algebras plays an important role in many mathematical and physical subjects ([5],[10]-[13], [15], [26], [28], [29], [38], [47], [48], [52], [55], [56]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([1]-[4], [6]-[8], [20], [23], [24], [33], [38], [43]-[46], [49], [51]) by using the standard embedding method ([22], [41], [42], [50], [54]). In particular, the generalized Jordan triple system of second order, or $(-1, 1)$-Freudenthal Kantor triple system (for short $(-1, 1)$-FKTS), is a useful concept ([13]-[21], [34]-[37], [40], [53]) for the constructions of simple Lie algebras, while the $(-1, -1)$-FKTS plays the same role ([6], [22], [25], [27]) for the construction of Lie superalgebras, while the δ-Jordan Lie triple systems act similarly for that of Jordan superalgebras ([23], [24], [49]). Specially, we have constructed a model of Lie superalgebras $D(2, 1; \alpha)$, $G(3)$ and $F(4)$ ([25]).

The purpose of this paper is to study applications of triple systems. First, we give several examples of triple systems defined by bilinear forms, and second, we give the construction of examples of Lie algebras or superalgebras associated with the triple systems and furthermore the connection with extended Dynkin diagrams. As a final comment of the introduction, we summarize with the following scheme:

Bilinear forms

(Jordan algebras)

$$\swarrow \qquad\qquad \searrow$$

$$(\varepsilon, \delta) - FKTS \qquad \Longleftrightarrow \qquad Lie\ (super)algebra\ construction$$

(Jordan triple systems) (TKK construction)

We show how these generalized triple systems, (i.e., (ε, δ)-FKTS), correspond to certain bilinear forms $<, >$ in analogy to the case of Jordan algebras and the TKK construction.

2 Preamble and definitions

In this paper triple systems have finite dimension being defined over a field Φ of characteristic $\neq 2$ or 3, unless otherwise specified. In order to render the paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order).

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{1}$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) - K(a, b)L(x, y) = 0, \tag{2}$$

where $L(a, b)c := (abc)$ and $K(a, b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the following condition

$$(abc) = (cba). \tag{3}$$

We can generalize the concept of GJTS of 2nd order as follows (see [13], [14], [18], [22], [54] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{4}$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) + \varepsilon K(a, b)L(x, y) = 0, \tag{5}$$

where

$$L(a, b)c := (abc), \quad K(a, b)c := (acb) - \delta(bca), \tag{6}$$

is called an (ε, δ)-FKTS. An (ε, δ)-FKTS is said to be *unitary* if $Id \in \{K(a, b)\}_{span}$.

A triple system satisfying only the identity (4) is called a *generalized* FKTS (for short GFKTS), while the identity (5) is called the *second order condition.*

Remark. We note that

$$K(b, a) = -\delta K(a, b). \tag{7}$$

A triple system is called a (α, β, γ) *triple system associated with a bilinear form* if

$$(xyz) = \alpha < x, y > z + \beta < y, z > x + \gamma < z, x > y,$$

where $< x, y >$ is a bilinear form such that $< x, y >= \kappa < y, x >$, $\kappa = \pm 1$, $\alpha, \beta, \gamma \in \Phi$.

From now on we will mainly consider this type of triple system.

An (ε, δ)-FKTS is said to be *balanced* if there is a bilinear form $< x, y > \in \Phi^*$ such that $K(x, y) =< x, y > Id$.

Triple products are denoted by (xyz), $\{xyz\}$, $[xyz]$ and $< xyz >$ upon their suitability.

Remark. We note that the concept of GJTS of 2nd order coincides with that of $(-1, 1)$-FKTS. Thus we can construct the corresponding Lie algebras by means of the standard embedding method ([6], [13]-[18], [22], [25], [27], [36], [54]).
For $\delta = \pm 1$, a triple system $(a, b, c) \mapsto [abc], a, b, c \in V$ is called a *δ-Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{aligned} [abc] &= -\delta[bac], \\ [abc] + [bca] + [cab] &= 0, \\ [ab[xyz]] &= [[abx]yz] + [x[aby]z] + [xy[abz]], \end{aligned} \tag{8}$$

where $a, b, x, y, z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS*, by [14].

Proposition 2.1 ([14],[22]) *Let $U(\varepsilon, \delta)$ be an (ε, δ)-FKTS. If J is an endomorphism of $U(\varepsilon, \delta)$ such that $J < xyz >=< JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon, \delta), [xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] :=< xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > . \tag{9}$$

Corollary *Let $U(\varepsilon, \delta)$ be an (ε, δ)-FKTS. Then the vector space $T(\varepsilon, \delta) = U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L(a, d) - \delta L(c, b) & \delta K(a, c) \\ -\varepsilon K(b, d) & \varepsilon(L(d, a) - \delta L(b, c)) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} \tag{10}$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(\varepsilon, \delta) = D(T(\varepsilon, \delta), T(\varepsilon, \delta)) \oplus T(\varepsilon, \delta)$, associated to $T(\varepsilon, \delta)$ where $D(T(\varepsilon, \delta), T(\varepsilon, \delta))$ is the set of inner derivations of $T(\varepsilon, \delta)$, i.e.

$$D(T(\varepsilon, \delta), T(\varepsilon, \delta)) := \left\{\begin{pmatrix} L(a, b) & \delta K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{pmatrix}\right\}_{span},$$

$$T(\varepsilon, \delta) := \left\{\begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U(\varepsilon, \delta)\right\}_{span}.$$

Proposition 2.2 ([15],[31]) *Let U be a unitary (ε, δ)-FKTS and $L(U)$ be the standard embedding Lie (super)algebra associated with U. Then the following are equivalent:*

(i) U is simple,

(ii) the algebra L is simple,

(iii) the JTS $\mathbf{k} := \{K(a, b)\}_{span}$ is simple and nondegenerate.

3 Examples of (ε,δ)-JTS

We consider here the particular case when $K(x,y)\equiv 0$ (identically), that is of an (ε,δ)-JTS.

Example 3.1 Let V be a vector space with a symmetric bilinear form $<x,y>$. Then

$$<xyz>=<x,y>z+<y,z>x-<z,x>y$$

defines on V a $(-1,1)$-JTS.

Example 3.2 Let V be a vector space with an anti-symmetric bilinear form $<x,y>$. Then

$$<xyz>=<x,y>z+<y,z>x-<z,x>y$$

defines on V a $(1,-1)$-JTS.

Example 3.3 Let V be a vector space with a symmetric bilinear form $<x,y>$. Then

$$<xyz>=<x,y>z-<y,z>x$$

defines on V a $(-1,-1)$-JTS.

Example 3.4 Let V be a vector space with an anti-symmetric bilinear form $<x,y>$. Then

$$<xyz>=<x,y>z-<y,z>x$$

defines on V a $(1,1)$-JTS.

Proposition 3.1 *Let $(U,<xyz>)$ be an (ε,δ)-JTS. Then the triple system is a δ-LTS with respect to the new product*

$$[xyz]=<xyz>-\delta<yxz>.$$

In the next subsection we study the case of an (ε,δ)-FKTS, but we give first two examples which are not (ε,δ)-JTS as it follows.

Proposition 3.2 *Let $(U,<xyz>)$ be a triple system with $<xyz>=<y,z>x$ and $<x,y>=-\varepsilon<y,x>$. Then this triple system is an (ε,δ)-FKTS.*

Proposition 3.3 *Let U be a balanced $(1,1)$-FKTS satisfying $<<xxx>,x>\equiv 0$ (identically) and $<x,y>$ is nondegenerate. Then U has a triple product defined by*

$$<xyz>=\frac{1}{2}(<y,x>z+<y,z>x+<x,z>y).$$

4 Complex structure

We discuss here about a complex structure on the vector space $T(\varepsilon,\delta)=g_{-1}\oplus g_1$.

We set

$$E=\begin{pmatrix}0 & Id\\ 0 & 0\end{pmatrix}, F=\begin{pmatrix}0 & 0\\ Id & 0\end{pmatrix}, H=\begin{pmatrix}Id & 0\\ 0 & -Id\end{pmatrix}, J=\delta E-\varepsilon F.$$

Then, by straightforward calculations, follows

$H = [E, F],\ [H, E] = 2E,\ [H, F] = -2F,\ J^2 = -\delta\varepsilon \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}.$

Next, we define the Nijenhuis operator on $T(\varepsilon, \delta)$ by

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y], \quad X, Y \in T(\varepsilon, \delta).$$

We study here the cases $\varepsilon\delta = 1$, that is, the case of an almost complex structure, i.e. $J^2 = -Id$. The cases $\varepsilon\delta = -1$, i.e. of para complex structure, will be considered elsewhere.

Then, by straightforward but extensive calculations (to be omitted here) it follows:

Theorem 4.1 *Let U be a (ε, δ)-FKTS. Then the following identities are equivalent:*

(i) $N(X, Y) = 0$,

(ii) $L(y, x) - \delta L(x, y) = K(x, y)$.

From these results as well as differential geometry, we obtain that there exists a complex structure on $T(\varepsilon, \delta)$ if the identity $L(y, x) - \delta L(x, y) = K(x, y)$ holds.

Remark Following [30], we have examples of $(-1, -1)$-FKTSs with a complex structure, which are also anti-structurable algebras. Also we note that a generalization of this endomorphism J will be discussed in [32].

Remark. We note that if U is unitary then $L(\varepsilon, \delta)$ contains the subalgebra $sl_2 = \{H, E, F\}_{span}$, because $Id \in \mathbf{k} = \{K(a, b)\}_{span} = g_{-2}$.

5 Main results and examples

5.1 Main theorem

In this subsection we discuss about triple systems defined by bilinear forms.

Theorem 5.1 *Let U be an (α, β, γ) triple system associated with a bilinear form $< x, y >$ with $< x, y >= \kappa < y, x >$, where $\kappa = \pm 1$. If U is an (ε, δ)-FKTS then we have the following twelve cases:*

$$(\natural) \quad (\kappa, \varepsilon, \delta, \alpha, \beta, \gamma) = \begin{cases} (\pm 1, \mp 1, \pm 1, \alpha, \alpha, 0) \\ (\pm 1, \mp 1, \pm 1, 0, \beta, 0) \\ (\pm 1, \mp 1, \pm 1, \alpha, \alpha, -\alpha) \\ (\pm 1, \mp 1, \mp 1, \alpha, -\alpha, 0) \\ (\pm 1, \mp 1, \mp 1, 0, \beta, 0) \\ (\pm 1, \mp 1, \mp 1, \alpha, -\alpha, \alpha) \end{cases}$$

where $\alpha \neq 0$ and $\beta \neq 0$.

5.2 Types $B_n, B(0, n), C_n, C(n+1), B(n, 1)$ and $D(n, 1)$

From now on, we give several examples of (ε, δ)-FKTSs defined by bilinear forms and their associated Lie algebras or superalgebras of the following types:

a) B_n type Lie algebras, b) $B(0,n)$ type Lie superalgebras, c) C_n type Lie algebras, d) $C(n+1)$ type Lie superalgebras, e) $B(n,1) = osp(2n+1|2)$ or $D(n,1) = osp(2n,|2)$ type Lie superalgebras ([9]).

a) B_n type is of dimension $\dim B_n = n(n+1)$.

Let U be the set of matrices $M(1,n;\Phi)$. Then, by Proposition 3.2, it follows that the triple product

$$L(x,y) =< xyz >:=< y,z > x$$

such that the bilinear form fulfills

$$< x,y >= - < y,x >$$

is a (1,1)-FKTS. Furthermore, the standard embedding Lie algebra is 5-graded and of B_n type. For the extended Dynkin diagram, we obtain from the results of §2

$$L_{-2}\oplus L_0\oplus L_2 := D(T(1,1),T(1,1)) = \left\{ \left(\begin{array}{cc} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{array} \right| \varepsilon = 1 = \delta \Bigg) \right\}_{span} \cong$$

$$\begin{array}{ccccc} \alpha_1 & \alpha_2 & \alpha_3 & \quad\quad \alpha_{n-1} & \alpha_n \\ \circ - & \circ - & \circ - - - - - - & \circ => & \odot \\ & | & & & \\ & \circ\ \alpha_0 & & & \end{array}$$

$$= D_n\ type\ (\alpha_n\ \odot\ deleted).$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right| \varepsilon = 1 = \delta \Bigg) \right\}_{span} \cong$$

$$\begin{array}{ccccc} \alpha_1 & \alpha_2 & \alpha_3 & \quad\quad \alpha_{n-1} & \alpha_n \\ \circ - & \circ - & \circ - - - - - - & \circ => & \odot \end{array}$$

$$= A_{n-1} \oplus \Phi Id\ (\alpha_n\ \odot\ and\ \alpha_0\ \circ\ deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of D_n type by deleting α_n $\odot$ and α_0 $\circ$. We note that this triple system is not balanced.

b) $B(0,n)$ type is of dimension $\dim B(0,n) = 2n^2 + 3n$.

Let U be the set of matrices $M(1,n;\Phi)$. Then, by Proposition 3.2, it follows that the triple product

$$L(x,y)z =< xyz >:=< y,z > x$$

such that the bilinear form fulfills

$$< x,y >=< y,x >$$

is a $(-1,-1)$-FKTS. Furthermore, the standard embedding Lie superalgebra is 5-graded and of $B(0,n)$ type. For the extended Dynkin diagram, we obtain from the results of §2

$$L_{-2}\oplus L_0\oplus L_2 := D(T(-1,-1),T(-1,-1)) = \left\{\left(\begin{array}{cc} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{array}\right| \varepsilon=-1=\delta\Bigg)\right\}_{span} \cong$$

$$\begin{array}{c} \alpha_0 \quad \alpha_1 \quad \alpha_2 \qquad\quad \alpha_{n-1} \quad \alpha_n \\ \circ => \circ - \circ - - - - - - \circ => \odot \\ = C_n \ type\ (\alpha_n\ \odot\ deleted). \end{array}$$

Also, we obtain

$$L_0 := \left\{\left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array}\right| \varepsilon=-1=\delta\Bigg)\right\}_{span} \cong$$

$$\begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_3 \qquad\quad \alpha_{n-1} \quad \alpha_n \\ \circ - \circ - \circ - - - - - - \circ => \odot \\ = A_{n-1}\oplus \Phi Id\ (\alpha_n\ \odot\ and\ \alpha_0\ \circ\ deleted). \end{array}$$

Thus the last diagram is obtained from the extended Dynkin diagram of $B(0,n)$ type by deleting $\alpha_n\ \odot$ *and* $\alpha_0\ \circ$. We note that this triple system is unitary, but is not the balanced, since $K(y,\frac{y}{2<y,y>})y=y$ for any $y\in U$.

c) C_n type is of dimension $\dim C_n = n(n+1)$.

Let U be the set of matrices $M(1,2(n-1);\Phi)$. Then, by Proposition 3.2, it follows that the triple product

$$L(x,y) =< xyz >:= \frac{1}{2}(-<x,y>z+<y,z>x-<z,x>y)$$

such that the bilinear form fulfills

$$<x,y>=-<y,x>,$$

is a balanced $(1,1)$-FKTS. Indeed, we have $K(x,z)y=<x,z>y$ and $L(x,y)z-L(y,x)z=<y,x>z=-<x,y>z$. Thus this case has a complex structure by means of § 4. Furthermore, the standard embedding Lie superalgebra is 5-graded and of C_n type. For the extended Dynkin diagram, we obtain from the results of § 2

$$L_{-2}\oplus L_0\oplus L_2 := D(T(1,1),T(1,1)) = \left\{\left(\begin{array}{cc} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{array}\right| \varepsilon=1=\delta\Bigg)\right\}_{span} \cong$$

$$\begin{array}{c} \alpha_0 \quad \alpha_1 \quad \alpha_2 \qquad\quad \alpha_{n-1} \quad \alpha_n \\ \circ => \odot - \circ - - - - - - \circ <= \circ \\ = A_1\oplus C_{n-1}\ type\ (\alpha_1\ \odot\ deleted). \end{array}$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right| \varepsilon = 1 = \delta \right) \right\}_{span} \cong$$

$$\begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_3 \qquad\quad \alpha_{n-1} \quad \alpha_n \\ \odot - \circ - \circ - - - - - - \circ <= \circ \end{array}$$

$$= C_{n-1} \oplus \Phi Id \; (\alpha_1 \;\odot\; and\; \alpha_0 \;\circ\; deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of C_n type by deleting α_1 $\odot$ and α_0 $\circ$. We note that this triple system is coincides with the one defined in Proposition 3.3 due to the property $< x, y >= - < y, x >$ of the bilinear form.

d) $C(n+1)$ type is of dimension $\dim C(n+1) = 2n^2 + 5n + 1$.

Let U be the set of matrices $M(1, 2n; \Phi)$. Then, by Example 3.2, it follows that the triple product

$$L(x,y)z =< xyz >:=< x, y > z+ < y, z > x- < z, x > y$$

such that the bilinear form fulfills

$$< x, y >= - < y, x >,$$

is a $(1,-1)$-JTS since $K(x,y) \equiv 0$ (identically). Furthermore, the standard embedding Lie superalgebra is 3-graded and of $C(n+1)$ type. For the extended Dynkin diagram, we obtain

$$L_{-1} \oplus L_0 \oplus L_1 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right) \right| \varepsilon = 1 = -\delta \right\}_{span} \oplus \left\{ \left(\begin{array}{c} e \\ f \end{array} \right) \right\}_{span} \cong$$

$$\begin{array}{l} \otimes\, \alpha_1 \quad\; \alpha_2 \quad \alpha_3 \qquad\quad \alpha_n \quad \alpha_{n+1} \\ \;\; \| \;\; > \;\; \circ - \circ - - - - - - \circ <= \circ \\ \otimes\, \alpha_0 \end{array}$$

$$= C(n+1) \; type \; (\alpha_1 \;\otimes\; deleted).$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right) \right| \varepsilon = 1 = -\delta \right\}_{span} \cong$$

$$\begin{array}{c} \alpha_2 \quad \alpha_3 \qquad\quad \alpha_n \quad \alpha_{n+1} \\ \circ - \circ - - - - - - \circ <= \circ \end{array}$$

$$= C_n \oplus \Phi Id \; (\alpha_1 \;\otimes\; and\; \alpha_0 \;\otimes\; deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $C(n+1)$ type by deleting α_1 $\otimes$ and α_0 $\otimes$. We note that this triple system is not balanced.

e) $B(n,1)$ and $D(n,1)$ type are of dimension $\dim B(n,1) = 2n^2+5n+5$ and $\dim D(n,1) = 2n^2+3n+3$, respectively.

Let U be the set of matrices $M(1,l;\Phi)$. Then, by Proposition 3.2, it follows that the triple product

$$L(x,y) =< xyz >:= \frac{1}{2}(< x,y > z- < y,z > x+ < z,x > y)$$

such that the bilinear form fulfills

$$< x,y >=< y,x >$$

is a $(-1,-1)$-FKTS. Furthermore, the standard embedding Lie superalgebra is 5-graded and of $B(n,1)$ type if $l = 2n+1$, or of $D(n,1)$ type if $l = 2n$. For the extended Dynkin diagram, we obtain from the results of § 2 the following.

For the case of $B(n,1)$ type we have

$$L_{-2}\oplus L_0\oplus L_2 := D(T(-1,-1),T(-1,-1)) = \left\{ \left(\begin{array}{cc} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\begin{array}{c} \alpha_0 \quad \alpha_1 \quad \alpha_2 \qquad\qquad \alpha_n \quad \alpha_{n+1} \\ \circ => \otimes - \circ - - - - - - \circ => \circ \end{array}$$

$$= A_1 \oplus B_n \ type \ (\alpha_1 \ \otimes \ deleted).$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\begin{array}{c} \alpha_2 \quad \alpha_3 \qquad\qquad \alpha_n \quad \alpha_{n+1} \\ \circ - \circ - - - - - - \circ => \circ \end{array}$$

$$= B_n \oplus \Phi Id \ (\alpha_1 \ \otimes \ and \ \alpha_0 \ \circ \ deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $B(n,1)$ type by deleting α_1 $\otimes$ and α_0 $\circ$.

Similarly, for the case of $D(n,1)$ type we have $L_{-2}\oplus L_0\oplus L_2 \cong A_1\oplus D_n$, $L_0 \cong D_n \oplus \Phi Id$. We note that this triple system is balanced, since $K(x,y) =< x,y > Id = L(x,y) + L(y,x)$.

Remark. The examples a),b),c),d) and e) are simple triple systems, since the bilinear forms $< x,y >$ are nondegenerate.

Indeed, if $I \neq 0$ is an ideal of U then, by straightforward calculations, from the fact that $< I,U > \ U \subseteq I$ and $< \ , \ >$ is nondegenerate, we have $I = U$. Hence U is simple.

Concluding Remark. Briefly summarizing this section we have the following table:

	balanced	not balanced
complex structure	c) and e)	b)
no complex structure		a) and d)

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These references are mainly papers for our study fields.