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Classification of Zeropotent Algebras of Dimension 3 *

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1 Introduction

Let A be a (not necessarily associative) algebra over a field K. We call A *zeropotent* if $x^2 = 0$ for all $x \in A$. A zeropotent algebra A is *anti-commutative*, that is, $xy = -yx$ for all $x, y \in A$. The converse is true if the characteristic of A is not equal to 2.

In this note we discuss the classification problem of zeropotent algebras of dimension 3. In particular, we give a complete classification over an algebraically closed field of characteristic not equal to 2. We determine the isomorphism classes of algebras by determining the equivalence classes of structure matrices of algebras.

Let A be a zeropotent algebra over K of dimension 3 with a linear base $\{e_1, e_2, e_3\}$. Because A is zeropotent, $e_1^2 = e_2^2 = e_3^2 = 0$, $e_1e_2 = -e_2e_1$, $e_1e_3 = -e_3e_1$ and $e_2e_3 = -e_3e_2$. Write

$$\begin{array}{rcl} e_2e_3 & = & a_{11}e_1 + a_{12}e_2 + a_{13}e_3 \\ e_3e_1 & = & a_{21}e_1 + a_{22}e_2 + a_{23}e_3 \\ e_1e_2 & = & a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \end{array} \tag{1}$$

with $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} \in K$. With the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{2}$$

*This is a digest version of Kobayashi et al. [1].

we can rewrite (1) as

$$\begin{pmatrix} e_2e_3 \\ e_3e_1 \\ e_1e_2 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

We call (2) the *structure matrix* of the algebra A. We use the same A both for the matrix and for the algebra.

2 Matrix equation for isomorphism

Let A' be another zeropotent algebra on a base $\{e'_1, e'_2, e'_3\}$ given by

$$\begin{pmatrix} e'_2e'_3 \\ e'_3e'_1 \\ e'_1e'_2 \end{pmatrix} = A' \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix} \quad \text{with } A' = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{pmatrix}. \tag{3}$$

Let $\Phi : A \to A'$ be an isomorphism given by a transformation matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

that is,

$$\begin{pmatrix} \Phi(e_1) \\ \Phi(e_2) \\ \Phi(e_3) \end{pmatrix} = X \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}$$

Since Φ is an isomorphism, we have

$$\begin{pmatrix} \Phi(e_2)\Phi(e_3) \\ \Phi(e_3)\Phi(e_1) \\ \Phi(e_1)\Phi(e_2) \end{pmatrix} = \begin{pmatrix} \Phi(e_2e_3) \\ \Phi(e_3e_1) \\ \Phi(e_1e_2) \end{pmatrix} = A \begin{pmatrix} \Phi(e_1) \\ \Phi(e_2) \\ \Phi(e_3) \end{pmatrix} = AX \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}. \tag{4}$$

The left side of (4) is

$$\begin{pmatrix} \Phi(e_2)\Phi(e_3) \\ \Phi(e_3)\Phi(e_1) \\ \Phi(e_1)\Phi(e_2) \end{pmatrix} = Y \begin{pmatrix} e'_2e'_3 \\ e'_3e'_1 \\ e'_1e'_2 \end{pmatrix} = YA' \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}, \tag{5}$$

where Y is the cofactor matrix of X. Because $Y = |X|\,{}^tX^{-1}$, by (4) and (5) we get

$$A' = \frac{1}{|X|}\,{}^tXAX. \tag{6}$$

Theorem 2.1. *A and A' are isomorphic if and only if there is a nonsingular matrix X (transformation matrix) satisfying (6). If K is algebraically closed, we can choose X as $|X| = 1$.*

Cororally 2.2. *If A and A' are isomorphic, then*
(i) $\operatorname{rank} A = \operatorname{rank} A'$, *and*
(ii) *A is symmetric if and only if A' is symmetric.*

3 Jacobi elements

By Corollary 2.2, the rank and symmetry are invariants under isomorphism of algebras. Another important invariant is the *Jacobi element* $\mathrm{jac}(A)$ of A, which is defined, with respect to the base $\{e_1, e_2, e_3\}$, by

$$\mathrm{jac}(A) = e_1(e_2e_3) + e_2(e_3e_1) + e_3(e_1e_2).$$

Proposition 3.1. (i) *If A is symmetric, then* $\mathrm{jac}(A) = 0$.
(ii) *If A is a Lie algebra if and only if* $\mathrm{jac}(A) = 0$.
(iii) *When* $\mathrm{rank}(A) = 3$, *A is a Lie algebra if and only if A is symmetric.*

For algebras A and A' with structure matrices in (2) and (3) respectively, let

$$\mathrm{jac}(A) = a_1e_1 + a_2e_2 + a_3e_3 \quad \text{and} \quad \mathrm{jac}(A') = a_1'e_1' + a_2'e_2' + a_3'e_3'.$$

Then, we have

Proposition 3.2 (Invariance of Jacobi elements)**.** *If A and A' are isomorphic with a transformation matrix X, then*

$$(a_1, a_2, a_3)X = |X|(a_1', a_2', a_3').$$

4 Classification

We give a classification result over the complex number field $K = \mathbb{C}$. Let

$$\mathcal{H} = \{z \in \mathbb{C} \mid -\pi/2 < \arg(z) \leq \pi/2\}$$

be the half plane.

Theorem 4.1. *Up to isomorphism, zeropotent algebras of dimension 3 over $\mathbb{C}$ are classified into 10 families*

$$A_0, A_1, A_2, A_3, \{A_4(a)\}_{a\in\mathcal{H}}, A_5, A_6, \{A_7(a)\}_{a\in\mathcal{H}}, A_8, A_9$$

defined by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively. Among them, symmetric algebras are

$$A_0, A_1, A_4(0), A_7(0)$$

and asymmetric Lie algebras are

$$A_2, A_3, \{A_4(a)\}_{a\in(H)\setminus\{0\}}.$$

This classification is valid even over an arbitrary algebraically closed field of characteristic not equal to 2.

5 Transformation

Let us take a quick look at a part of the ways how general matrices are transformed to the forms listed in Theorem 4.1.

Let A be a matrix of rank 3 given in (2), and let

$$X = \begin{pmatrix} \sqrt{\dfrac{c_{11}}{\det A}} & 0 & 0 \\ \dfrac{c_{12}}{\sqrt{c_{11}\det A}} & \sqrt{\dfrac{a_{33}}{c_{11}}} & 0 \\ \dfrac{c_{13}}{\sqrt{c_{11}\det A}} & \dfrac{-a_{32}}{\sqrt{a_{33}c_{11}}} & \dfrac{1}{\sqrt{a_{33}}} \end{pmatrix}, \tag{7}$$

where c_{ij} is the (i,j)-cofactor of A, for example, $c_{11} = a_{22}a_{33} - a_{23}a_{32}$. Then, we have

$${}^tXAX = A(a,b,c) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$a = \frac{c_{12} - c_{21}}{\sqrt{a_{33}\det A}}, \quad b = \frac{a_{23}c_{12} + a_{13}c_{11} + a_{33}c_{13}}{\sqrt{a_{33}c_{11}\det \mathrm{A}}}, \quad c = \frac{a_{23} - a_{32}}{\sqrt{c_{11}}}.$$

Thus, A is isomorphic to an algebra with upper-triangular structure matrix by the transformation matrix X in (7).

Next, with the matrix

$$Y = \begin{pmatrix} 0 & \dfrac{h}{d} & \dfrac{c}{d} \\ -\dfrac{a}{h} & \dfrac{bc - ad^2}{hd} & -\dfrac{b}{d} \\ \dfrac{ac - b}{h} & \dfrac{(ac-b)d^2 - ac}{hd} & \dfrac{a}{d} \end{pmatrix}, \tag{8}$$

where $h = \sqrt{a^2 + b^2 - abc}$ and $d = \sqrt{a^2 + b^2 + c^2 - abc}$, we have

$${}^tY\,A(a,b,c)\,Y = A(d,0,0) = \begin{pmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, $A(a,b,c)$ is isomorphic to the algebra $A_7(d)$ in Theorem 4.1 with the transformation matrix Y in (8), if $h \neq 0$ and $d \neq 0$. Consequently, an algebra of rank 3 is isomorphic to $A_7(d)$ in a generic case.

References

[1] Y. Kobayashi, K. Shirayanagi, S.-E. Takahasi and M. Tsukada, *Classification of three-dimensional zeropotent algebras over an algebraically closed field*, Comm. Algebra, Vol. 45, Iss. 12, 5037–5052, 2017.