Notes on Self-Commuting Functions on a Finite Set

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Abstract

A multi-variable function is said to be self-commuting if it commutes with itself. As the first step toward the characterization of self-commuting functions defined on a finite set this article studies very basic facts on them. We restrict our attention to self-commuting and conservative binary functions. Such functions on a three-element set are characterized and some examples of such functions on arbitrary finite set are presented.

Keywords: commutation; self-commuting function; conservative function

1 Introduction

Ever since the concept of commutation was introduced on the set of finitary functions over some set, commutation and its related topics have been studied by many authors. (See, e.g., [BW87], [Da79], [Her08], [MR04], [MR11], [Sza85].)

When two functions f and g commute we write $f \perp g$. (The definition of commutation will appear in the next section.) The binary relation induced by $\perp$ is symmetric, but it is not reflexive, that is, there exists a function which does not commute with itself. We call a function *self-commuting* if it commutes with itself.

Our goal is the characterization of self-commuting functions defined on arbitrary finite set. As the first step toward this goal, we take up in this note conservative binary functions and present some preparatory results on self-commuting and conservative binary functions. In Section 4 such functions on a three-element set are characterized and in Section 5 an attempt is made to generalize the result on a three-element set to that on any finite set.[1]

2 Definitions and Notations

Let $E_k = \{0, 1, \ldots, k-1\}$ for finite $k > 1$. We denote by $\mathcal{O}_k^{(n)}$ for $n \geq 1$ the set of n-variable functions defined on E_k, that is, the set of maps from E_k^n into E_k, and by $\mathcal{O}_k$ the set of functions defined over E_k, i.e., $\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}$. A *projection* e_i^n, for $n > 0$ and $1 \leq i \leq n$, is the function in $\mathcal{O}_k^{(n)}$ which always takes the value of the i-th variable. The set of all

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[1]After finishing this article, the complete characterization of self-commuting and conservative binary functions on arbitrary finite set was obtained (P. P. Pálfy and H. Machida).

projections is denoted by $\mathcal{J}_k$. A *clone* over E_k is a subset C of $\mathcal{O}_k$ which is closed under (functional) composition and includes $\mathcal{J}_k$.

Commutation for two multi-variable functions are defined in the following way.

Definition 2.1 *For $f \in \mathcal{O}_k^{(n)}$ and $g \in \mathcal{O}_k^{(m)}$, f and g* commute *if*

$$\begin{aligned} & g\left(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn})\right) \\ = \; & f\left(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn})\right) \end{aligned}$$

holds for all $a_{ij} \in E_k$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$.

Let us introduce the operation $\diamondsuit : \mathcal{O}_k^{(m)} \times \mathcal{O}_k^{(n)} \longrightarrow E_k^{\mathcal{M}_{m,n}(E_k)}$ in the following way: For $f \in \mathcal{O}_k^{(n)}$, $g \in \mathcal{O}_k^{(m)}$ and $M \in \mathcal{M}_{m,n}(E_k)$ define

$$(g \diamondsuit f)(M) \quad = \quad g\left(f(\boldsymbol{r}_1), \ldots, f(\boldsymbol{r}_m)\right)$$

where $\boldsymbol{r}_i$ is the i-th row of M $(1 \le i \le m)$. Then, f and g commute if

$$(g \diamondsuit f)(M) \quad = \quad (f \diamondsuit g)({}^tM)$$

holds for all $M \in \mathcal{M}_{m,n}(E_k)$.

When f and g commute, we write $f \perp g$. Obviously, the binary relation $\perp$ is symmetric. However, the relation $\perp$ is *not* reflexive, that is, $f \not\perp f$ holds for some f in $\mathcal{O}_k$.

Definition 2.2 *For $f \in \mathcal{O}_k$, we say f is* self-commuting *if $f \perp f$ holds.*

There are plenty of functions which are not self-commuting. We draw some such examples from $\mathcal{O}_2^{(2)}$, i.e., binary functions on E_2. The binary function f will be denoted by f_i $(0 \le i \le 15)$ when $i = f(0,0) \cdot 2^3 + f(0,1) \cdot 2^2 + f(1,0) \cdot 2 + f(1,1)$. Thus, for example, f_0 is the constant function taking 0, f_1 is AND and f_3 is the projection e_1^2.

In Table 1, the last column shows whether a function f is self-commuting or not. The symbol **o** in the i-th row indicates the function f_i is self-commuting and **x** indicates it is not self-commuting. Among 16 functions, 10 are self-commuting and 6 are not self-commuting. In order to verify the results in the table, we pick up just one case: f_{14} (= NAND). Consider the following $(2,2)$-matrix M.

$$M = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is immediate to see that $f_{14} \not\perp f_{14}$ is verified by this M.

3 Some Simple Properties

(1) For $f \in \mathcal{O}_k^{(n)}$, $g \in \mathcal{O}_k^{(m)}$ and $M \in \mathcal{M}_{m,n}(E_k)$, we shall write

$$f \perp g \quad \text{on} \quad M$$

if $(g \diamondsuit f)(M) = (f \diamondsuit g)({}^tM)$ holds. Thus, $f \perp g$ if and only if "$f \perp g$ on M" holds for all M in $\mathcal{M}_{m,n}(E_k)$.

	$f(0,0)$	$f(0,1)$	$f(1,0)$	$f(1,1)$	$\perp$
f_0 (c_0)	0	0	0	0	o
f_1 (AND)	0	0	0	1	o
f_2	0	0	1	0	x
f_3 (e_1^2)	0	0	1	1	o
f_4	0	1	0	0	x
f_5 (e_2^2)	0	1	0	1	o
f_6 $(\oplus)$	0	1	1	0	o
f_7 (OR)	0	1	1	1	o
f_8 (NOR)	1	0	0	0	x
f_9	1	0	0	1	o
f_{10} $(\neg e_2^2)$	1	0	1	0	o
f_{11}	1	0	1	1	x
f_{12} $(\neg e_1^2)$	1	1	0	0	o
f_{13}	1	1	0	1	x
f_{14} (NAND)	1	1	1	0	x
f_{15} (c_1)	1	1	1	1	o

Table 1: Binary Functions on E_2

Lemma 3.1 *For $f \in \mathcal{O}_k^{(n)}$ and $M \in \mathcal{M}_n(E_k)$ we have:*

(1) If M is symmetric then $f \perp f$ on M

(2) $f \perp f$ on M if and only if $f \perp f$ on ${}^tM$ (Here, ${}^tM$ is the transposed matrix of M.)

(2) Let us define $\#M = |\{a_{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}|$ for $M = (a_{ij}) \in \mathcal{M}_{m,n}(E_k)$. For example, let

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Then we have $\#M_1 = 1$, $\#M_2 = 2$, $\#M_3 = 3$.

Next, $f \in \mathcal{O}_k^{(n)}$ is said to be *conservative* if $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ holds for all $a_1, \ldots, a_n \in E_k$.

Lemma 3.2 *For $f \in \mathcal{O}_k^{(2)}$ and $M \in \mathcal{M}_2(E_k)$, if f is conservative and $\#M \le 2$ then $f \perp f$ on M.*

Proof When $\#M = 1$ the assertion is trivial. Suppose $\#M = 2$. Then, due to Lemma 3.1, among the following matrices where $a \neq b$ we need to consider only two matrices (the second and the fifth).

$$\begin{pmatrix} a & a \\ a & b \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{a} \end{pmatrix} \begin{pmatrix} a & b \\ a & a \end{pmatrix} \begin{pmatrix} b & a \\ a & a \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{b} \end{pmatrix} \begin{pmatrix} a & b \\ a & b \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Case 1: $M_1 = \begin{pmatrix} a & a \\ b & a \end{pmatrix}$

Let $\spadesuit = f(f(a,a), f(b,a)) = f(a, f(b,a))$ and $\heartsuit = f(f(a,b), f(a,a)) = f(f(a,b), a)$. If

$f(a,b) = a$ then it is clear that $\spadesuit = \heartsuit = a$. Next, suppose $f(a,b) = b$. Then, $f(b,a) = a$ implies $\spadesuit = \heartsuit = a$ and $f(b,a) = b$ implies $\spadesuit = \heartsuit = b$. Hence $f \perp f$ on M_1.

Case 2: $M_2 = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$

Clearly we have $f(f(a,a), f(b,b)) = f(a,b) = f(f(a,b), f(a,b))$, showing $f \perp f$ on M_2. □

Corollary 3.3 *For a binary function f on E_2, i.e., $f \in \mathcal{O}_2^{(2)}$, if f is conservative then f is self-commuting.*

4 Binary Functions on E_3

In this section we consider 2-variable functions on $E_3 = \{0, 1, 2\}$. Our aim in this section is to find all conservative $f \in \mathcal{O}_3^{(2)}$ satisfying $f \perp f$.

In order to find all such functions, we need to consider M only of the following type, because of Lemma 3.1 again.

$$\text{(A)} \begin{pmatrix} a_1 & 0 \\ 1 & a_2 \end{pmatrix} \qquad \text{(B)} \begin{pmatrix} b_1 & 0 \\ 2 & b_2 \end{pmatrix} \qquad \text{(C)} \begin{pmatrix} c_1 & 1 \\ 2 & c_2 \end{pmatrix}$$

First, take, for example, a matrix M_0 of type (A):

$$M_0 = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

In this case, in order to have the condition "$f \perp f$ on M_0", f must satisfy

$$f(0, f(1,2)) \quad = \quad f(f(0,1), f(0,2)).$$

It follows that

- $f(0,1) = 0,\ f(0,2) = 2 \implies f(1,2) = 2$
- $f(0,1) = 1,\ f(0,2) = 0,\ f(1,2) = 1 \implies f(1,0) = 1$
  $f(0,1) = 1,\ f(0,2) = 0,\ f(1,2) = 2 \implies f(1,0) = 0$

In other words, each row in the following table gives a forbidden combination for M_0:

$f(0,1)$	$f(0,2)$	$f(1,0)$	$f(1,2)$	$f(2,0)$	$f(2,1)$
0	2		1		
1	0	0	1		
1	0	1	2		

By applying the similar consideration to other matrices of type (A) we get the following table of forbidden combinations.

(A)

$f(0,1)$	$f(0,2)$	$f(1,0)$	$f(1,2)$	$f(2,0)$	$f(2,1)$
0	2		1		
1	0	0	1		
	0	1	2		
0		1		0	1
		0		2	1
1				0	2
	2		1	0	2
	0		2	2	1

The forbidden combinations for types (B) and (C) are the following:

(B)

$f(0,1)$	$f(0,2)$	$f(1,0)$	$f(1,2)$	$f(2,0)$	$f(2,1)$
1	0				2
0	2			0	2
	0	0	2	2	
		1	2	0	
1		0	1		2
0		1	2		1
	2	0	1		
0				2	1

(C)

$f(0,1)$	$f(0,2)$	$f(1,0)$	$f(1,2)$	$f(2,0)$	$f(2,1)$
0	2	1		0	
1	0	0		2	
0	2				1
1	2		1		2
1	0		2		
		0	1	2	
		1	2	2	1
		1		0	2

Now, the list of the binary functions on E_3 which are self-commuting and conservative is:

	$f(0,1)$	$f(0,2)$	$f(1,0)$	$f(1,2)$	$f(2,0)$	$f(2,1)$
K_0	**0**	**0**	**0**	1	**0**	1
	0	**0**	**0**	1	**0**	2
	0	**0**	**0**	2	**0**	1
	0	**0**	**0**	2	**0**	2
K_1	**1**	0	**1**	**1**	0	**1**
	1	0	**1**	**1**	2	**1**
	1	2	**1**	**1**	0	**1**
	1	2	**1**	**1**	2	**1**
K_2	0	**2**	0	**2**	**2**	**2**
	0	**2**	1	**2**	**2**	**2**
	1	**2**	0	**2**	**2**	**2**
	1	**2**	1	**2**	**2**	**2**
$\mathcal{J}_3$	0	0	1	1	2	2
	1	2	0	2	0	1

To rephrase, we conclude as follows.

Proposition 4.1 *If $f \in \mathcal{O}_3^{(2)}$ is self-commuting and conservative, then f is a projection or one of the following shape. (Here the symbol $*$ indicates arbitrary element provided that conservativeness is preserved.)*

$x\backslash y$	0	1	2
0	0	0	0
1	0	1	$*$
2	0	$*$	2

$x\backslash y$	0	1	2
0	0	1	$*$
1	1	1	1
2	$*$	1	2

$x\backslash y$	0	1	2
0	0	$*$	2
1	$*$	1	2
2	2	2	2

5 Binary Functions on E_k

In this section we consider more general case: E_k for any $k > 1$. We present some examples of self-commuting and conservative binary functions on E_k.

Notice that if $f \in \mathcal{O}_k^{(2)}$ is self-commuting and conservative then the following property, which we may call the 3-element property, must be satisfied for f.

"The 3-element property": For *every* 3-element subset $\{a, b, c\}$ of E_k, the 'sub-tables' of Cayley table of f consisting of the rows and the columns corresponding to a, b and c are one of the following forms. (Here the symbol $*$ is arbitrary up to preserving conservativeness.)

$x\backslash y$	a	b	c
a	a	a	a
b	a	b	$*$
c	a	$*$	c

$x\backslash y$	a	b	c
a	a	b	$*$
b	b	b	b
c	$*$	b	c

$x\backslash y$	a	b	c
a	a	$*$	c
b	$*$	b	c
c	c	c	c

Now, the following Cayley tables are three examples of self-commuting and conservative binary functions f ($\in \mathcal{O}_k^{(2)}$) on E_k. (Again, $*$ is arbitrary up to preserving conservativeness.)

(1)

$x\backslash y$	0	1	2	3	$\cdots$	$k-2$	$k-1$
0	0	$*$	2	3	$\cdots$	$k-2$	$k-1$
1	$*$	1	2	3	$\cdots$	$k-2$	$k-1$
2	2	2	2	3	$\cdots$	$k-2$	$k-1$
3	3	3	3	3	$\cdots$	$k-2$	$k-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$k-2$	$k-2$	$k-2$	$k-2$	$k-2$	$\cdots$	$k-2$	$k-1$
$k-1$	$k-1$	$k-1$	$k-1$	$k-1$	$\cdots$	$k-1$	$k-1$

(2)

$x\backslash y$	0	1	2	3	$\cdots$	$k-2$	$k-1$
0	0	0	0	3	$\cdots$	$k-2$	$k-1$
1	0	1	$*$	3	$\cdots$	$k-2$	$k-1$
2	0	$*$	2	3	$\cdots$	$k-2$	$k-1$
3	3	3	3	3	$\cdots$	$k-2$	$k-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$k-2$	$k-2$	$k-2$	$k-2$	$k-2$	$\cdots$	$k-2$	$k-1$
$k-1$	$k-1$	$k-1$	$k-1$	$k-1$	$\cdots$	$k-1$	$k-1$

(3)

$x\backslash y$	0	1	2	$\cdots$	3	$k-2$	$k-1$
0	0	0	0	$\cdots$	0	0	0
1	0	1	1	$\cdots$	1	1	1
2	0	1	2	$\cdots$	2	2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$k-3$	0	1	2	$\cdots$	$k-3$	$k-3$	$k-3$
$k-2$	0	1	2	$\cdots$	$k-3$	$k-2$	$*$
$k-1$	0	1	2	$\cdots$	$k-3$	$*$	$k-1$

We remark that the 3-element property can be effectively used to obtain these examples.

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