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ON THE RELATION BETWEEN UNIFORM K-STABILITY AND CHOW STABILITY OF TORIC VARIETIES

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Abstract. The aim of this note is to announce the current research progress in [HIY18] concerning the relation between uniform K-stability and asymptotic Chow stability for polarized toric varieties.

1. Preliminary toric GIT stability

1.1. **Polarized toric varieties.** Let Δ be an n-dimensional lattice polytope in $\mathbb{R}^n$ which is described by an intersection of half spaces:

$$\Delta = \bigcap_{i=1}^{d} \{ x \in \mathbb{R}^n \mid \ell_i(x) := \langle x, v_i \rangle + \lambda_i \geqslant 0 \} \tag{1.1}$$

where $\lambda_i \in \mathbb{Z},\ v_i \in \mathbb{Z}^n$ is primitive and d is the number of facets of Δ. We denote the interior and the boundary of Δ by Δ° and $\partial\Delta$ respectively. The set of vertices of Δ is written by $\mathcal{V}(\Delta)$. Analogously $\mathcal{F}(\Delta)$ denotes the set of facets. For a finite set S, a *convex polyhedral cone* of S is a set of the form

$$\mathrm{Cone}(S) := \left\{ \sum_{u\in S} c_u u \;\middle|\; c_u \geqslant 0 \right\}.$$

We observe that a convex polytope Δ gives a convex polyhedral cone $C(\Delta) \subseteq \mathbb{R}^n \times \mathbb{R}$, called the *cone of* Δ and defined by

$$C(\Delta) := \{ c \cdot (u, 1) \in \mathbb{R}^n \times \mathbb{R} \mid u \in \Delta,\ c \geqslant 0 \}.$$

If $\Delta = \mathrm{Conv}(S)$, then this is described as $C(\Delta) = \mathrm{Cone}(S \times \{1\})$. Defining $S_\Delta = C(\Delta) \subset \mathbb{Z}^{n+1}$, one can see that S_Δ is an affine (finitely generated) semigroup by Gordan's lemma. Let us denote its semigroup ring by $\mathbb{C}[S_\Delta]$. The character corresponding to $(m,k) \in S_\Delta$ is $\chi^m t^k$, and $\mathbb{C}[S_\Delta]$ is

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graded by height, i.e., $\deg(\chi^m t^k) = k$. Consequently, we obtain the graded $\mathbb{C}$-algebra

$$\mathbb{C}[S_\Delta] = \bigoplus_{k \in \mathbb{Z}} R_k, \qquad R_k := \{ f \in \mathbb{C}[S_\Delta] \mid \deg f = k \},$$

from a polytope Δ. We define the polarized toric variety $(X_\Delta, \mathbb{L}_\Delta)$ by

$$(X_\Delta, \mathbb{L}_\Delta) := (\operatorname{Proj} \mathbb{C}[S_\Delta], \mathcal{O}_{X_\Delta}(1)).$$

Observe that the semigroup S_Δ is generated by $(\Delta \cap \mathbb{Z}^n) \times \{ 1 \}$ in the above construction. This implies that the line bundle $\mathbb{L}_\Delta = \mathcal{O}_{X_\Delta}(1)$ is very ample.

1.2. **Set up and notation.** We fix our notation as follows.

- Throughout the paper $\Delta \subset \mathbb{R}^n$ denotes an n-dimensional convex lattice polytope in $\mathbb{R}^n$ with the form (1.1).
- $(X_\Delta, \mathbb{L}_\Delta)$ is the associated polarized toric variety as constructed in Section 1.1.
- For $i \in \mathbb{Z}_{\geq 0}$, $E_\Delta(i)$ denotes the Ehrhart polynomial of Δ satisfying

$$E_\Delta(i) := \# \left(\sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \mathbf{a} \right) = \dim H^0(X_\Delta, \mathbb{L}_\Delta^i).$$

- For $i \in \mathbb{Z}_{\geq 0}$, we set the *sum polynomial of* Δ which is the $\mathbb{R}^n$-valued polynomial and is given by

$$s_\Delta(i) := \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \mathbf{a} = \frac{1}{i} \sum_{\mathbf{a} \in i\Delta \cap \mathbb{Z}^n} \mathbf{a}.$$

Let $\operatorname{Aut}^0(X_\Delta)$ denote the identity component of the automorphism group of X_Δ. Then there is a maximal torus $T = (\mathbb{C}^\times)^n < \operatorname{Aut}^0(X_\Delta)$ by Demazure's structure theorem. Denoting the normalizer of T in $\operatorname{Aut}^0(X_\Delta)$ by $N(T)$, we define the *Weyl group* $W(X) := N(T)/T$.

1.3. **Chow stability.** Next we define Chow form and Chow stability of irreducible projective varieties. See [GKZ94] for more details. Let $X \subset \mathbb{C}P^N$ be an n-dimensional irreducible complex projective variety of degree d. Recall that the Grassmann variety $\mathbb{G}(k, \mathbb{C}P^N)$ parameterizes k-dimensional projective linear subspaces of $\mathbb{C}P^N$. The *associated hypersurface* of $X \subset \mathbb{C}P^N$ is the subvariety in $\mathbb{G}(N-n-1, \mathbb{C}P^N)$ which is given by

$$Z_X := \{ W \in \mathbb{G}(N-n-1, \mathbb{C}P^N) \mid W \cap X \neq \emptyset \}.$$

It is known that is Z_X is an irreducible hypersurface with $\deg Z_X = d$ in the Plücker coordinates. In particular, Z_X is given by the vanishing of a section $R_X^* \in H^0(\mathbb{G}(N-n-1, \mathbb{C}P^N), \mathcal{O}(d))$. We call R_X^* the *Chow form* of

X. Note that R_X^* is well defined up to a multiplicative constant. Let $\mathbf{V} := H^0(\mathbb{G}(N-n-1, \mathbb{C}P^N), \mathcal{O}(d))$ and $R_X \in \mathbb{P}(\mathbf{V})$ be the projectivization of R_X^*. We call R_X the *Chow point* of X. Since we have the natural action of $G = \mathrm{SL}(N+1, \mathbb{C})$ into $\mathbb{P}(\mathbf{V})$, we can define stabilities of R_X as follows.

Definition 1.1. Let $X \subset \mathbb{C}P^N$ be an irreducible, n-dimensional complex projective variety. Then X is said to be *Chow polystable* if the Chow point R_X of X is $\mathrm{SL}(N+1, \mathbb{C})$-polystable i.e., the $\mathrm{SL}(N+1, \mathbb{C})$-orbit of R_X in $\mathbf{V}$ is a closed orbit.

Definition 1.2. Let $(X, \mathbb{L})$ be a polarized variety. For $i \gg 0$, let $\Psi_i : X \longrightarrow \mathbb{P}(H^0(X, \mathbb{L}^i)^*)$ be the Kodaira embedding.

(1) Suppose that $\mathbb{L}$ is very ample. $(X, \mathbb{L})$ is said to be *Chow polystable* if $\Psi_1(X) \subset \mathbb{P}(H^0(X, \mathbb{L})^*)$ is Chow polystable.
(2) $(X, \mathbb{L})$ is called *asymptotically Chow polystable* if there is an i_0 such that $\Psi_i(X)$ is Chow polystable for each $i \geqslant i_0$.

1.4. **Chow weight of** $(X_\Delta, \mathbb{L}_\Delta^i)$**.** Let $(X_\Delta, \mathbb{L}_\Delta)$ be a polarized toric variety with the moment polytope $\Delta \subset M_{\mathbb{R}}$. We fix any $i \in \mathbb{Z}_{>0}$. Let $(\mathscr{X}, \mathscr{L}) \to \mathbb{P}^1$ be any T-equivariant test configuration of $(X_\Delta, \mathbb{L}_\Delta^i)$.

Theorem 1.3 (Theorem 1.1 [Ono13], Corollary 2.7, [LLSW17])**.** *In the above, the Chow weight for the degeneration $(\mathscr{X}, \mathscr{L}) \to \mathbb{P}^1$ is given by*

$$Q_\Delta(i, g) := E_\Delta(i) \int_\Delta g \, dx - \mathrm{vol}(\Delta) \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} g(\mathbf{a}) \tag{1.2}$$

where g is the corresponding rational piece-wise linear concave function over Δ. In particular, $(X_\Delta, \mathbb{L}_\Delta^i)$ is Chow polystable iff $Q_\Delta(i, g) \geq 0$ holds for any Weyl group invariant concave piece-wise linear function

$$g \in \mathrm{PL}(\Delta; i)^{W(X)} = \{ g \in \mathrm{PL}(\Delta; i) \mid g(w \cdot x) = g(x) \quad \forall w \in W \},$$

and equality holds when and only when g is an affine linear.

Applying (1.2) to linear functions, one can see the following.

Corollary 1.4 (Corollary 4.7 [Ono13])**.** *If $(X_\Delta, \mathbb{L}_\Delta^i)$ is Chow semistable for $i \in \mathbb{Z}_{>0}$, then*

$$\mathrm{Chow}_\Delta(i) := E_\Delta(i) \int_\Delta \mathbf{x} \, dx - \mathrm{vol}(\Delta) \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \mathbf{a} \equiv 0.$$

In short, the equality

$$\sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} \mathbf{a} = \frac{E_\Delta(i)}{\mathrm{vol}(\Delta)} \int_\Delta \mathbf{x} \, dx \tag{1.3}$$

holds.

By the equality (1.3), one can see that $Q_\Delta(i,g)$ is invariant when adding an affine linear function to g, and is homogeneous with respect to g.

Proof of Theorem 1.3. Since $Q_\Delta(i,g)$ is invariant under adding a constant, we may assume $g \geqslant 0$. Let $(\mathscr{X}, \mathscr{L}) \to \mathbb{P}^1$ be a T-equivariant toric test configuration of $(X_\Delta, \mathbb{L}_\Delta^i)$ and g be the corresponding piece-wise linear function. Hence $\mathscr{X}$ is an $(n+1)$-dimensional toric variety with the moment polytope

$$Q_g := \{ (x,\lambda) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mid 0 \leq \lambda \leq g(x) \}$$

We observe that

$$\text{vol}(Q_g) = \int_\Delta g(x)\, dx, \qquad E_{Q_g}(i) - E_\Delta(i) = \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} g(\mathbf{a}) \tag{1.4}$$

In the proof of Proposition 4.2.1 in [Dona02], the weight of $\mathbb{C}^\times$-action on $\bigwedge^{E_\Delta(m)} H^0(\mathscr{X}_0, \mathscr{L}^m\big|_{\mathscr{X}_0})$ is given by

$$\begin{aligned} w_m &= \dim H^0(\mathscr{X}_{Q_g}, \mathscr{L}_{Q_g}^m) - \dim H^0(X_\Delta, \mathbb{L}_\Delta^m) \\ &= E_{Q_g}(m) - E_\Delta(m) \\ &= a_{n+1}(i) m^{n+1} + a_n(i) m^n + \dots \end{aligned}$$

where

$$a_k(i) = a_{kn} i^n + a_{k,n-1} i^{n-1} + \dots$$

Note that there are asymptotic expansions

$$E_{Q_g}(m) = \text{vol}(Q_g) m^{n+1} + \mathcal{O}(m^n), \qquad E_\Delta(m) = \text{vol}(\Delta) m^n + \mathcal{O}(m^{n-1})$$

by the Ehrhart theorem. As in [RT07], the Chow weight for the degeneration $(\mathscr{X}, \mathscr{L}) \to \mathbb{P}^1$ is given by the *normalized* leading coefficient of $a_{n+1}(i)$, we compute

$$\begin{aligned} & w_m - m E_\Delta(m) \frac{w_i}{E_\Delta(i)} \\ &= (E_{Q_g}(m) - E_\Delta(m)) - m E_\Delta(m) \frac{E_{Q_g}(i) - E_\Delta(i)}{E_\Delta(i)} \\ &= \text{vol}(Q_g) m^{n+1} - \text{vol}(\Delta) \frac{E_{Q_g}(i) - E_\Delta(i)}{E_\Delta(i)} m^{n+1} + \mathcal{O}(m^n) \\ &= m^{n+1} \left(\int_\Delta g\, dx - \frac{\text{vol}(\Delta)}{E_\Delta(i)} \sum_{\mathbf{a} \in \Delta \cap (\mathbb{Z}/i)^n} g(\mathbf{a}) \right) + \mathcal{O}(m^n) \end{aligned}$$

Here we used (1.4) in the last equality. The assertion is verified. □

1.5. **Uniform K-stability.** It is crucial to see the coercivity of the K-energy map when we consider the existence problem of constant scalar curvature Kähler metrics on a certain polarized manifold $(X, \mathbb{L})$. It has been conjectured that the coercivity property of the K-energy map is corresponding to **uniform K-stability** of $(X, \mathbb{L})$. In [Hisa16], this conjecture was justified in the case where $(X, \mathbb{L})$ is a polarized toric manifold. The toric reduction of uniform K-stability is the following.

Definition 1.5 (Hisamoto, [Hisa16])**.** Let $(X_\Delta, \mathbb{L}_\Delta)$ be a polarized toric variety with the moment polytope $\Delta \subset M_\mathbb{R}$. For a rational piece-wise linear convex function u over Δ, we define

$$\mathcal{L}_\Delta(u) := \int_{\partial\Delta} u \, d\sigma - \frac{\mathrm{vol}(\partial\Delta)}{\mathrm{vol}(\Delta)} \int_\Delta u \, dx.$$

Then $(X_\Delta, \mathbb{L}_\Delta)$ is said to be *uniformly K-polystable in the toric sense* if there exists a constant $\delta_\Delta > 0$ such that

$$\mathcal{L}_\Delta(u) \geqslant \delta_\Delta \left\| u \right\|_J \tag{1.5}$$

where $\left\| u \right\|_J$ is the J-norm defined as

$$\left\| u \right\|_J := \inf_\ell \left\{ \frac{1}{\mathrm{vol}(\Delta)} \int_\Delta (u + \ell) \, dx - \min_\Delta \{ u + \ell \} \right\},$$

and ℓ runs over all the affine functions.

2. The Main Result

The main result of this note is stated as follows.

Theorem 2.1. *Suppose* $\mathrm{Chow}_\Delta(i) \equiv 0$ *holds for* $i \gg 0$*. If a polarized toric variety* $(X_\Delta, \mathbb{L}_\Delta)$ *is uniformly K-polystable in the toric sense, then* $(X_\Delta, \mathbb{L}_\Delta)$ *is asymptotically Chow polystable in the toric sense.*

2.1. **Approach.** One can see that $Q_\Delta(i, g) = 0$ for affine linear functions by the assumption of $\mathrm{Chow}_\Delta(i) \equiv 0$ in Theorem 2.1. Hence it suffices to show that for $i \gg 0$, $Q_\Delta(i, g) > 0$ when $g \in PL(\Delta; i)^{W(X)}$ is NOT affine linear, in order to prove Theorem 2.1.

2.2. **Sketch of the proof of the main Theorem.** Since $Q_\Delta(i, g)$ is invariant when adding an affine linear function to g, we may assume that $u = -g$ is a rational piece-wise linear convex function normalized at 0 in the sense that

$$\inf_{x \in \Delta} u(x) = u(0) = 0, \qquad \text{and} \qquad \int_{\partial\Delta} u \, d\sigma = 1.$$

The key lemma below is an improvement of Lemma 3.3 of [ZZ08], not only it has estimates on the coefficients but also it holds for general rational piecewise linear functions. We summarize key ideas of the proof of Lemma 2.2 in Appendix.

Lemma 2.2 (Euler-Maclaurin Formula)**.** *Assume Δ is a lattice polytope and u is a nonnegative rational piece-wise linear function, then*

$$\sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} u(\mathbf{a}) = i^n \int_\Delta u\,dx + \frac{i^{n-1}}{2}\int_{\partial\Delta} u\,d\sigma + \sum_{k=0}^{n-2}\alpha_k i^k,$$

where

$$\alpha_k \geqslant -C_{n,k;\Delta}\left(\int_{\partial\Delta} u\,d\sigma + \int_\Delta u\,dx\right), \quad k = 0,...,n-2$$

for some constant $C_{n,k;\Delta} > 0$ depending only on n, k, and Δ.

3. Appendix: Bounds on coefficients of Ehrhart polynomial and their applications

In this section, we discuss some application of the bounds on coefficients of Ehrhart polynomial.

First, we recall some useful results on Ehrhart polynomial

$$E_\Delta(i) = \sum_{k=0}^{n} e_k i^k. \tag{3.1}$$

Recall that one has

$$e_n = \mathrm{vol}(\Delta),\ e_{n-1} = \frac{\mathrm{vol}(\partial\Delta)}{2}, e_0 = 1.$$

No convex geometric meaning is known for the rest coefficients. However, the upper and lower bounds for them have been established by [BM85, HT09], respectively. We conclude them as follows

Theorem 3.1. *Let Δ be an n-dimensional lattice polytope and e_k are given by (3.1). Then*

(1)

$$e_k \leqslant (-1)^{n-k}\mathfrak{s}(n,k)\mathrm{vol}(\Delta) + \frac{(-1)^{n-k-1}\mathfrak{s}(n,k+1)}{(n-1)!},$$

where $\mathfrak{s}_{n,k}$ denotes the Stirling numbers of the first kind which can be defined via the identity

$$\prod_{k=0}^{n-1}(z-k) = \sum_{k=1}^{n}\mathfrak{s}(n,k)z^k.$$

(2) *If* $n \geqslant 3$*, then for* $k = 1, ..., n-1$*, we have*

$$e_k \geqslant \frac{1}{n!}\left[(-1)^{n-k}\mathfrak{s}(n+1, k+1) + (n!\mathrm{vol}(\Delta) - 1)\mathfrak{m}_{k,n}\right].$$

Here $\mathfrak{m}_{k,n}$ *is given by*

$$\mathfrak{m}_{k,n} = \min\{C^n_{k,j} : 1 \leqslant j \leqslant n-2\},$$

where $C^n_{k,j}$ *is the* k*-th coefficient of the polynomial*

$$(z+j)(z+j-1)\cdots(z+j-(n-1))$$

with variable z*.*

The following fact will be frequently used later.

Lemma 3.2. *If* Δ *is an* n*-dimensional lattice polytope, then*

$$\mathrm{vol}(\Delta) \geqslant \frac{1}{n!}. \tag{3.2}$$

In order to prove Lemma 2.2 used in Section 2.2, we need the following Lemma 3.3. In fact, Lemma 2.2 follows by an approximation argument of the following lemma, because a general nonnegative continuous function can be approximated by nonnegative rational piecewise linear function.

Lemma 3.3. *Assume* Δ *is a lattice polytope and* u *is a nonnegative rational piecewise linear function, then*

$$\sum_{\mathbf{a}\in\Delta\cap(\mathbb{Z}/i)^n} u(\mathbf{a}) = i^n \int_\Delta u\, dx + \frac{i^{n-1}}{2}\int_{\partial\Delta} u\, d\sigma + \sum_{k=0}^{n-2}\alpha_k i^k, \tag{3.3}$$

where

$$\alpha_k \geqslant -C_k\left(\int_{\partial\Delta} u\, d\sigma + \int_\Delta u\, dx\right),\ k = 0, ..., n-2$$

for some $C_k > 0$ *depending on* n*,* k*, and* Δ*.*

Remark 3.4. Our current approach for proving Lemma 3.3 is the following: since u is a piecewise linear function, we shall consider a decomposition of $\Delta = \cup_{s=1}^p \Delta_s$ such that u is linear on each piece Δ_s. Then we assume that each Δ_s is a lattice polytope. Otherwise it suffices to consider a dilation of Δ. Setting an $(n+1)$-dimensional convex polytope $\mathcal{D}_s := \mathrm{graph}(\Delta_s)$, we further assume that all $\mathcal{D}_s$ are lattice polytopes. Otherwise we shall consider an $i_0 u$ for some $i_0 \in \mathbb{Z}$ since (3.3) is homogeneous with respect to u. However the main difficulty in this approach is that (3.3) does not have good invariance under scaling of domain. Hence we may need another approach for dealing with this difficulty.

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