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A note on the paper “A knot with destabilized bridge spheres of arbitrarily high bridge number”

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1 Introduction

In a recent paper [1], the authors gave an interesting example of bridge spheres of knots as follows. Let n be an integer with $n \geq 4$. Let K_n, S_n, $S_n^{\perp}$ be the knot and the two spheres, respectively, shown in Figure 1. They showed that S_n is a destabilized n-bridge sphere of K_n, and $S_n^{\perp}$ is a destabilized 3-bridge sphere of K_n.

In this paper, we prove the following fact about stabilizations for the above bridge spheres.

Fact 1. *There exists an $(n+1)$-bridge sphere of the knot K_n which is obtained from S_n by one stabilization and bridge isotopies, and from $S_n^{\perp}$ by $n-2$ stabilizations and bridge isotopies.*

In the paper [1], the authors also gave an interesting example of bridge positions of knot types as follows. Let p_i be an integer with $|2p_i+1| \geq 5$ for each $i \in \{1,2,3,4\}$, let q be an even integer with $|q| \geq 12$, and let k be a non-negative integer. Let $K_{p_1,p_2,p_3,p_4,q,k}$ be the knot shown in Figure 2. In fact, the knot type of $K_{p_1,p_2,p_3,p_4,q,k}$ does not depend on k. Let $\mathcal{K}_{p_1,p_2,p_3,p_4,q}$ denote the knot type. They showed that $K_{p_1,p_2,p_3,p_4,q,k}$ is a destabilized $(2k+5)$-bridge position of $\mathcal{K}_{p_1,p_2,p_3,p_4,q}$.

In this paper, we also prove the following fact about stabilizations for the above bridge positions.

Fact 2. *For non-negative integers k_+ and k_- with $k_+ \geq k_-$, there exists a $(2k_+ + 6)$-bridge position of the knot type $\mathcal{K}_{p_1,p_2,p_3,p_4,q}$ which is obtained from $K_{p_1,p_2,p_3,p_4,q,k_+}$ by one stabilization and bridge isotopies, and from $K_{p_1,p_2,p_3,p_4,q,k_-}$ by $2(k_+ - k_-)+1$ stabilizations and bridge isotopies.*

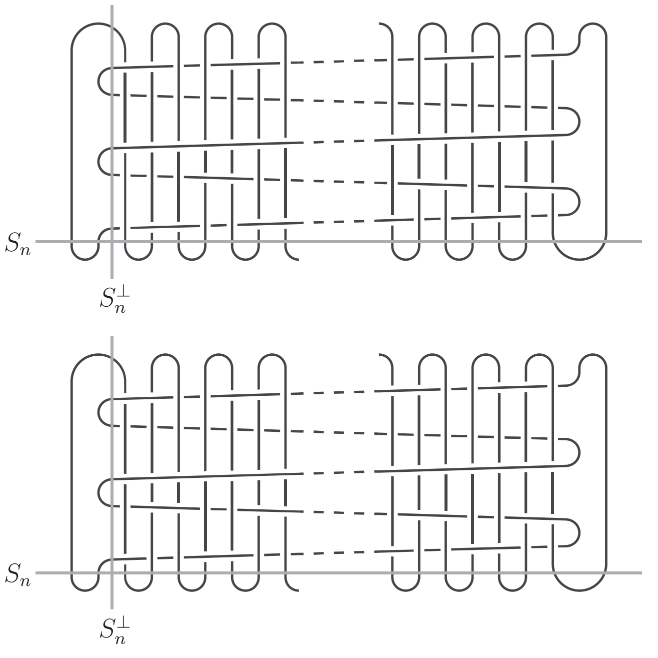

Figure 1: A knot K_n and spheres S_n and $S_n^\perp$. The top shows the case where n is even, and the bottom shows the case where n is odd. The knot K_n intersects S_n in $2n$ points. The second top weft runs under the warps, and the third weft runs over the warps. Each of the other wefts threads across the warps by repeating "over, over, under, under" with the exception in the leftmost and rightmost parts.

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2 Preliminaries

In this section, we review basic definitions concerning bridge spheres and bridge positions. We work in the smooth category.

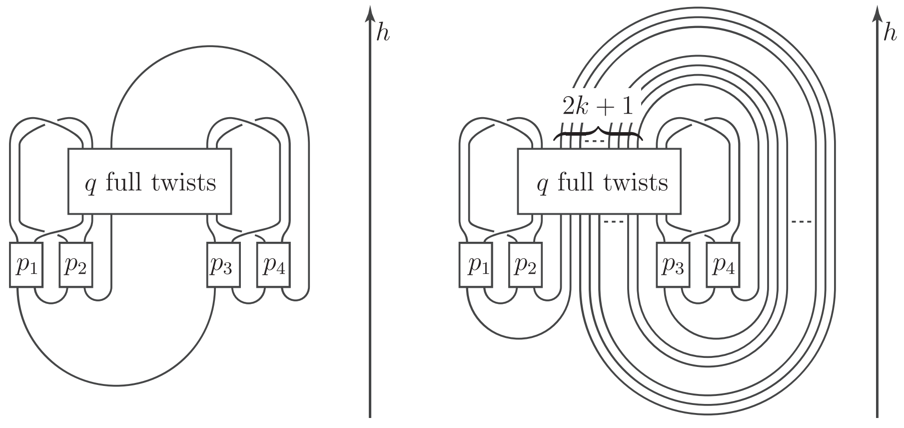

Figure 2: A knot $K_{p_1,p_2,p_3,p_4,q,k}$ and the standard height function $h\colon S^3 \to \mathbb{R}$. The left shows the case where $k = 0$, and the right shows the case where $k > 0$. The box labelled "p_i" represents the p_i left-handed half-twists.

We do not confuse the notions of knot and knot type. A *knot* is a circle embedded in the 3-sphere S^3. Two knots are said to be *isotopic* if there is an ambient isotopy of S^3 which takes one to the other. A *knot type* is an isotopy class of knots.

The notion of bridge sphere for knots is defined as follows. Let K be a knot, and n be a positive integer. An *n-bridge sphere* of K is a 2-sphere S in S^3 which is transverse to K and decomposes (S^3, K) into two n-string trivial tangles. That is to say, letting B_+ and B_- denote the 3-balls divided by S in S^3, the intersection $K \cap B_\varepsilon$ is a collection of n arcs which are simultaneously parallel to ∂B_ε for each $\varepsilon \in \{+, -\}$. A *bridge sphere* of K is an m-bridge sphere of K for some positive integer m.

The notion of bridge isotopy for bridge spheres is defined as follows. Let S be a bridge sphere of a knot K. Let $\{\varphi_t\colon S^3 \to S^3\}_{t\in[0,1]}$ be an ambient isotopy such that $\varphi_0 = \mathrm{id}_{S^3}$, and $\varphi_t(S)$ is a bridge sphere of K for every $t \in [0, 1]$. We say that $\varphi_1(S)$ is obtained from S by a *bridge isotopy*. For two bridge spheres S_1 and S_2 of K, by $S_1 \underset{K}{\sim} S_2$ we mean that S_2 is obtained from S_1 by a bridge isotopy.

The notion of stabilization for bridge spheres is defined as follows. Let S be a bridge sphere of a knot K. Let S' be a sphere obtained from S by a local deformation near a point in $K \cap S$ as in Figure 3. One can see that if S is an n-bridge sphere of K for a positive integer n, then S' is an $(n + 1)$-bridge sphere of K. We say that S' is obtained from S by a *stabilization*. A bridge sphere is said to be *destabilized* if it cannot be obtained from any bridge sphere by a stabilization and a bridge isotopy.

We let h denote the standard height function of the 3-sphere throughout this paper. To

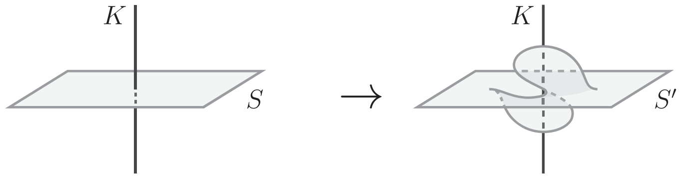

Figure 3: A stabilization for a bridge sphere.

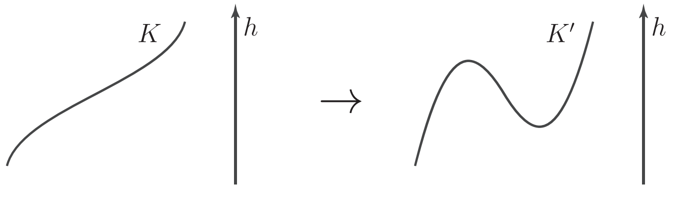

Figure 4: A stabilization for a bridge position.

be specific, one may regard S^3 as the unit sphere in $\mathbb{R}^4$ and $h\colon S^3 \to \mathbb{R}$ as the restriction of the projection.

The notion of bridge position for knot types is defined as follows. Let $\mathcal{K}$ be a knot type, and n be a positive integer. An *n-bridge position* of $\mathcal{K}$ is a knot K in $\mathcal{K}$ such that the function $h|_K$ has exactly $2n$ critical points, they are all non-degenerate, and any locally maximal value is higher than any locally minimal value. A *bridge position* of $\mathcal{K}$ is an m-bridge position of $\mathcal{K}$ for some positive integer m.

The notion of bridge isotopy for bridge positions is defined as follows. Let K be a bridge position of a knot type $\mathcal{K}$. Let $\{\varphi_t\colon S^3 \to S^3\}_{t\in[0,1]}$ be an ambient isotopy such that $\varphi_0 = \mathrm{id}_{S^3}$, and $\varphi_t(K)$ is a bridge position of $\mathcal{K}$ for every $t \in [0,1]$. We say that $\varphi_1(K)$ is obtained from K by a *bridge isotopy*. For two bridge positions K_1 and K_2 of $\mathcal{K}$, by $K_1 \underset{h}{\sim} K_2$ we mean that K_2 is obtained from K_1 by a bridge isotopy.

The notion of stabilization for bridge positions is defined as follows. Let K be a bridge position of a knot type $\mathcal{K}$. Let v_+ denote the minimum of the locally maximal values of $h|_K$, and v_- denote the maximum of the locally minimal values of $h|_K$. Let K' be a knot obtained from K by a local deformation near a point in $K \cap h^{-1}((v_-, v_+))$ as in Figure 4. One can see that if K is an n-bridge position of $\mathcal{K}$ for a positive integer n, then K' is an $(n+1)$-bridge position of $\mathcal{K}$. We say that K' is obtained from K by a *stabilization*. A bridge position is said to be *destabilized* if it cannot be obtained from any bridge position by a stabilization and a bridge isotopy.

3 Proofs

We prove Fact 1 by a concrete construction. Let n, K_n, S_n and $S_n^{\perp}$ be as in Introduction. Figures 5–7 show a sequence of bridge spheres which represents a deformation of S_n to a sphere S'_n in the case where n is even. Note that the deformation is composed of one stabilization and bridge isotopies. Figures 8–10 show a sequence of bridge spheres which represents a deformation of $S_n^{\perp}$ to S'_n in the case where n is even and $n > 4$. Note that the deformation is composed of $n-2$ stabilizations and bridge isotopies. Similar figures work in the case where n is odd or $n = 4$. These give a proof of Fact 1.

We also prove Fact 2 by a concrete construction. Let p_1, p_2, p_3, p_4, q, k, $K_{p_1,p_2,p_3,p_4,q,k}$ and $\mathcal{K}_{p_1,p_2,p_3,p_4,q}$ be as in Introduction, and let $K'_{p_1,p_2,p_3,p_4,q,k}$ be the knot shown in Figure 11. Note that $K'_{p_1,p_2,p_3,p_4,q,k}$ is a $(2k+6)$-bridge position of $\mathcal{K}_{p_1,p_2,p_3,p_4,q}$ obtained from $K_{p_1,p_2,p_3,p_4,q,k}$ by one stabilization. Figures 12–15 show a sequence of bridge positions which represents a deformation of $K'_{p_1,p_2,p_3,p_4,q,k}$ to $K'_{p_1,p_2,p_3,p_4,q,k+1}$ in the case where $k = 3$ and p_4 is even. Note that the deformation is composed of two stabilizations and bridge isotopies. Similar figures work in the case where $k \neq 3$, and simpler figures work in the case where p_4 is odd. These give a proof of Fact 2.

Reference

[1] Y. Jang, T. Kobayashi, M. Ozawa and K. Takao, *A knot with destabilized bridge spheres of arbitrarily high bridge number*, J. London Math. Soc. (2) **93** (2016), no. 2, 379–396.

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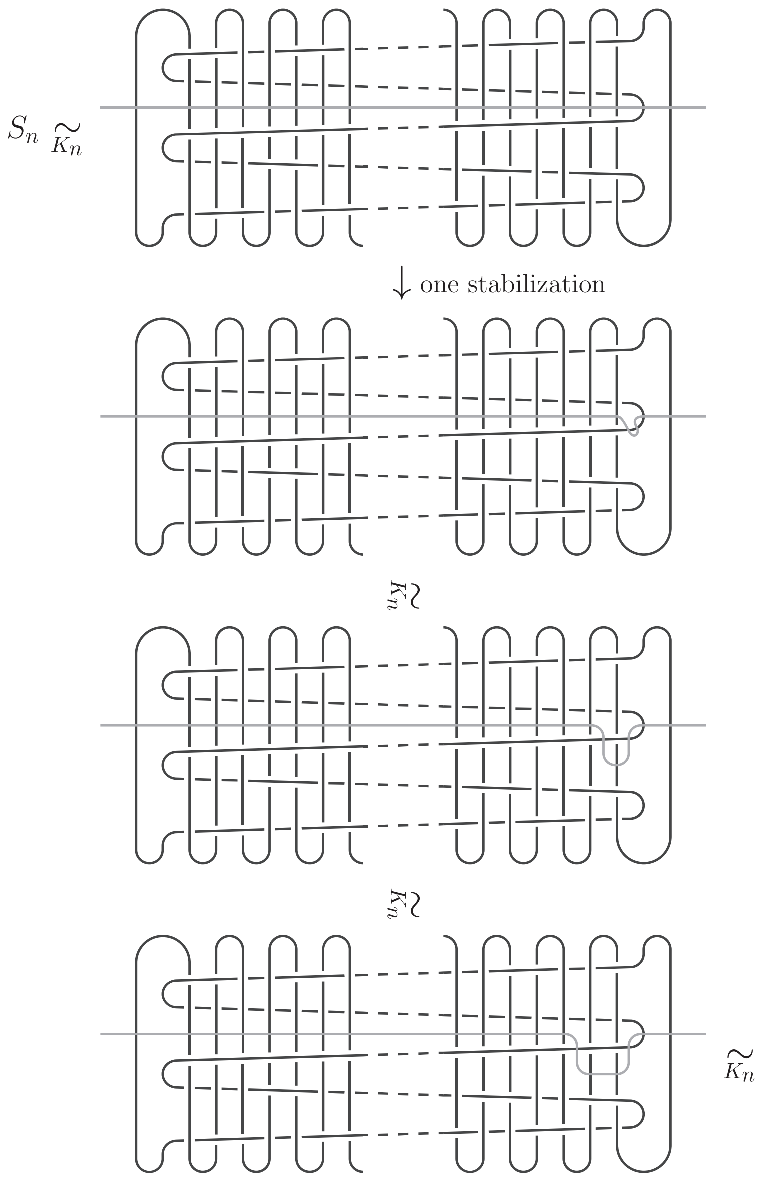

Figure 5: A sequence of bridge spheres, part 1.

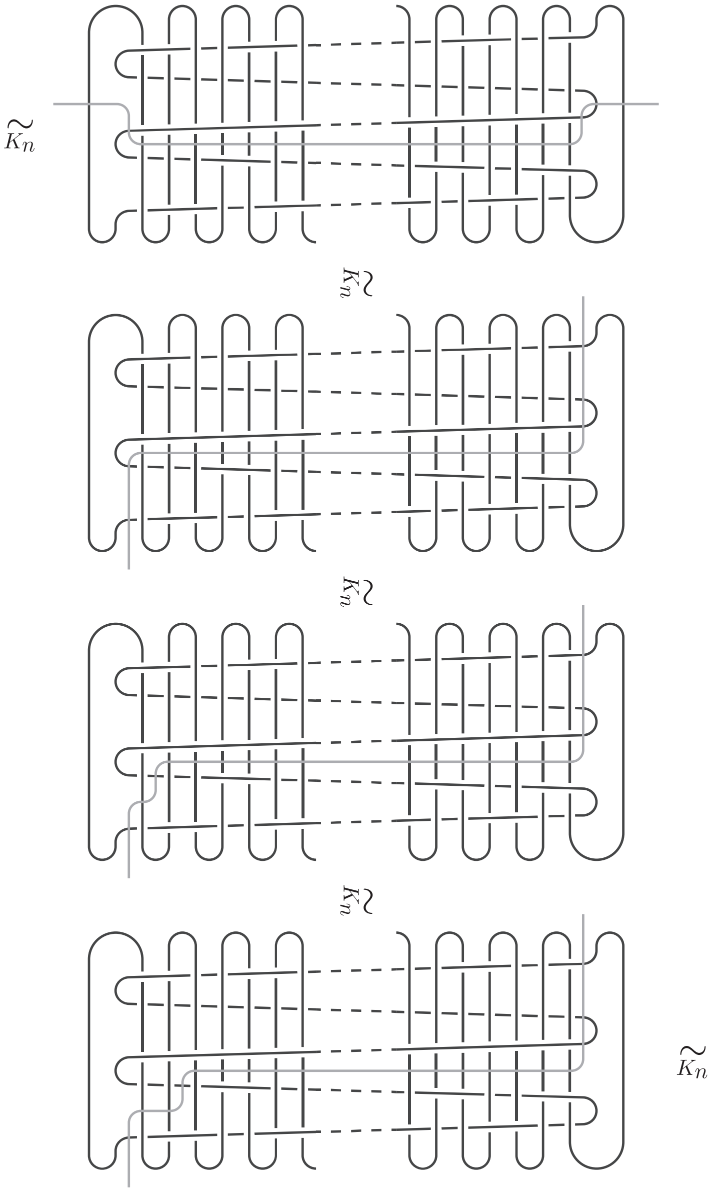

Figure 6: A sequence of bridge spheres, part 2.

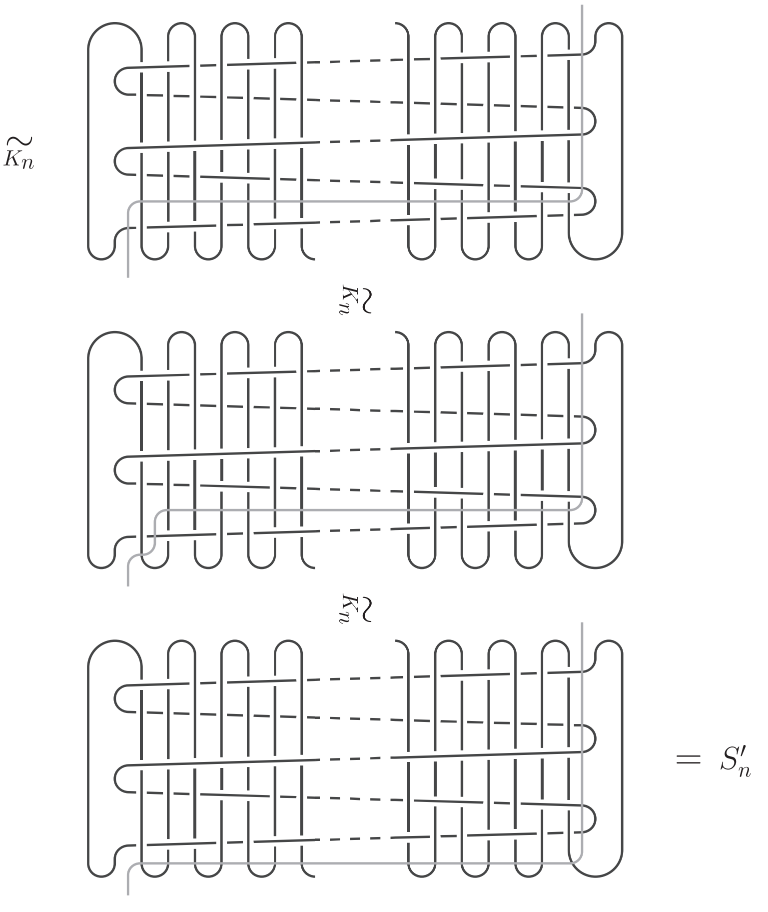

Figure 7: A sequence of bridge spheres, part 3.

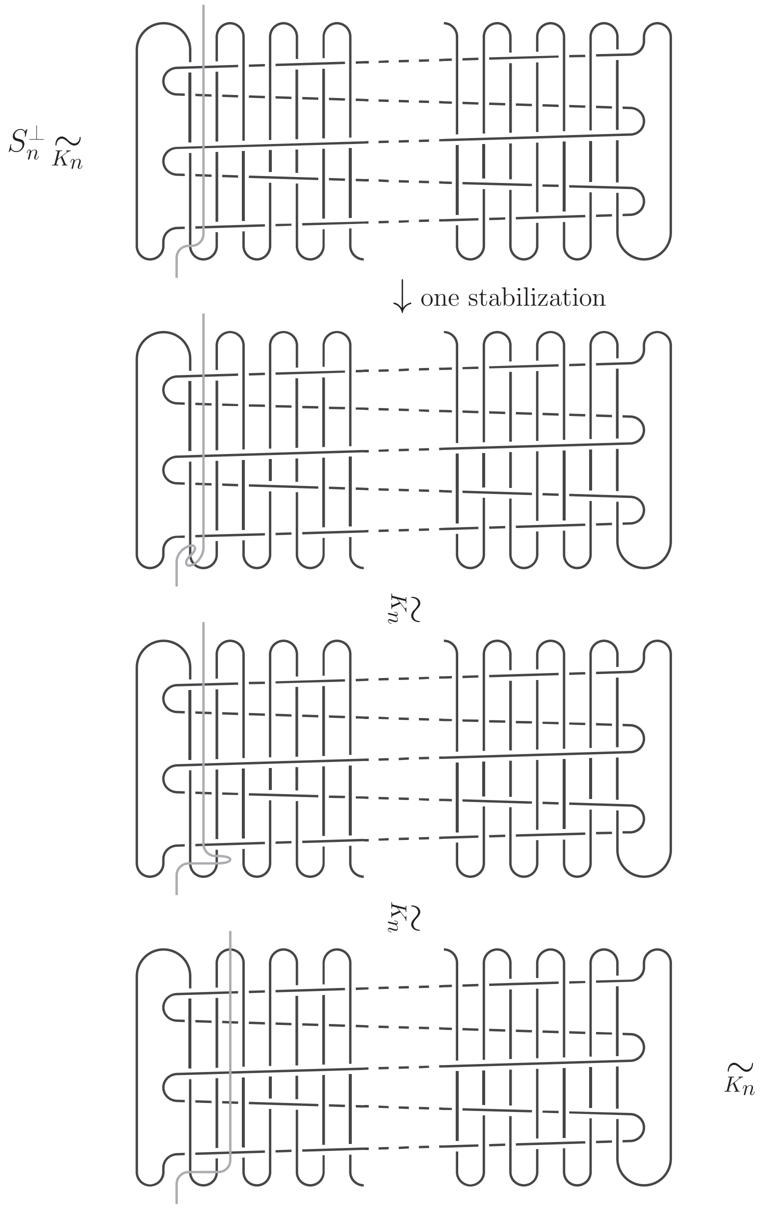

Figure 8: A sequence of bridge spheres, part 1.

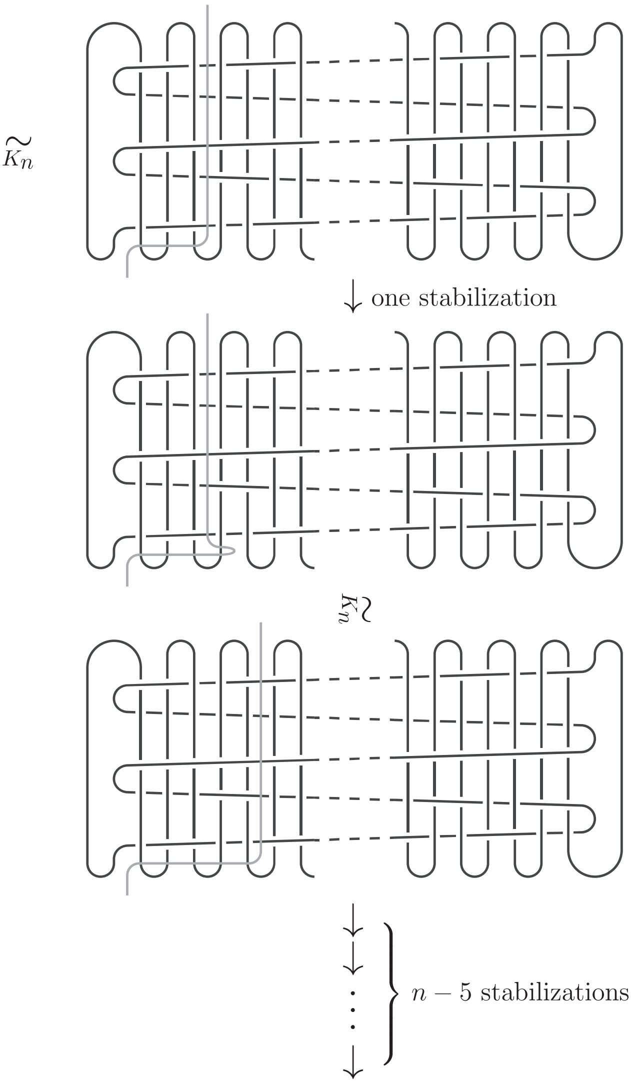

Figure 9: A sequence of bridge spheres, part 2.

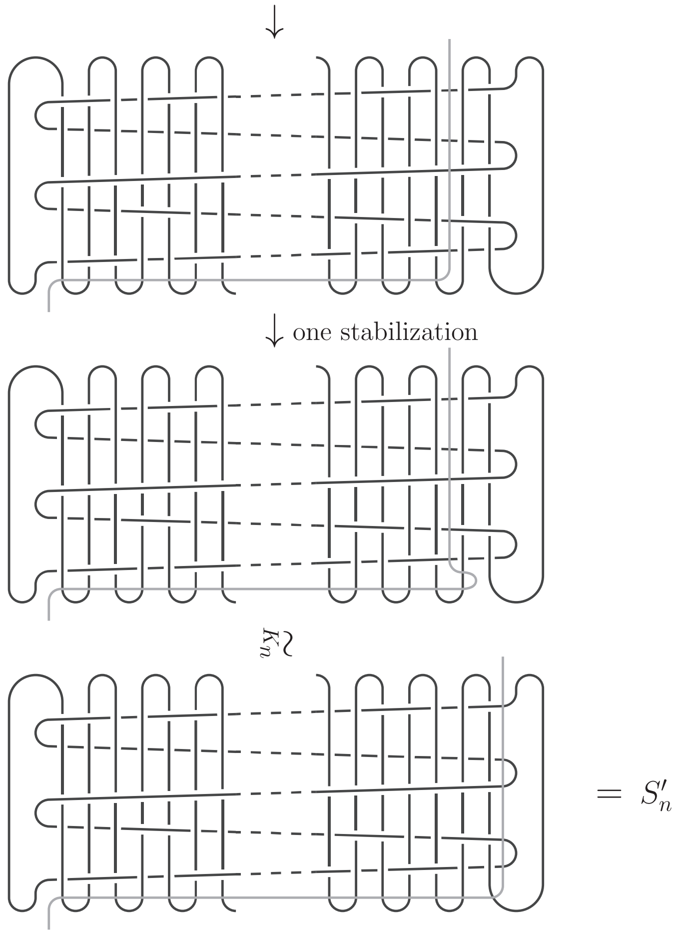

Figure 10: A sequence of bridge spheres, part 3.

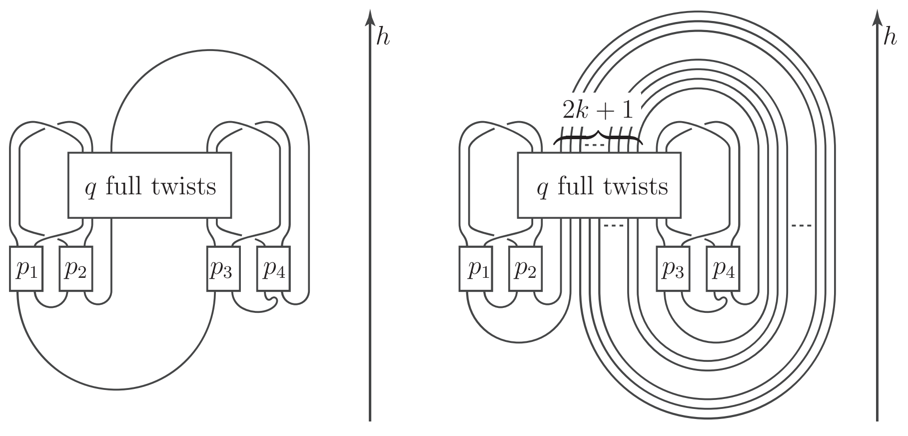

Figure 11: A knot $K'_{p_1,p_2,p_3,p_4,q,k}$ and the standard height function $h\colon S^3 \to \mathbb{R}$. The left shows the case where $k = 0$, and the right shows the case where $k > 0$.

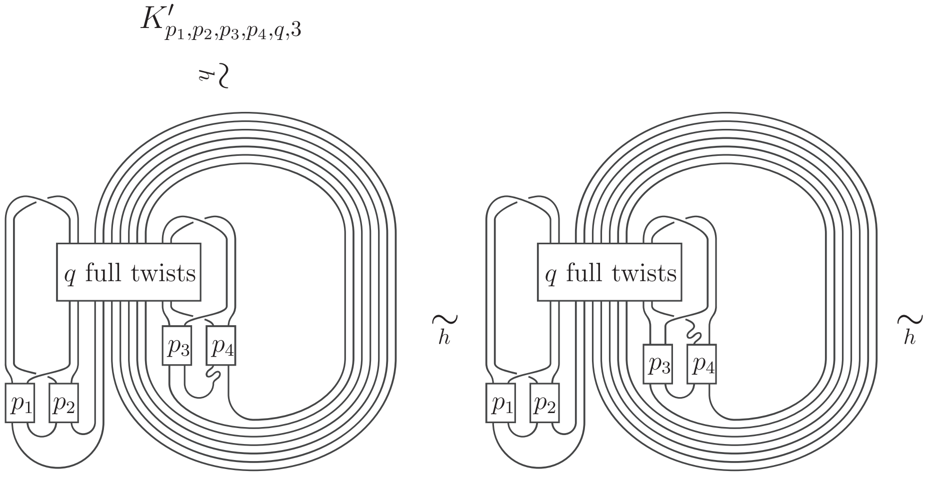

Figure 12: A sequence of bridge positions, part 1. The height function h is similar to that in the above figures.

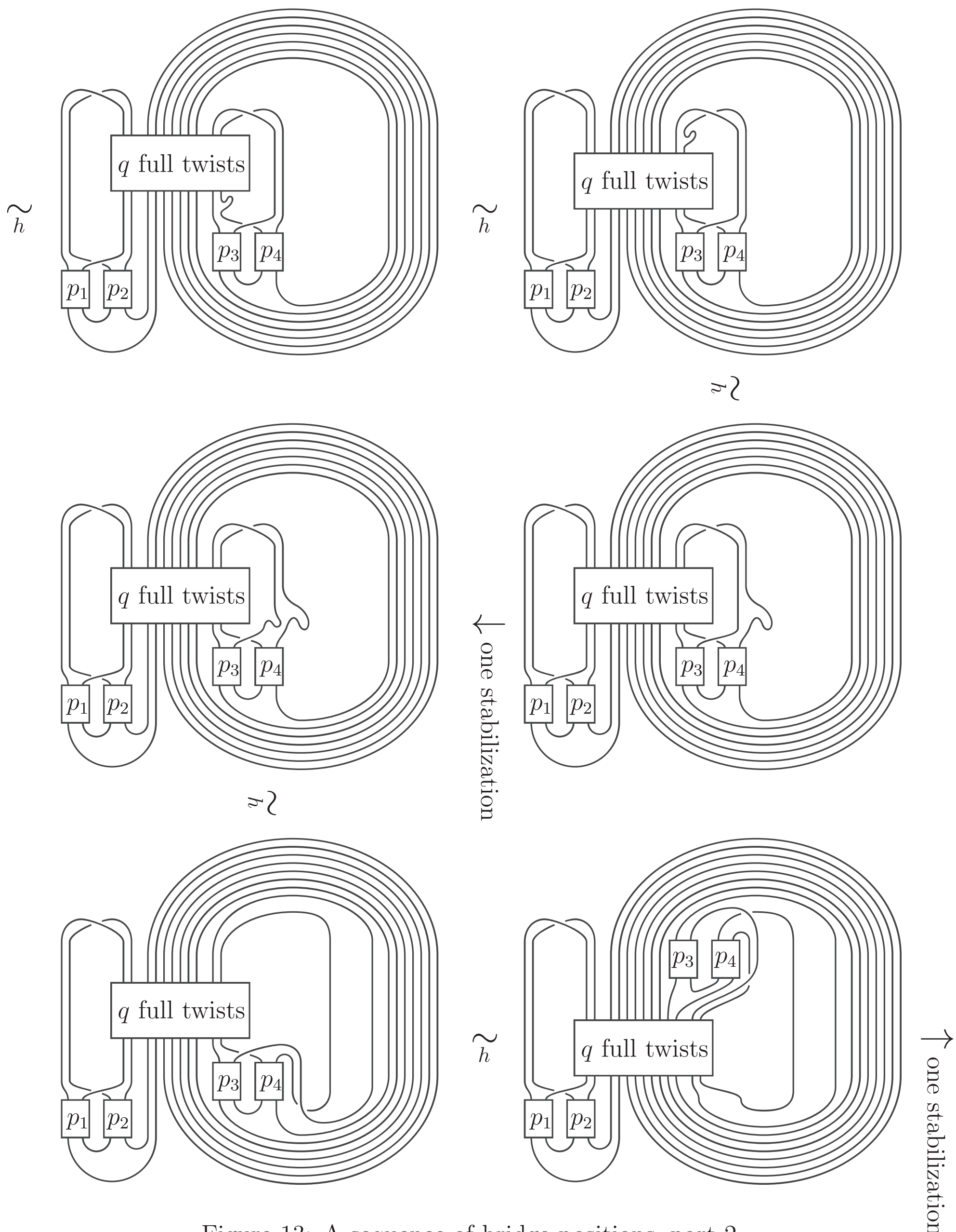

Figure 13: A sequence of bridge positions, part 2.

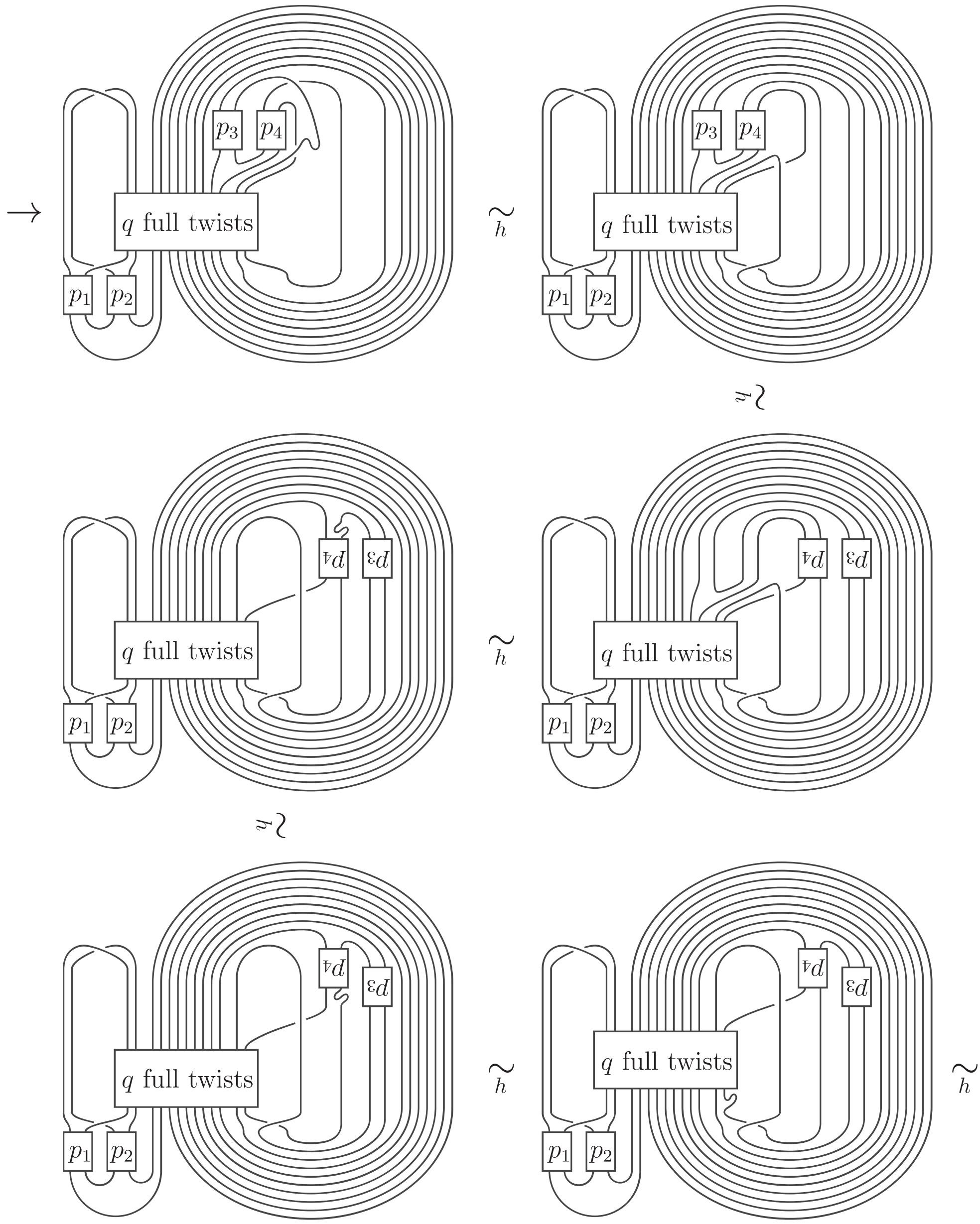

Figure 14: A sequence of bridge positions, part 3.

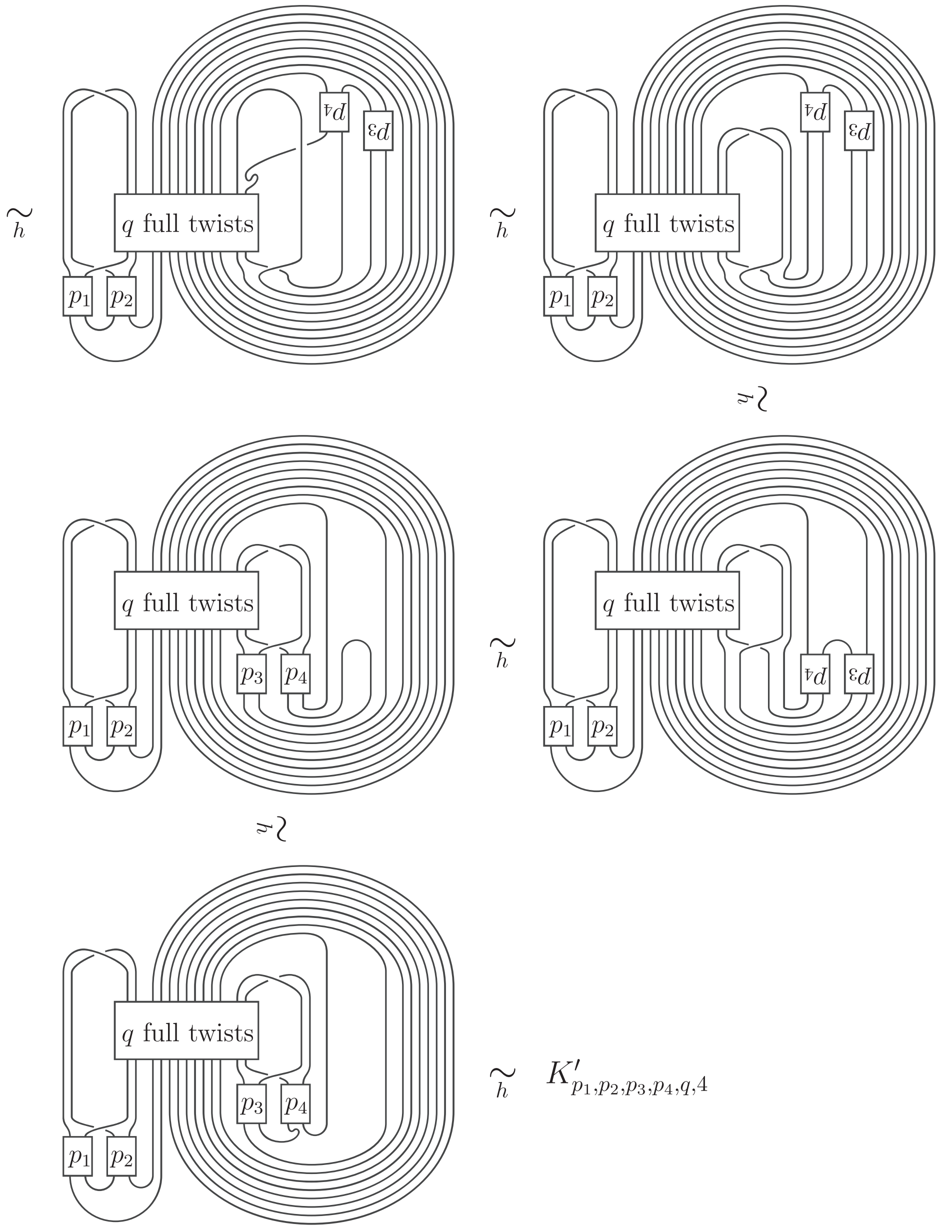

Figure 15: A sequence of bridge positions, part 4.

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