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A survey on second-order conditions for nonlinear symmetric cone programming via squared slack variables

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Abstract

The nonlinear symmetric cone programming (NSCP) problems contain as special cases the nonlinear semidefinite programming, the nonlinear second-order cone programming and the nonlinear programming problems. In this survey, we first explain the basics about Euclidean Jordan algebras and symmetric cones. Then, we observe that NSCP problems can be reformulated as nonlinear programming problems with the use of squared slack variables. Using such reformulations, we show how to obtain second-order optimality conditions for NSCP problems. In particular, under the strict complementarity condition, we lead to a description of the so-called sigma-term.

Keywords: Nonlinear symmetric cone programming, optimality conditions, second-order conditions, sigma-term.

1 Introduction

In this work, we are interested in *nonlinear symmetric cone programming* (NSCP) problems, that give an unified framework for a number of different problems. In particular, they include the classical nonlinear programming (NLPs), the nonlinear second-order cone programming (NSOCPs), the nonlinear semidefinite programming (NSDPs), and any mixture of those three. In this survey, we will show a way to obtain a workable description of second-order conditions for NSCP problems. First, let us recall that in NLP, the second-order conditions require positive semidefiniteness/definiteness of the Hessian of the Lagrangian function over the critical cone. These optimality conditions are called of *zero-gap*, in the sense that the change from "necessary" to "sufficient" involves only a change from "$\geq$" to "$>$". For NSOCPs and NSDPs, the usual zero-gap condition needs an extra term, that

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appears together with the Hessian of the Lagrangian. This term is called *sigma-term*, and it is said to model the curvature of the underlying cone.

Typically, there are two ways to obtain zero-gap second-order conditions. One approach consists in computing directly the so-called *second-order tangent sets* of the cone. This was done, for instance, by Bonnans and Ramírez [1] for NSOCP. Another way consists in expressing the cone using an appropriate convex function, and using its second-order directional derivative to compute the second-order tangent sets. This approach was chosen by Shapiro [10] for NSDPs. For the general symmetric cone case, it seems complicated to describe the second-order tangent sets directly. For the second approach, it is known that the appropriate convex function is the minimum eigenvalue function. However, it is still an open problem to give explicit descriptions of higher-order directional derivatives for this minimum eigenvalue function.

Here, we bypass all these difficulties by transforming the NSCP into an ordinary NLP with equality constraints, by using squared slack variables. The derivation of optimality conditions using reformulations with squared slack variables was proposed originally by Fukuda and Fukushima for NLP and NSOCP problems [5, 6]. After that, Lourenço, Fukuda and Fukushima extended the idea to the more general NSDP [8] and NSCP [9] problems. This survey is based on this latter work. We show that by writing down the second-order conditions of the reformulated problem, and eliminating the slack variable, we can obtain second-order conditions for the original NSCP problem. The drawback of this approach is that the resulting second-order conditions require strict complementarity.

This survey is organized as follows. In Section 2, we review basic notions related to Euclidean Jordan algebras. Section 3 is devoted to the symmetric cone programming problem, the reformulation problem using slack variables, as well as their Karush-Kuhn-Tucker (KKT) conditions. In Section 4, we provide sufficient conditions that guarantee equivalence between the KKT points of the original and the reformulated problems, and we present the zero-gap second-order conditions for NSCP. We conclude in Section 5, with final remarks.

2 Euclidean Jordan algebras

In this section, we first establish some notations and recall the basics about Euclidean Jordan algebras. We refer to the book by Faraut and Korányi [3] and the paper by Faybusovich [4] to more details. Let $\mathcal{E}$ be a finite dimensional space equipped with an inner product $\langle \cdot, \cdot \rangle$. We will use $\|\cdot\|$ to indicate the norm induced by $\langle \cdot, \cdot \rangle$. We say that $\mathcal{K} \subseteq \mathcal{E}$ is a *symmetric cone* if (i) it is self-dual; (ii) it is full-dimensional, i.e., the interior of $\mathcal{K}$ is not empty; and (iii) it is homogeneous, i.e., for every x, y in the interior of $\mathcal{K}$, there is a linear bijection Φ such that $\Phi(x) = y$ and $\Phi(\mathcal{K}) = \mathcal{K}$.

Since $\mathcal{K}$ is a symmetric cone, we may assume that $\mathcal{E}$ is equipped with a bilinear map $\circ : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ such that $\mathcal{K}$ is the corresponding *cone of squares*, that is,

$$\mathcal{K} = \{y \circ y \mid y \in \mathcal{E}\}. \tag{1}$$

For all $y, w, z \in \mathcal{E}$, we assume that this map possesses the following three properties:

(a) $y \circ z = z \circ y$,

(b) $y \circ (y^2 \circ z) = y^2 \circ (y \circ z)$, where $y^2 = y \circ y$,

(c) $\langle y \circ z, w \rangle = \langle y, z \circ w \rangle$.

Under these conditions, $(\mathcal{E}, \circ)$ is called an *Euclidean Jordan algebra*. It can be shown that every symmetric cone arises as the cone of squares of some Euclidean Jordan algebra [3, Theorems III.2.1 and III.3.1]. Moreover, we can assume that $\mathcal{E}$ has an unit element e satisfying $y \circ e = y$ for all $y \in \mathcal{E}$. The map $\circ$, which is also called *Jordan product*, is associated to the linear operator L_y defined by

$$L_y(w) = y \circ w \qquad \text{for all } w \in \mathcal{E},$$

where $y \in \mathcal{E}$ is given. In what follows, we say that c is an *idempotent* if $c \circ c = c$. Furthermore, c is *primitive* if it is nonzero and there is no way of writing $c = \tilde{c} + c'$, with nonzero idempotents $\tilde{c}$ and c' satisfying $\tilde{c} \circ c' = 0$.

Theorem 1. *[3, Theorem III.1.2] Let $(\mathcal{E}, \circ)$ be an Euclidean Jordan algebra and let $y \in \mathcal{E}$. Then there are primitive idempotents $c_1, \ldots, c_r$ satisfying*

$$\begin{aligned} &c_i \circ c_j = 0, && i \neq j, \\ &c_i \circ c_i = c_i, && i = 1, \ldots, r, \\ &c_1 + \cdots + c_r = e, && \end{aligned}$$

and unique real numbers $\sigma_1, \ldots, \sigma_r$ satisfying $y = \sum_{i=1}^{r} \sigma_i c_i$. This sum is called spectral decomposition *of y.*

We say that $c_1, \ldots, c_r$ in Theorem 1 form a *Jordan frame* for y, and $\sigma_1, \ldots, \sigma_r$ are the *eigenvalues* of y. We remark that r only depends on the algebra $\mathcal{E}$. Given $y \in \mathcal{E}$, we define its trace by $\mathrm{tr}(y) := \sigma_1 + \cdots + \sigma_r$, where $\sigma_1, \ldots, \sigma_r$ are the eigenvalues of y. As in the case of matrices, it turns out that the trace function is linear. It can also be used to define an inner product compatible with the Jordan product, and so henceforth we will assume that $\langle x, y \rangle = \mathrm{tr}(x \circ y)$. We define the *rank* of $y \in \mathcal{E}$ as the number of its nonzero eigenvalues. Then, the rank of $\mathcal{K}$ is defined by $\mathrm{rank}\,\mathcal{K} = \max\{\mathrm{rank}\, y \mid y \in \mathcal{K}\} = r = \mathrm{tr}(e)$. We will also say that the rank of $\mathcal{E}$ is $r = \mathrm{tr}(e)$. Now, if $y \in \mathcal{E}$ and $a \in \mathbb{R}$, we define the following set:

$$V(y, a) := \{z \in \mathcal{E} \mid y \circ z = az\}.$$

For any $V, V' \subseteq \mathcal{E}$, we write $V \circ V' = \{y \circ z \mid y \in V, z \in V'\}$.

Theorem 2 (Peirce decomposition — 1st version)**.** *[3, Proposition IV.1.1] Let $c \in \mathcal{E}$ be an idempotent. Then $\mathcal{E}$ is decomposed as the orthogonal direct sum*

$$\mathcal{E} = V(c, 1) \bigoplus V\left(c, \frac{1}{2}\right) \bigoplus V(c, 0).$$

Also, $V(c, 1)$ and $V(c, 0)$ are Euclidean Jordan algebras satisfying $V(c, 1) \circ V(c, 0) = \{0\}$. Moreover, the following inclusions hold: $(V(c, 1) + V(c, 0)) \circ V(c, 1/2) \subseteq V(c, 1/2)$ and $V(c, 1/2) \circ V(c, 1/2) \subseteq V(c, 1) + V(c, 0)$.

The Peirce decomposition has another version, with detailed information on the way that the algebra is decomposed.

Theorem 3 (Peirce decomposition — 2nd version)**.** *[3, Theorem IV.2.1] Let $c_1, \dots, c_r$ be a Jordan frame for $y \in \mathcal{E}$. Then $\mathcal{E}$ is decomposed as the orthogonal sum*

$$\mathcal{E} = \bigoplus_{1 \le i \le j \le r} V_{ij},$$

where $V_{ii} = V(c_i, 1) = \{\alpha c_i \mid \alpha \in \mathbb{R}\}$, and $V_{ij} = V\left(c_i, \frac{1}{2}\right) \cap V\left(c_j, \frac{1}{2}\right)$ for $i \neq j$. Moreover, (a) the V_{ii}'s are subalgebras of $\mathcal{E}$, and (b) the following relations hold:

$$\begin{aligned} V_{ij} \circ V_{ij} &\subseteq V_{ii} + V_{jj} && \text{for all } i, j, \\ V_{ij} \circ V_{jk} &\subseteq V_{ik} && \text{if } i \neq k, \\ V_{ij} \circ V_{kl} &= \{0\} && \text{if } \{i, j\} \cap \{k, l\} = \emptyset. \end{aligned}$$

The algebra $(\mathcal{E}, \circ)$ is said to be *simple* if it is not possible to write $\mathcal{E} = V \oplus V'$, where V and V' are both nonzero subalgebras of $\mathcal{E}$. We will say that $\mathcal{K}$ is *simple* if it is the cone of squares of a simple algebra. It turns out that every Euclidean Jordan algebra can be decomposed as a direct sum of simple Euclidean Jordan algebras, which then induces a decomposition of $\mathcal{K}$ in simple symmetric cones. This means that we can write $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell$, and $\mathcal{K} = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_\ell$, where the $\mathcal{E}_i$'s are simple Euclidean Jordan algebras of rank r_i and $\mathcal{K}_i$ is the cone of squares of $\mathcal{E}_i$. Note that orthogonality expressed by this decomposition is not only with respect to the inner product $\langle \cdot, \cdot \rangle$ but also with respect to the Jordan product $\circ$.

We now recall the following properties of $\mathcal{K}$. See [3, 7] for detailed proofs.

Proposition 4. *Let $y, w \in \mathcal{E}$.*

(a) $y \in \mathcal{K}$ if and only if its eigenvalues are nonnegative.

(b) $y \in \operatorname{int} \mathcal{K}$ if and only if its eigenvalues are positive.

(c) $y \in \operatorname{int} \mathcal{K}$ if and only if $\langle y, w \circ w \rangle > 0$ for all nonzero $w \in \mathcal{E}$.

(d) Suppose $y, w \in \mathcal{K}$. Then, $y \circ w = 0$ if and only if $\langle y, w \rangle = 0$.

From item (d) of Proposition 4, we have that if c and c' are two idempotents belonging to distinct blocks, we also have $c \circ c' = 0$ in addition to $\langle c, c' \rangle = 0$. Since this holds for all idempotents, we have $\mathcal{E}_i \circ \mathcal{E}_j = 0$, whenever $i \neq j$. From the same proposition, if $y \in \mathcal{K}$, then the eigenvalues of y are nonnegative, and so we can define the square root of y as $\sqrt{y} = \sum_{i=1}^{r} \sqrt{\sigma_i} c_i$, where $\{c_1, \dots, c_r\}$ is a Jordan frame for y.

3 Symmetric cone programming problems

In this paper, we consider the following *symmetric cone programming* problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & h(x) = 0, \\ & g(x) \in \mathcal{K}, \end{array} \tag{P1}$$

where $f \colon \mathbb{R}^n \to \mathbb{R}$, $h \colon \mathbb{R}^n \to \mathbb{R}^m$ and $g \colon \mathbb{R}^n \to \mathcal{E}$ are twice continuously differentiable functions, $\mathcal{E}$ is defined similarly to the previous section, i.e., it is a finite dimensional space equipped with inner product $\langle \cdot, \cdot \rangle$, and $\mathcal{K} \subseteq \mathcal{E}$ is a symmetric cone. The Lagrangian function $L \colon \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E} \to \mathbb{R}$ associated with problem (P1) is given by

$$L(x, \mu, \lambda) := f(x) - \langle h(x), \mu \rangle - \langle g(x), \lambda \rangle.$$

We say that $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ is a *Karush-Kuhn-Tucker* (KKT) triple of problem (P1) if the following conditions are satisfied:

$$\begin{aligned} \nabla f(x) - Jh(x)^* \mu - Jg(x)^* \lambda &= 0, \\ \lambda &\in \mathcal{K}, \\ g(x) &\in \mathcal{K}, \\ \lambda \circ g(x) &= 0, \\ h(x) &= 0, \end{aligned}$$

where $\nabla f(x)$ is the gradient of f at x, $Jg(x)$ is the Jacobian of g at x and $Jg(x)^*$ denotes the adjoint of $Jg(x)$. Usually, instead of $\lambda \circ g(x) = 0$, we would have $\langle \lambda, g(x) \rangle = 0$, but in view of Proposition 4(d), they are equivalent. Note also that the first equality is equivalent to $\nabla L_x(x, \mu, \lambda) = 0$, where ∇L_x denotes the gradient of L with respect to x.

Now, recalling (1), we add an slack variable $y \in \mathcal{E}$ into (P1), in order to obtain the following optimization problem:

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & f(x) \\ \text{subject to} & h(x) = 0, \\ & g(x) = y \circ y. \end{array} \tag{P2}$$

Note that is is just an equality constrained NLP problem. We observe that $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ is a KKT quadruple of (P2) if the conditions below are satisfied:

$$\begin{aligned} \nabla_{(x,y)} \mathcal{L}(x, y, \mu, \lambda) &= 0, \\ h(x) &= 0, \\ g(x) - y \circ y &= 0, \end{aligned}$$

where $\mathcal{L} \colon \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E} \to \mathbb{R}$ is the Lagrangian function associated with (P2), i.e.,

$$\mathcal{L}(x, y, \mu, \lambda) := f(x) - \langle h(x), \mu \rangle - \langle g(x) - y \circ y, \lambda \rangle,$$

and $\nabla_{(x,y)}\mathcal{L}$ denotes the gradient of $\mathcal{L}$ with respect to (x, y). The KKT conditions for (P2) can be easily rewritten as

$$\begin{aligned} \nabla f(x) - Jh(x)^*\mu - Jg(x)^*\lambda = 0,\\ \lambda \circ y = 0,\\ g(x) - y \circ y = 0,\\ h(x) = 0. \end{aligned}$$

Checking the KKT conditions for (P1) and (P2), we note that they are equivalent, except that it is not required that λ belongs to $\mathcal{K}$ for the reformulated problem (P2).

We end this section by recalling the strict complementarity condition and some constraint qualifications. If $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ is a KKT triple of (P1) such that

$$\operatorname{rank} g(x) + \operatorname{rank} \lambda = r,$$

then (x, λ) is said to satisfy the *strict complementarity* condition. For (P1), we say that $x \in \mathcal{K}$ is *nondegenerate* if

$$\begin{aligned} \mathbb{R}^m &= \operatorname{Im} Jh(x),\\ \mathcal{K} &= \operatorname{lin} \mathcal{T}_{\mathcal{K}}(g(x)) + \operatorname{Im} Jg(x), \end{aligned}$$

where $\operatorname{Im} Jg(x)$ denotes the image of the linear map $Jg(x)$, $\mathcal{T}_{\mathcal{K}}(g(x))$ denotes the tangent cone of $\mathcal{K}$ at $g(x)$, and $\operatorname{lin} \mathcal{T}_{\mathcal{K}}(g(x))$ is the lineality space of $\mathcal{T}_{\mathcal{K}}(g(x))$, i.e., $\operatorname{lin} \mathcal{T}_{\mathcal{K}}(g(x)) = \mathcal{T}_{\mathcal{K}}(g(x)) \cap -\mathcal{T}_{\mathcal{K}}(g(x))$ (see [10, Definition 4] and [2, Section 4.6.1]). Finally, for (P2), we say that the *linear independence constraint qualification* (LICQ) is satisfied at a point (x, y) if the gradients of the active constraints are linearly independent.

4 KKT conditions and second-order optimality conditions

As we mentioned before, the KKT points of (P1) and (P2) are not necessarily the same. However, if (x, μ, λ) is a KKT triple for (P1), it is easy to construct a KKT quadruple for (P1). However, the opposite does not necessarily hold because λ might fail to belong to $\mathcal{K}$.

Proposition 5. *[9, Propositions 4.1 and 4.2] If $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ is a KKT triple for* (P1)*, then $(x, \sqrt{g(x)}, \mu, \lambda)$ is a KKT quadruple for* (P2)*. If $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ is a KKT quadruple for* (P2) *and $\lambda \in \mathcal{K}$ holds, then (x, μ, λ) is a KKT triple for* (P1).

Let us recall that for (P2), the second-order sufficient condition (SOSC-NLP) holds if $\langle \nabla^2_{(x,y)}\mathcal{L}(x, y, \mu, \lambda)(v, w), (v, w)\rangle > 0$ for every nonzero $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2y \circ w = 0$ and $Jh(x)v = 0$, where $\nabla^2_{(x,y)}\mathcal{L}$ denotes the Hessian of $\mathcal{L}$ with respect to (x, y). We can also present the SOSC-NLP in terms of the Lagrangian of (P1) (see [9, Proposition 2.3] for the calculations). More precisely, the SOSC-NLP holds if

$$\langle \nabla^2_x L(x, \mu, \lambda)v, v\rangle + 2\langle w \circ w, \lambda\rangle > 0$$

for every nonzero $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2y \circ w = 0$ and $Jh(x)v = 0$.

Similarly, if (x, y) is a local minimum for (P2) and $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ is a KKT quadruple such that LICQ holds, then the following second-order necessary condition holds:

$$\langle \nabla_x^2 L(x, \mu, \lambda)v, v\rangle + 2\langle w \circ w, \lambda\rangle \geq 0$$

for every $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2y \circ w = 0$ and $Jh(x)v = 0$.

We observe that in the above second-order conditions, an extra term appears together with the Lagrangian of (P1). This term is connected with the so-called *sigma-term* that appears in second-order optimality conditions for optimization problems over general closed convex cones. It plays an important role in the construction of no-gap optimality conditions.

Now, note that since $\mathcal{K}$ is self-dual, we have that $\lambda \in \mathcal{K}$ if and only if $\langle \lambda, w \circ w\rangle \geq 0$ for all $w \in \mathcal{E}$. The next result shows another criterion for membership in $\mathcal{K}$ that involves rank information. Let us just consider $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell$ and $\mathcal{K} = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_\ell$, where the $\mathcal{E}_i$ are simple Euclidean Jordan algebras of rank r_i and $\mathcal{K}_i$ is the cone of squares of $\mathcal{E}_i$. The rank of $\mathcal{E}$ is $r = r_1 + \cdots + r_\ell$.

Theorem 6. *[9, Theorem 3.1] Let $(\mathcal{E}, \circ)$ be an Euclidean Jordan algebra of rank r and $\lambda \in \mathcal{E}$. The following statements are equivalent:*

(i) $\lambda \in \mathcal{K}$.

(ii) There exists $y \in \mathcal{E}$ such that $y \circ \lambda = 0$ and $\langle w \circ w, \lambda\rangle > 0$ for every $w \in \mathcal{E}$ satisfying $y \circ w = 0$ and $w \neq 0$.

Moreover, any y satisfying (ii) is such that

(a) $\operatorname{rank} y = r - \operatorname{rank} \lambda$, *i.e., y and λ satisfy strict complementarity,*

(b) if σ and σ' are non-zero eigenvalues of y belonging to the same block, then $\sigma + \sigma' \neq 0$.

The proposition below extend previous results obtained in [5, Section 3] for NSOCPs and in [8, Section 3] for NSDPs. It is also a consequence of Proposition 5.

Proposition 7. *[9, Proposition 4.3] Let $(x, y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^m \times \mathcal{E}$ be a KKT quadruple for* (P2). *(a) If y and λ satisfy the assumptions of Theorem 6(ii), then (x, μ, λ) is a KKT triple for* (P1) *satisfying strict complementarity. (b) If SOSC-NLP holds at (x, y, μ, λ), then (x, μ, λ) is a KKT triple for* (P1) *satisfying strict complementarity.*

Theorem 8. *[9, Proposition 6.1] Let $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ be a KKT triple of problem* (P1). *Suppose that*

$$\langle \nabla_x^2 L(x, \mu, \lambda)v, v\rangle + 2\langle w \circ w, \lambda\rangle > 0,$$

for every nonzero $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2\sqrt{g(x)} \circ w = 0$ and $Jh(x)v = 0$. Then, x is a local minimum for (P1), *$\lambda \in \mathcal{K}$, and strict complementarity is satisfied.*

We observe that the condition in Theorem 8 is strong enough to ensure strict complementarity. And, in fact, when strict complementarity holds and $\mathcal{K}$ is either the cone of positive semidefinite matrices or a product of Lorentz cones, the condition in Proposition 8 is equivalent to the second-order sufficient conditions described in [1, 10]. We also have the following necessary condition.

Theorem 9. *[9, Proposition 6.2] Let $x \in \mathbb{R}^n$ be a local minimum of* (P1). *Assume that $(x, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{E}$ is a KKT triple for* (P1) *satisfying nondegeneracy. Then the following condition holds:*

$$\langle \nabla_x^2 L(x, \mu, \lambda)v, v\rangle + 2\langle w \circ w, \lambda\rangle \geq 0,$$

for every $(v, w) \in \mathbb{R}^n \times \mathcal{E}$ such that $Jg(x)v - 2\sqrt{g(x)} \circ w = 0$ and $Jh(x)v = 0$.

5 Final remarks

In this survey, we presented a discussion on optimality conditions for nonlinear symmetric cone programs through slack variables. For more details, we refer to the original paper [9], where discussions about constraint qualifications and augmented Lagrangian methods are also made. Although the idea of using squared slack variables is simple, the obtained second-order sufficient conditions make the strict complementarity to be automatically satisfied. Therefore, an interesting research topic would be to find out whether the NSCP admits another reformulation as a nonlinear programming problem without this deficiency.

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