

Optimality conditions for quasi (ϵ, α) -solutions in convex optimization problems under data uncertainty

Liguo Jiao¹ and Do Sang Kim²

¹Finance·Fishery·Manufacture Industrial Mathematics Center on Big Data
Pusan National University, Busan 46241, Korea
E-mail: hanchezi@163.com

²Department of Applied Mathematics
Pukyong National University, Republic of Korea
E-mail: dskim@pknu.ac.kr

Abstract. Under the robust characteristic cone constraint qualification (RC-CCQ), a represented form of the ϵ -normal set to a convex set C at $\bar{x} \in C$ is proposed, where $C := \{x \in \mathbb{R}^n : g(\cdot, v) \leq 0, \forall v \in \mathcal{V}\}$, here $g: \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ is a continuous function such that for all $v \in \mathbb{R}^p$, $g(\cdot, v)$ is a convex function, and $\mathcal{V} \subset \mathbb{R}^p$ is a compact and convex uncertain set. Then, the proposed result is applied to studying approximate optimality theorems for a quasi (α, ϵ) -solution for the robust counterpart of a convex optimization problem in the face of data uncertainty.

1 Introduction

The problem to examine a point in a given set, which minimizes a given numerical function over that set, is an interesting mathematical model, and we say it as *mathematical optimization problems*. In particular, if the given numerical function is a convex function and the given set is a convex set, it is then said to be *convex programming*; see, for example, [4, 5, 15] for a more detail study. The fact that an optimal solution of a mathematical programming problem may not be exact but very near to it leads to the concepts of approximate solutions, which play an important role in algorithmic study of optimization problems. It is worth mentioning that among them, the concept of the so-called *quasi ϵ -solution* introduced by Loridan [14] is motivated by the well-known Ekeland's Variational Principle [6].

In 2008, Beldiman *et al.* [2] introduced a new class of approximate solutions in scalar/vector optimization problems, and discussed the relationship among the introduced approximate solutions. However, they did not explore the optimality conditions for such a class of approximate solutions. In this paper, we aim to study the approximate optimality conditions for such a class of generalized approximate solutions in robust convex optimization problems. An overview of the optimization problems with data uncertainty and the method (robust approach) to deal with such problems is stated in Section 3, as we shall see.

In the face of approximate solutions in the robust convex optimization, Lee and Lee [12] and Lee and Jiao [11] employed the robust version of Farkas's lemma to study some characterizations of ϵ -solutions and quasi ϵ -solutions, respectively. Besides, Lee and Lee [13] and Jiao and Lee [10] proposed approximate optimality conditions for ϵ -solutions and quasi ϵ -solutions in the robust convex *semidefinite programs* according to its special structure, respectively.

Apart from the Farkas's lemma approach, Strodiot *et al.* [16] pointed out an effective method, i.e., by analysing ϵ -normal set when the feasible set was explicitly expressed in terms of (convex) inequality systems under Slater's constraint qualification; then they formulated approximate optimality for a convex programming problem.

In the present paper, we focus on the study of a generalized approximate solution in the sense of Beldiman *et al.* [2] for robust convex optimization problems in the face of uncertainty data. First, we study the ϵ -normal set, which is explicitly expressed in terms of robust (convex) inequality systems, under a constraint qualification called *robust characteristic cone constraint qualification* [7], which is weaker than the Slater condition. Then, we examine approximate optimality theorems on the generalized approximate solution for the robust convex optimization problem.

The rest of this paper is organized as follows. Section 2 states some notations and preliminaries. In Sect. 3, we examine the approximate optimality theorem on the generalized approximate solution for the robust convex optimization problem. Finally, Conclusions are given in Sect. 4.

2 Preliminaries

This section gives some notations and preliminary results that will be used in the paper. We abbreviate (x_1, x_2, \dots, x_n) by x . The Euclidean space \mathbb{R}^n is equipped with the usual Euclidean norm $\|\cdot\|$. The nonnegative orthant of \mathbb{R}^n is defined by $\mathbb{R}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$. The inner product in \mathbb{R}^n is defined by $\langle x, y \rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say that a set A in \mathbb{R}^n is convex whenever $\mu a_1 + (1 - \mu)a_2 \in A$ for all $\mu \in [0, 1]$, $a_1, a_2 \in A$. For a given set $A \subset \mathbb{R}^n$, we denote the interior, closure and convex hull generated by A , by $\text{int } A$, $\text{cl } A$ and $\text{conv } A$, respectively.

Let f be a function from \mathbb{R}^n to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, +\infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\text{dom } f$, that is, $\text{dom } f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$.

The epigraph f , $\text{epi } f$, is defined by $\text{epi } f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$. The function f is said to be convex if for all $\mu \in [0, 1]$, $f((1 - \mu)x + \mu y) \leq (1 - \mu)f(x) + \mu f(y)$ for all $x, y \in \mathbb{R}^n$; equivalently, $\text{epi } f$ is convex. The function

f is said to be concave whenever $-f$ is convex. Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function, the (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle \leq f(y) - f(x), \forall y \in \mathbb{R}^n\}, & \text{if } x \in \text{dom } f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

More generally, for any $\epsilon \geq 0$, the ϵ -subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle \leq f(y) - f(x) + \epsilon, \forall y \in \mathbb{R}^n\}, & \text{if } x \in \text{dom } f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We say f is a lower semicontinuous function if $\liminf_{y \rightarrow x} f(y) \geq f(x)$ for all $x \in \mathbb{R}^n$. Let δ_C be the *indicator function* with respect to a closed convex subset C of \mathbb{R}^n , that is, $\delta_C(x) = 0$ if $x \in C$, and $\delta_C(x) = +\infty$ if $x \notin C$.

Definition 2.1 (ϵ -Normal Set) Let $C \subset \mathbb{R}^n$ be a closed convex set and $x \in C$. Then $N_\epsilon(x, C) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x \rangle \leq \epsilon, \forall y \in C\}$ is called the ϵ -normal set to C at x .

The following two lemmas are the *sum rule* and the *scalar product rule* of the ϵ -subdifferential that will be used in the sequel.

Lemma 2.1 [5, Theorem 2.115] *Consider two proper convex functions $f_1, f_2 : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ such that $\text{ri dom } f_1 \cap \text{ri dom } f_2 \neq \emptyset$. Then for $\epsilon > 0$,*

$$\partial_\epsilon(f_1 + f_2)(\bar{x}) = \bigcup_{\substack{\epsilon_1 \geq 0, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 = \epsilon}} (\partial_{\epsilon_1} f_1(\bar{x}) + \partial_{\epsilon_2} f_2(\bar{x}))$$

Lemma 2.2 [5, Theorem 2.117] *For a proper convex function $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ and any $\epsilon \geq 0$,*

$$\partial_\epsilon(\lambda f)(\bar{x}) = \lambda \partial_{\epsilon/\lambda} f(\bar{x}), \forall \lambda > 0.$$

3 Main Results

3.1 Model Statements

A standard form of a convex programming problem [4, 5] is the one:

$$\min f(x) \quad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \dots, m, \quad (\text{CP})$$

where $f, g_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, m$ are convex functions.

The convex programming problem (CP) in the face of data uncertainty in the constraints can be captured by the one:

$$\min f(x) \quad \text{subject to} \quad g_i(x, v_i) \leq 0, \quad i = 1, \dots, m, \quad (\text{UCP})$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$ is continuous, $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q$, $i = 1, \dots, m$. The problem (UCP) is to optimize convex optimization problems with data uncertainty (incomplete data), which means that input parameters of these problems are not known exactly at the time when solution has to be determined [3].

In the present paper, we explore approximate optimality theorem for problem (UCP) by examining its robust (worst-case) counterpart [1, 3]:

$$\min f(x) \quad \text{subject to} \quad g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i, \quad i = 1, \dots, m. \quad (\text{RCP})$$

Denote by $F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i, \quad i = 1, \dots, m\}$ as the feasible set of problem (RCP).

3.2 Solution Concepts

Consider the robust counterpart (RCP) of problem (UCP).

Definition 3.1 Let $\alpha \geq 0$ and $\epsilon \geq 0$ be given, then \bar{x} is said to be

- (i) an ϵ -solution of problem (RCP), if $f(\bar{x}) \leq f(x) + \epsilon, \quad \forall x \in F$;
- (ii) a quasi α -solution of problem (RCP), if $f(\bar{x}) \leq f(x) + \alpha\|x - \bar{x}\|, \quad \forall x \in F$;
- (iii) a regular (α, ϵ) -solution of problem (RCP), if for any $x \in F$, \bar{x} is an ϵ -solution of problem (RCP) as well as a quasi α -solution.

Now, we introduce a generalized approximate solution, i.e., the so-called quasi (α, ϵ) -solution, for problem (RCP).

Definition 3.2 [2] Let $\alpha \geq 0$ and $\epsilon \geq 0$ be given, then \bar{x} is said to be a quasi (α, ϵ) -solution of problem (RCP), if

$$f(\bar{x}) \leq f(x) + \alpha\|x - \bar{x}\| + \epsilon, \quad \forall x \in F.$$

3.3 Approximate Optimality Theorem

In this subsection, we establish approximate optimality theorem for (RCP) under a robust characteristic cone constraint qualification [7], that is, the cone

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \text{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^*$$

is closed and convex.

Note that the set $D := \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \text{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^*$ is a cone in \mathbb{R}^{n+1} , which is called the *robust characteristic cone* [7].

Definition 3.3 We say a *robust characteristic cone constraint qualification* (RC-CCQ) holds if the robust characteristic cone D is closed and convex.

Jeyakumar and Li [7] have shown that the robust characteristic cone D is convex whenever $g_i(x, \cdot)$ is concave and $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \dots, m$, is convex; in addition, they also proved that the robust characteristic cone D is closed whenever the robust Slater condition holds, that is, $\{x \in \mathbb{R}^n : g_i(x_0, v_i) < 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\} \neq \emptyset$.

3.3.1 ϵ -Optimality conditions for an unconstrained problem

Lemma 3.1 Consider the unconstrained convex programming problem:

$$\min f(x) \quad \text{subject to} \quad x \in \mathbb{R}^n. \quad (\text{CP}_u)$$

Then \bar{x} is an ϵ -solution of (CP_u) if and only if $0 \in \partial_\epsilon f(\bar{x})$.

Theorem 3.1 Let \bar{x} be a quasi (α, ϵ) -solution for (CP_u) . Then there exist $\bar{\epsilon}_f \geq 0$ and $\bar{\epsilon}_b \geq 0$ with $\bar{\epsilon}_f + \bar{\epsilon}_b = \epsilon$, such that

$$0 \in \partial_{\bar{\epsilon}_f} f(\bar{x}) + \alpha \partial_{\bar{\epsilon}_b/\alpha} \|\cdot - \bar{x}\|(\bar{x}).$$

Remark 3.1 For any $\epsilon \geq 0$, let $\bar{x} = 0$, $\partial_\epsilon \|\bar{x}\| = \partial \|\bar{x}\| = \mathbb{B}$.

With the help of Remark 3.1, we get the restatement of Theorem 3.1.

Theorem 3.2 (Restatement of Theorem 3.1) Consider the unconstrained convex programming problem (CP_u) . Let \bar{x} be a quasi (α, ϵ) -solution for (CP_u) . Then there exist $\bar{\epsilon}_f \geq 0$ and $\bar{\epsilon}_b \geq 0$ with $\bar{\epsilon}_f + \bar{\epsilon}_b = \epsilon$, such that

$$0 \in \partial_{\bar{\epsilon}_f} f(\bar{x}) + \alpha \mathbb{B}.$$

3.3.2 Representation of the ϵ -normal set

In order to obtain the approximate optimality condition in terms of the constraint functions $g_i(x, v_i) \leq 0$, $\forall v_i \in \mathcal{V}_i$, $i = 1, \dots, m$, the ϵ -normal set (see Definition 2.1) must be explicitly expressed in their terms. Below, we present such a result, which modifies the one studied by Strodiot *et al* [16], under the robust characteristic cone constraint qualification (see Definition 3.3) rather than the Slater's constraint qualification.

Proposition 3.1 Let $\epsilon \geq 0$ be given. Let $g : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ be continuous function such that for all $v \in \mathbb{R}^p$, $g(\cdot, v)$ is a convex function. Suppose that $\mathcal{V} \subset \mathbb{R}^p$ is compact and convex, and (RCCCQ) holds. Let $\bar{x} \in C := \{x \in \mathbb{R}^n : g(\cdot, v) \leq 0, \forall v \in \mathcal{V}\}$. Then $\xi \in N_\epsilon(\bar{x}, C)$ if and only if there exist $\bar{\lambda} \geq 0$, $\bar{v} \in \mathcal{V}$ and $\bar{\epsilon} \geq 0$ such that

$$\bar{\epsilon} \leq \bar{\lambda} g(\bar{x}, \bar{v}) + \epsilon \quad \text{and} \quad \xi \in \partial_\epsilon(\bar{\lambda} g)(\bar{x}, \bar{v}).$$

Remark 3.2 We mention here that if $\epsilon = 0$, then the ϵ -normal set at \bar{x} to C becomes a normal cone at \bar{x} to C . With the same condition (RCCCQ), Jiao *et al* [9] obtained a result, which is the representation of the normal cone \bar{x} to C .

3.3.3 Approximate optimality conditions

Now, we are ready to give the main theorem in the paper, which is the approximate optimality condition for a quasi (α, ϵ) -solution of problem (RCP) under the fulfilment of the robust characteristic cone constraint qualification.

Theorem 3.3 (Approximate Optimality Theorem) *Let $\alpha \geq 0$ and $\epsilon \geq 0$ be given. Let $g_i : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$, $i = 1, \dots, m$ be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is convex on \mathbb{R}^n . Suppose that the (RCCCQ) condition holds. Then, \bar{x} is a quasi (α, ϵ) -solution of problem (RCP) if and only if there exist $\bar{\epsilon}_0 \geq 0$, $\bar{\epsilon}_i \geq 0$, $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0$, $i = 1, \dots, m$ such that*

$$0 \in \partial_{\bar{\epsilon}_0} f(\bar{x}) + \sum_{i=1}^m \partial_{\bar{\epsilon}_i} (\bar{\lambda}_i g_i(\cdot, \bar{v}_i))(\bar{x}) + \alpha \mathbb{B},$$

$$\sum_{i=0}^m \bar{\epsilon}_i - \epsilon \leq \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i).$$

4 Conclusions

In this paper, we studied a generalized approximate solution, i.e., quasi (α, ϵ) -solution for a robust convex optimization problem. Approximate optimality theorems for the generalized approximate solution in the robust convex optimization problem under the (RCCCQ) condition were studied, after we explored the representation of the ϵ -normal set to a convex set (see Proposition 3.1). It is worth mentioning that Proposition 3.1 played a key role in establishing our main results, which informed us that our approach in this paper was different from [10–12]. Besides, we claim that most of the results of this paper are presented in the paper [8] by the authors.

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