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EXISTNECE AND CONVERGENCE THEOREMS FOR NORMALLY 2-GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

ATSUMASA KONDO AND WATARU TAKAHASHI

Abstract. This article reviews the existence and convergence results for normally 2-generalized hybrid mappings in Hilbert spaces. The results generalize many existing theorems simultaneously. Special attention will be paid to lemmas that play important roles for proving the results.

1. Introduction

Throughout this paper, we denote a real Hilbert space by H and its inner product and norm by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. We denote the set of natural and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let C be a nonempty subset of H and let T be a mapping from C to H. The set of fixed and attractive points of T are denoted by

$$\begin{aligned} F(T) &= \{u \in H : Tu = u\} \text{ and} \\ A(T) &= \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for any } y \in C\}, \end{aligned}$$

respectively. The concept of attractive points was introduced by Takahashi and Takeuchi [19]. A mapping $T : C \to H$ is called:

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(ii) *nonspreading* [6] if $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2$ for all $x, y \in C$;

(iii) *hybrid* [18] if $3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|x - Ty\|^2$ for all $x, y \in C$;

(iv) *generalized hybrid* [5] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$.

It is well-known that the class of nonexpansive mappings plays an important role in optimization theory in Hilbert spaces. The generalized hybrid mappings contain all mappings (i)–(iii) as special cases. For generalized hybrid mappings, fixed and attractive point approximation methods have been extensively studied. The next theorem of Baillon's type was proved by Takahashi and Takeuchi in 2011; see also [2], [15] and [5].

Theorem 1.1 ([19]). *Let C be a nonempty subset of H and let $T : C \to C$ be a generalized hybrid mapping with $A(T) \neq \emptyset$. For any $x \in C$, define a sequence as*

$$x_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k x \in H$$

for $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to an attractive point of T.

The following theorem of Mann's type was demonstrated by [5]; see also [11] and [14].

Theorem 1.2 ([5]). *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $P_{F(T)}$ be the metric projection from H onto $F(T)$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $0 \leq \lambda_n \leq 1$ and $\liminf_{n\to\infty} \lambda_n (1-\lambda_n) > 0$. Define a sequence $\{x_n\}$ in C as*

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n \in C$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to an element u of $F(T)$, where $u = \lim_{n\to\infty} P_{F(T)} x_n$.

The following strong convergence theorem of Halpern's type for generalized hybrid mappings has been established by Takahashi, Wong and Yao in 2015; see also [3], [22].

Theorem 1.3 ([21]). *Let C be a nonempty and convex subset of H and $T : C \to C$ be a generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\eta_n\}$ be sequences of real numbers in the interval $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\liminf\limits_{n\to\infty} \eta_n (1-\eta_n) > 0$. Given $z, x_1 \in C$, define a sequence $\{x_n\}$ in C as*

$$x_{n+1} = \lambda_n z + (1-\lambda_n)(\eta_n x_n + (1-\eta_n) T x_n) \in C$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to an attractive point of T.

In 2011, Hojo and Takahashi proved the following strong convergence theorem. An important precursor is Kurokawa and Takahashi [9].

Theorem 1.4 ([4]). *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Given $x_1, z \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = \lambda_n z + (1-\lambda_n)\frac{1}{n}\sum_{k=0}^{n-1} T^k x_n \in C$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\overline{z} \equiv P_{F(T)} z$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

The generalized hybrid mapping is further extended. Takahashi, Wong and Yao [20] and Maruyama, Takahashi and Yao [13] introduced new types of nonlinear mappings, which are more general than generalized hybrid mappings. A mapping $T : C \to C$ is called

(v) *normally generalized hybrid* [20] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \leq 0$$

for all $x, y \in C$, where (a) $\alpha + \beta + \gamma + \delta \geq 0$ and (b) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$;

(vi) 2-*generalized hybrid* [13] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1 \left\|T^2x - Ty\right\|^2 + \alpha_2 \|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2) \|x - Ty\|^2 \\ \leq \beta_1 \left\|T^2x - y\right\|^2 + \beta_2 \|Tx - y\|^2 + (1 - \beta_1 - \beta_2) \|x - y\|^2 \end{aligned}$$

for all $x, y \in C$.

Very recently, Kondo and Takahashi [7] introduced a new type of nonlinear mappings that contains all the mappings (i)–(vi) as special cases, and proved the existence and weak convergence results that extended Theorem 1.1 and 1.2. In their succeeding paper [8], they demonstrated two types of strong convergence theorems that extended Theorem 1.3 and 1.4. This article briefly reviews these results with paying careful attention to crucial lemmas by which these results were demonstrated.

2. Preliminaries

This section briefly offers background information and lemmas. It holds that

$$2\langle x - y,\ y\rangle \leq \|x\|^2 - \|y\|^2 \leq 2\langle x - y,\ x\rangle \tag{2.1}$$

for all $x, y \in H$. We know from [17] that

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda (1 - \lambda) \|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. We also know from [13] that

(2.3)

$$\begin{aligned} &\|ax + by + cz\|^2 \\ &\quad = a \|x\|^2 + b \|y\|^2 + c \|z\|^2 - ab \|x - y\|^2 - bc \|y - z\|^2 - ca \|z - x\|^2 \end{aligned}$$

for all $x, y, z \in H$ and $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. It is clear that the equation (2.3) is a generalization of (2.2). Let A be a nonempty, closed and convex subset of H. We know that for any $z \in H$, there exists a unique nearest point $\overline{z} \in A$, that is, $\|z - \overline{z}\| = \inf_{u \in A} \|z - u\|$. This correspondence is called the *metric projection* from H onto A, and is denoted by P_A. Since the set of attractive points is closed and convex (see [19]), the metric projection from H onto $A(T)$ exists if $A(T)$ is nonempty.

Let l^∞ be the Banach space of bounded real sequences with the supremum norm. Let $\mu \in (l^\infty)^*$, where $(l^\infty)^*$ is the dual space of l^∞. For simplicity, we often denote $\mu(\{x_n\})$ by $\mu_n x_n$ if no ambiguity arises. A linear continuous functional $\mu \in (l^{\infty})^{*}$ is called a *mean* if $\mu(\{1,1,1,\cdots\}) = \|\mu\| = 1$. When a mean additionally satisfies $\mu_n(x_n) = \mu_n(x_{n+1})$, it is called a *Banach limit* on l^{∞}. It is well-known that a Banach limit exists, which is demonstrated by using the Hahn–Banach theorem. For more details, see Takahashi [16].

The following lemma, Lemma 2.1, is utilized in the proofs of Theorem 4.3 and 4.4 while Lemma 2.2 is applied to the proof of Theorem 4.3.

Lemma 2.1 ([1], see also [23]). *Let $\{X_n\}$ be a sequence of nonnegative real numbers, let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n = \infty$. Let $\{Y_n\}$ be a sequence of real numbers such that $\limsup_{n\to\infty} Y_n \le 0$, and let $\{Z_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} Z_n < \infty$. If $X_{n+1} \le (1-\lambda_n)X_n + \lambda_n Y_n + Z_n$ for all $n \in \mathbb{N}$, then $X_n \to 0$ as $n \to \infty$.*

Lemma 2.2 ([12]). *Let $\{X_n\}$ be a sequence of real numbers. Assume that $\{X_n\}$ is not monotone decreasing for sufficiently large $n \in \mathbb{N}$, in other words, there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $\{k \le n_0 : X_k < X_{k+1}\} \neq \emptyset$. Define a sequence $\{\tau(n)\}_{n\ge n_0}$ of natural numbers as follows:*

$$\tau(n) = \max\{k \le n : X_k < X_{k+1}\} \quad \text{for } n \ge n_0.$$

Then, the followings hold:

(a) $\tau(n) \to \infty$ *as* $n \to \infty$;

(b) $X_n \le X_{\tau(n)+1}$ *and* $X_{\tau(n)} < X_{\tau(n)+1}$ *for* $n \ge n_0$.

3. Existence Results

In this section, we define a new type of mappings called normally 2-generalized hybrid mappings [7], and review theorems that guarantee the existence of attractive or fixed points. A mapping $T : C \to C$ is called

(vii) *normally 2-generalized hybrid* [7] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\alpha_2\left\|T^2x - Ty\right\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 + \beta_2\left\|T^2x - y\right\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \le 0$$

for all $x, y \in C$, where (a) $\sum_{n=0}^{2}(\alpha_n + \beta_n) \ge 0$ and (b) $\alpha_2+\alpha_1+\alpha_0 > 0$. It is easily ascertained that the class of normally 2-generalized hybrid mappings contains both normally generalized hybrid mappings and 2-generalized hybrid mappings as special cases. Thus, it includes all the mappings (i)–(vi) simultaneously.

The following lemma is essential for proving Theorem 3.1 and 3.2.

Lemma 3.1 ([10], [15]). *Let μ be a mean on l^{∞}. Then, for any bounded sequence $\{x_n\}$ in H, there is a unique element $u \in \overline{co}\{x_n\}$ such that*

$$\mu_n\langle x_n,\ v\rangle = \langle u,\ v\rangle$$

for any $v \in H$, where $\overline{co}\{x_n\}$ is the closure of the convex hull of $\{x_n : n \in \mathbb{N}\}$.

Theorem 3.1 ([7]). *Let C be a nonempty subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Then, the following three statements are equivalent:*

(I) *for any $x \in C$, $\{T^n x\}$ is a bounded sequence in C;*
(II) *there exists $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C;*
(III) *$A(T) \neq \emptyset$.*

It is obvious that (I)$\Longrightarrow$(II) in Theorem 3.1 holds. The proof of (III)$\Longrightarrow$(I) is easy. Lemma 2.1 is useful to prove (II)$\Longrightarrow$(III); For the bounded sequence $\{T^n z\}\,(\subset C)$, there is a unique element $u \in \overline{co}\,\{T^n z\}\,(\subset H)$ such that $\mu_n \langle T^n z,\ v\rangle = \langle u,\ v\rangle$ for any $v \in H$, where $\mu \in (l^\infty)^*$ is a Banach limit. We can show that the element $u\,(\in H)$ is an attractive point of T, and thus, $A(T) \neq \emptyset$. With additionally supposing that C is closed and convex, we obtain the following fixed point theorem.

Theorem 3.2 ([7]). *Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Then, the following four statements are equivalent:*

(I) *for any $x \in C$, $\{T^n x\}$ is a bounded sequence in C;*
(II) *there exists $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C;*
(III) *$A(T) \neq \emptyset$;*
(IV) *$F(T) \neq \emptyset$.*

The proof of (III)$\Longrightarrow$(IV) in Theorem 3.2 immediately follows from the next lemma.

Lemma 3.2 ([19]). *Let C be a nonempty, closed and convex subset of H and let T be a mapping from C to itself. Suppose that $A(T) \neq \emptyset$. Then, $F(T) \neq \emptyset$.*

The proof of Lemma 3.2 can be sketched as follows: From the assumptions on C, there exists the metric projection P_C from H onto C. Since $A(T) \neq \emptyset$ is assumed, we can take an element $u \in A(T)$. Map it by P_C onto C. It can be shown that $P_C u \in F(T)$. In other words, any nearest points in C from $A(T)$ are fixed points of T.

4. Convergence Results

This section reorganizes the convergence theorems demonstrated by [7] and [8]. For proving the theorems, certain types of lemmas that guarantee that any weak limits are attractive points are useful. To our best knowledge, there are at least three types of such lemmas, Lemma 4.1–4.3. First, by using Lemma 4.1, we can obtain the weak convergence theorem that extends Theorem 1.1. The basic technique of the proof is developed by Takahashi [15].

Lemma 4.1 ([7]). *Let C be a nonempty subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Suppose that there is an element $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. Define $S_n z \equiv$*

$\frac{1}{n}\sum_{k=0}^{n-1} T^k z\,(\in H)$ *and assume that* $S_{n_i}z \rightharpoonup u$, *where* $\{S_{n_i}z\}$ *is a subsequence of* $\{S_n z\}$. *Then,* $u \in A(T)$.

Theorem 4.1 ([7]). *Let* C *be a nonempty subset of* H *and let* $T : C \to C$ *be a normally* 2*-generalized hybrid mapping with* $A(T) \neq \emptyset$. *Let* $P_{A(T)}$ *be the metric projection from* H *onto* $A(T)$. *Then, for any* $x \in C$, *the sequence* $\left\{S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ *converges weakly to* $u \in A(T)$, *where* $u = \lim_{n\to\infty} P_{A(T)}T^n x$.

The second type lemma that guarantees that any weak limits are attractive points is as follows:

Lemma 4.2 ([7]). *Let* C *be a nonempty subset of* H, *let* $T : C \to C$ *be a normally* 2*-generalized hybrid mapping and let* $\{x_n\}$ *be a sequence in* C *satisfying* $x_n - Tx_n \to 0$, $T^2x_n - x_n \to 0$ *and* $x_n \rightharpoonup v$. *Then,* $v \in A(T)$.

If the conclusion in Lemma 4.2 is $v \in F(T)$ instead of $v \in A(T)$ and the assumptions do not include $T^2x_n - x_n \to 0$, then the mapping $I - T$ is called demiclosed, where I is the identity mapping. By using Lemma 4.2 and (2.3), we can prove Theorem 4.2 and 4.3, which extend Theorem 1.2 and 1.3, respectively.

Theorem 4.2 ([7]). *Let* C *be a nonempty and convex subset of* H *and let* $T : C \to C$ *be a normally* 2*-generalized hybrid mapping with* $A(T) \neq \emptyset$. *Let* $P_{A(T)}$ *be the metric projection from* H *onto* $A(T)$. *Let* $\{a_n\}, \{b_n\}, \{c_n\}$ *be real sequences in the interval* $(0,1)$ *such that* $a_n + b_n + c_n = 1$ *and* $0 < a \leq a_n, b_n, c_n \leq b < 1$ *for all* $n \in \mathbb{N}$. *Define a sequence* $\{x_n\}$ *in* C *as*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n \in C$$

for all $n \in \mathbb{N}$, *where* $x_1 \in C$ *is given. Then, the sequence* $\{x_n\}$ *converges weakly to an element* u *of* $A(T)$, *where* $u = \lim_{n\to\infty} P_{A(T)}x_n$.

To prove the next strong convergence theorem (Theorem 4.3), Lemma 2.1, 2.2 and the equation (2.1) are required.

Theorem 4.3 ([8]). *Let* C *be a nonempty and convex subset of* H *and let* $T : C \to C$ *be a normally* 2*-generalized hybrid mapping with* $A(T) \neq \emptyset$. *Let* $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be sequences of real numbers in the interval* $(0,1)$ *such that*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty, \quad a_n + b_n + c_n = 1 \ \ (n \in \mathbb{N}),$$

$$\liminf_{n\to\infty} a_n b_n > 0, \quad \liminf_{n\to\infty} b_n c_n > 0 \quad \text{and} \quad \liminf_{n\to\infty} c_n a_n > 0.$$

Given $x_1, z \in C$, *define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1} = \lambda_n z + (1-\lambda_n)\left(a_n x_n + b_n T x_n + c_n T^2 x_n\right)$$

for all $n \in \mathbb{N}$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $\overline{z} \equiv P_{A(T)}z$, *where* $P_{A(T)}$ *is the metric projection from* H *onto* $A(T)$.

Finally, by using Lemma 4.3 (and Lemma 2.1), we can prove Theorem 4.4. Please refer to Kurokawa and Takahashi [9].

Lemma 4.3 ([8]). *Let C be a nonempty subset of H, let $T : C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$, and let $\{x_n\}$ be a bounded sequence in C. Define $z_n \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n \,(\in H)$ and assume that $z_{n_i} \rightharpoonup v$, where $\{z_{n_i}\}$ is a subsequence of $\{z_n\}$. Then, $v \in A(T)$.*

Theorem 4.4 ([8]). *Let C be a nonempty and convex subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Given $x_1, z \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = \lambda_n z + (1 - \lambda_n) \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\overline{z} \equiv P_{A(T)} z$, where $P_{A(T)}$ is the metric projection from H onto $A(T)$.

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(Atsumasa Kondo) Department of Economics, Shiga University, Banba 1-1-1, Hikone, Shiga 522-0069, Japan

E-mail address: a-kondo@biwako.shiga-u.ac.jp

(Wataru Takahashi) Center for Fundamental Science, and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 80708, Taiwan; Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan

E-mail address: wataru@is.titech.ac.jp; wataru@a00.itscom.net