数理解析研究所講究録 第2112巻 2019年 195-200

A Characterization of Comparison Indices for Fuzzy Sets Based on Possibility Theory

Koichiro Ike and Tamaki Tanaka

Graduate School of Science and Technology
Niigata University

1 Introduction

Fuzzy set [6] is an extended concept of set and one of the most effective tools for mathematically modeling various real-world things containing uncertainty or vagueness. Specifically, a fuzzy set $\tilde{A}$ in a space X is a pair $(X, \mu_{\tilde{A}})$ where $\mu_{\tilde{A}}$ is a function from X to the unit interval $[0,1]$ and is called the membership function of $\tilde{A}$. Every (crisp) set $A \subset X$ can, of course, be considered as a fuzzy set in the sense that its membership function coincides with the characteristic function $\chi_A \colon X \to \{0,1\}$. In this paper, we focus on the following question: "How should two or more fuzzy sets be compared?" For fuzzy numbers (i.e., fuzzy sets in $\mathbb{R}$ with certain restrictions), a large number of studies about comparison or ranking methods have been done; see [1] for an overview of them. By contrast, there are not so many papers on those for general fuzzy sets owing in part to the complexity of their practical applications. Given this situation, we here aim at establishing reasonable and easy-to-handle comparison indices for general fuzzy sets in an arbitrary dimensional vector space.

In introducing the comparison indices, two approaches can naturally be considered: using (crisp) binary relations and employing fuzzy relations [7] (where a fuzzy relation on X means a fuzzy set in the product $X \times X$). The former approach has been adopted, for example, in [3] by the present authors. However, the latter is expected to be more suitable because fuzzy set and fuzzy relation are both "fuzzy" notions, meshing with each other. We thus mainly discuss the fuzzy relational approach in this paper. The key ideas of the discussion are based on the knowledge of possibility theory, especially possibility measure and necessity measure, and also seen as a generalization of those in [2, 5]. As a collateral result, it is shown that the obtained fuzzy relations can be fully characterized by set relations [4] widely used in the area of set optimization.

2 Preliminaries

Throughout the paper, let X be a real topological vector space equipped with the preorder $\leq_C$ induced by a convex cone C, i.e., $x \leq_C y \;:\Longleftrightarrow\; y - x \in C$ for $x, y \in X$.

Let $\mathcal{F}(X)$ denote the set of all fuzzy sets in X and $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$. The inclusion and the equality of fuzzy sets are defined by

$$\tilde{A} \subset \tilde{B} \;:\Longleftrightarrow\; \forall x \in X\colon \mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x),$$
$$\tilde{A} = \tilde{B} \;:\Longleftrightarrow\; \forall x \in X\colon \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x).$$

The complement $\tilde{A}^{\mathrm{c}}$ of $\tilde{A}$ is given by $\mu_{\tilde{A}^{\mathrm{c}}}(x) := 1 - \mu_{\tilde{A}}(x)$, $x \in X$. For each $\alpha \in [0,1]$, the α-level set of $\tilde{A}$ is defined as

$$[\tilde{A}]_\alpha := \begin{cases} \{x \in X \mid \mu_{\tilde{A}}(x) \ge \alpha\} & (\alpha \in (0,1]) \\ \mathrm{cl}\,\{x \in X \mid \mu_{\tilde{A}}(x) > 0\} & (\alpha = 0), \end{cases}$$

where cl represents the topological closure operator. The fuzzy set $\tilde{A}$ is said to be normal if there exists $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$ (or equivalently, $[\tilde{A}]_1 \ne \emptyset$).

Definition 1 (Kuroiwa, Tanaka, Ha [4])**.** Six types of set relations are defined by

$$A \le_C^{(1)} B \;:\Longleftrightarrow\; \forall a \in A \; \forall b \in B\colon a \le_C b,$$
$$A \le_C^{(2L)} B \;:\Longleftrightarrow\; \exists a \in A \; \forall b \in B\colon a \le_C b,$$
$$A \le_C^{(2U)} B \;:\Longleftrightarrow\; \exists b \in B \; \forall a \in A\colon a \le_C b,$$
$$A \le_C^{(3L)} B \;:\Longleftrightarrow\; \forall b \in B \; \exists a \in A\colon a \le_C b,$$
$$A \le_C^{(3U)} B \;:\Longleftrightarrow\; \forall a \in A \; \exists b \in B\colon a \le_C b,$$
$$A \le_C^{(4)} B \;:\Longleftrightarrow\; \exists a \in A \; \exists b \in B\colon a \le_C b$$

for $A, B \subset X$.

When A and B are nonempty, from the definition we obtain

$$A \le_C^{(1)} B \implies A \le_C^{(2L)} B \implies A \le_C^{(3L)} B \implies A \le_C^{(4)} B,$$
$$A \le_C^{(1)} B \implies A \le_C^{(2U)} B \implies A \le_C^{(3U)} B \implies A \le_C^{(4)} B.$$

The following is a crisp relational answer to the comparison problem of fuzzy sets using the above set relations.

Definition 2 (Ike, Tanaka [3])**.** Let $\Omega \subset [0,1]$. For each $i = 1, 2L, 2U, 3L, 3U, 4$, a fuzzy-set relation $\le_C^{\Omega(i)}$ is defined by

$$\tilde{A} \le_C^{\Omega(i)} \tilde{B} \;:\Longleftrightarrow\; \forall \alpha \in \Omega\colon [\tilde{A}]_\alpha \le_C^{(i)} [\tilde{B}]_\alpha$$

for $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$.

When $\tilde{A}$ and $\tilde{B}$ are normal, by analogy with the set relations we immediately have

$$\tilde{A} \le_C^{\Omega(1)} \tilde{B} \implies \tilde{A} \le_C^{\Omega(2L)} \tilde{B} \implies \tilde{A} \le_C^{\Omega(3L)} \tilde{B} \implies \tilde{A} \le_C^{\Omega(4)} \tilde{B},$$
$$\tilde{A} \le_C^{\Omega(1)} \tilde{B} \implies \tilde{A} \le_C^{\Omega(2U)} \tilde{B} \implies \tilde{A} \le_C^{\Omega(3U)} \tilde{B} \implies \tilde{A} \le_C^{\Omega(4)} \tilde{B}.$$

As for other properties and results related to the fuzzy-set relations, see [3].

3 Comparison Indices Based on Possibility Theory

For $A, B \subset X$, define

$$\Pi_A(B) := \begin{cases} 1 & (A \cap B \neq \emptyset) \\ 0 & (A \cap B = \emptyset), \end{cases} \qquad N_A(B) := \begin{cases} 1 & (A \subset B) \\ 0 & (A \not\subset B). \end{cases}$$

The quantity $\Pi_A(B)$ indicates whether it possibly holds $x \in B$ or not when $x \in A$ is known, and thus $\Pi_A(\cdot)$ is called a possibility measure. The quantity $N_A(B)$ indicates whether it necessarily holds $x \in B$ or not when $x \in A$ is known, and thus $N_A(\cdot)$ is called a necessity measure. Using deformations

$$\begin{aligned} \Pi_A(B) &= \sup_{x \in X} \min \{\chi_A(x), \chi_B(x)\}, \\ N_A(B) &= \inf_{x \in X} \max \{1 - \chi_A(x), \chi_B(x)\} \end{aligned}$$

with the characteristic functions, we extend these measures to the case of fuzzy sets.

Definition 3 (Dubois, Prade [2]). Let $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$. A possibility measure $\Pi_{\tilde{A}}(\cdot)$ and a necessity measure $N_{\tilde{A}}(\cdot)$ are defined by

$$\begin{aligned} \Pi_{\tilde{A}}(\tilde{B}) &:= \sup_{x \in X} \min \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \\ N_{\tilde{A}}(\tilde{B}) &:= \inf_{x \in X} \max \{1 - \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}. \end{aligned}$$

It is clear that $\Pi_{\tilde{A}}(\tilde{B}), N_{\tilde{A}}(\tilde{B}) \in [0, 1]$. In addition, the properties below hold:

(i) $\Pi_{\tilde{A}}(\tilde{B}) = 1 - N_{\tilde{A}}(\tilde{B}^{c})$, $N_{\tilde{A}}(\tilde{B}) = 1 - \Pi_{\tilde{A}}(\tilde{B}^{c})$ (Duality);

(ii) $\tilde{B}_1 \subset \tilde{B}_2 \implies \Pi_{\tilde{A}}(\tilde{B}_1) \leq \Pi_{\tilde{A}}(\tilde{B}_2)$, $N_{\tilde{A}}(\tilde{B}_1) \leq N_{\tilde{A}}(\tilde{B}_2)$ (Monotonicity);

(iii) If $\tilde{A}$ is normal, then $N_{\tilde{A}}(\tilde{B}) \leq \Pi_{\tilde{A}}(\tilde{B})$ (Necessity implies possibility).

In the following, we give a specific procedure for constructing some fuzzy relations on $\mathcal{F}(X)$. Note that this procedure is reduced to that in [2], where four indices for comparing fuzzy numbers were originally proposed. For $x, y \in X$, let $[x, +\infty)_C := x + C$ and $(-\infty, y]_C := y - C$, i.e., $[x, +\infty)_C$ (or $(-\infty, y]_C$) denotes the set of elements greater than x (or less than y) with respect to the vector ordering $\leq_C$. We define for $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$:

(i) $[\tilde{A}, +\infty)_C^{\Pi}$: the fuzzy set of elements possibly greater than $\tilde{A}$

$$\mu_{[\tilde{A}, +\infty)_C^{\Pi}}(y) := \Pi_{\tilde{A}}((-\infty, y]_C) = \sup_{\substack{x \in X \\ x \leq_C y}} \mu_{\tilde{A}}(x), \ y \in X;$$

(ii) $[\tilde{A}, +\infty)_C^{N}$: the fuzzy set of elements necessarily greater than $\tilde{A}$

$$\mu_{[\tilde{A}, +\infty)_C^{N}}(y) := N_{\tilde{A}}((-\infty, y]_C) = \inf_{\substack{x \in X \\ x \not\leq_C y}} (1 - \mu_{\tilde{A}}(x)), \ y \in X;$$

(iii) $(-\infty, \tilde{B}]^{\Pi}_{C}$: the fuzzy set of elements possibly less than $\tilde{B}$

$$\mu_{(-\infty,\tilde{B}]^{\Pi}_{C}}(x) := \Pi_{\tilde{B}}([x, +\infty)_C) = \sup_{\substack{y \in X \\ x \le_C y}} \mu_{\tilde{B}}(y),\ x \in X;$$

(iv) $(-\infty, \tilde{B}]^{N}_{C}$: the fuzzy set of elements necessarily less than $\tilde{B}$

$$\mu_{(-\infty,\tilde{B}]^{N}_{C}}(x) := N_{\tilde{B}}([x, +\infty)_C) = \inf_{\substack{y \in X \\ x \not\le_C y}} (1 - \mu_{\tilde{B}}(y)),\ x \in X.$$

By using these interval-like fuzzy sets and the possibility and necessity measures for fuzzy sets, we consider the following eight quantities that each represent the degree of "$\tilde{A} \le \tilde{B}$" in a certain sense:

(i) the possibility of $\tilde{B}$ being possibly greater than $\tilde{A}$

$$\Pi_{\tilde{B}}([\tilde{A}, +\infty)^{\Pi}_{C}) = \sup_{\substack{x,y \in X \\ x \le_C y}} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\};$$

(ii) the necessity of $\tilde{B}$ being possibly greater than $\tilde{A}$

$$N_{\tilde{B}}([\tilde{A}, +\infty)^{\Pi}_{C}) = \inf_{y \in X} \sup_{\substack{x \in X \\ x \le_C y}} \max\{\mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(y)\};$$

(iii) the possibility of $\tilde{B}$ being necessarily greater than $\tilde{A}$

$$\Pi_{\tilde{B}}([\tilde{A}, +\infty)^{N}_{C}) = \sup_{y \in X} \inf_{\substack{x \in X \\ x \not\le_C y}} \min\{1 - \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\};$$

(iv) the necessity of $\tilde{B}$ being necessarily greater than $\tilde{A}$

$$N_{\tilde{B}}([\tilde{A}, +\infty)^{N}_{C}) = \inf_{\substack{x,y \in X \\ x \not\le_C y}} \max\{1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(y)\};$$

(v) the possibility of $\tilde{A}$ being possibly less than $\tilde{B}$

$$\Pi_{\tilde{A}}((-\infty, \tilde{B}]^{\Pi}_{C}) = \sup_{\substack{x,y \in X \\ x \le_C y}} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\};$$

(vi) the necessity of $\tilde{A}$ being possibly less than $\tilde{B}$

$$N_{\tilde{A}}((-\infty, \tilde{B}]^{\Pi}_{C}) = \inf_{x \in X} \sup_{\substack{y \in X \\ x \le_C y}} \max\{1 - \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\};$$

(vii) the possibility of $\tilde{A}$ being necessarily less than $\tilde{B}$

$$\Pi_{\tilde{A}}((-\infty, \tilde{B}]_C^N) = \sup_{x \in X} \inf_{\substack{y \in X \\ x \not\leq_C y}} \min \{\mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(y)\};$$

(viii) the necessity of $\tilde{A}$ being necessarily less than $\tilde{B}$

$$N_{\tilde{A}}((-\infty, \tilde{B}]_C^N) = \inf_{\substack{x,y \in X \\ x \not\leq_C y}} \max \{1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(y)\}.$$

Definition 4. Fuzzy relations $\precsim_C^{(i)}$ $(i = 1, 2L, 2U, 3L, 3U, 4)$ on $\mathcal{F}(X)$ are defined by

$$\begin{aligned}
\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) &:= N_{\tilde{B}}([\tilde{A}, +\infty)_C^N) = N_{\tilde{A}}((-\infty, \tilde{B}]_C^N), \\
\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) &:= \Pi_{\tilde{A}}((-\infty, \tilde{B}]_C^N), \\
\mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) &:= \Pi_{\tilde{B}}([\tilde{A}, +\infty)_C^N), \\
\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) &:= N_{\tilde{B}}([\tilde{A}, +\infty)_C^\Pi), \\
\mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) &:= N_{\tilde{A}}((-\infty, \tilde{B}]_C^\Pi), \\
\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) &:= \Pi_{\tilde{B}}([\tilde{A}, +\infty)_C^\Pi) = \Pi_{\tilde{A}}((-\infty, \tilde{B}]_C^\Pi)
\end{aligned}$$

for $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$.

The next theorem shows why we employ the same numbering as the set relations in the above definition.

Theorem 1. *Let $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$. Then, the following equalities hold:*

$$\begin{aligned}
\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) &= \sup \left\{ \alpha \in [0,1] \,\middle|\, [\tilde{A}]_{1-\alpha} \leq_C^{(1)} [\tilde{B}]_{1-\alpha} \right\}; \\
\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) &= \sup \left\{ \alpha \in [0,1] \,\middle|\, [\tilde{A}]_{\alpha} \leq_C^{(2L)} [\tilde{B}]_{1-\alpha} \right\}; \\
\mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) &= \sup \left\{ \alpha \in [0,1] \,\middle|\, [\tilde{A}]_{1-\alpha} \leq_C^{(2U)} [\tilde{B}]_{\alpha} \right\}; \\
\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) &= \sup \left\{ \alpha \in [0,1] \,\middle|\, [\tilde{A}]_{\alpha} \leq_C^{(3L)} [\tilde{B}]_{1-\alpha} \right\}; \\
\mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) &= \sup \left\{ \alpha \in [0,1] \,\middle|\, [\tilde{A}]_{1-\alpha} \leq_C^{(3U)} [\tilde{B}]_{\alpha} \right\}; \\
\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) &= \sup \left\{ \alpha \in [0,1] \,\middle|\, [\tilde{A}]_{\alpha} \leq_C^{(4)} [\tilde{B}]_{\alpha} \right\}.
\end{aligned}$$

When $\tilde{A}$ and $\tilde{B}$ are normal, from this theorem we deduce

$$\begin{aligned}
\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \leq \mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) \leq \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) \leq \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}), \\
\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \leq \mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) \leq \mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) \leq \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}).
\end{aligned}$$

4 Conclusion

In this paper, we have utilized the knowledge of possibility theory to propose six types of fuzzy relations between fuzzy sets and found that they can be characterized by well-known set relations. The obtained fuzzy relations have certain advantages in theoretical research in the sense that they are strongly connected to the set relations and hence have potential to access fruitful results in set optimization. It is expected that they will hold a position as preferable comparison indices for fuzzy sets through further investigations.

References

[1] M. Brunelli and J. Mezei, *How different are ranking methods for fuzzy numbers? A numerical study*, Internat. J. Approx. Reason. **54** (2013), 627–639.

[2] D. Dubois and H. Prade, *Ranking fuzzy numbers in the setting of possibility theory*, Inform. Sci. **30** (1983), 183–224.

[3] K. Ike and T. Tanaka, *Convex-cone-based comparisons of and difference evaluations for fuzzy sets*, Optimization **67** (2018), 1051–1066.

[4] D. Kuroiwa, T. Tanaka, and T. X. D. Ha, *On cone convexity of set-valued maps*, Nonlinear Anal. **30** (1997), 1487–1496.

[5] H. Kuwano, *A consideration about ordering indices between fuzzy sets based on possibility theory* (Japanese), RIMS Kokyuroku **1194** (2001), 67–72.

[6] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338–353.

[7] L. A. Zadeh, *Similarity relations and fuzzy orderings*, Inform. Sci. **3** (1971), 177–200.