Robust minimax optimization problems with applications

Liguo Jiao[1] and Do Sang Kim[2]

[1]Department of Mathematics
Yanbian University, Jilin Province, China
E-mail: hanchezi@163.com

[2]Department of Applied Mathematics
Pukyong National University, Republic of Korea
E-mail: dskim@pknu.ac.kr

Abstract. In this paper, we study the optimality conditions and duality in minimax programming problems in the face of data uncertainty. Following the robust optimization approach (worst-case approach), we formulate its robust counterpart of the minimax programming problems under data uncertainty. A representation of the normal cone to a convex set is established under the robust characteristic cone constraint qualification. Then, by using the obtained result, we propose the necessary condition for optimal solutions of the considered problem; moreover, a dual problem in term of Wolfe type to the primal one is stated; and weak and strong duality relations between them are explored. Finally, some of these results are applied to a robust multiobjective optimization problem.

1 Introduction and Preliminaries

We use the following notation and terminology. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. We say that a set Γ in $\mathbb{R}^n$ is *convex* whenever $\mu a_1 + (1-\mu)a_2 \in \Gamma$ for all $\mu \in [0,1]$, $a_1, a_2 \in \Gamma$. We denote the domain of f by $\operatorname{dom} f$, that is, $\operatorname{dom} f := \{x \in \mathbb{R}^n \colon f(x) < +\infty\}$. f is said to be *convex* if for all $\lambda \in [0,1]$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in \mathbb{R}^n$. The function f is said to be *concave* whenever $-f$ is convex. The (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x \rangle \leq f(y) - f(x), \ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \operatorname{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Lemma 1.1 [6] (Moreau–Rockafellar sum rule) *Consider two proper convex functions* $f_1, f_2 : \mathbb{R}^n \to \bar{\mathbb{R}}$ *such that* $\operatorname{ri} \operatorname{dom} f_1 \cap \operatorname{ri} \operatorname{dom} f_2 \neq \emptyset$. *Then*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$$

for every $x \in \operatorname{dom}(f_1 + f_2)$.

Proposition 1.1 (max-function rule) *Consider convex functions* $f_k : \mathbb{R}^n \to \mathbb{R}, k = 1, \ldots, l,$ *and let* $\varphi(x) = \max\{f_1(x), \ldots, f_l(x)\}$. *then*

$$\partial\varphi(\bar{x}) = \operatorname{conv} \bigcup_{k \in \mathbb{K}(\bar{x})} \partial f_k(\bar{x}),$$

where $\mathbb{K}(\bar{x}) := \{k \in \mathbb{K} := \{1, \ldots, l\} : \varphi(\bar{x}) = f_k(\bar{x})\}$ *denotes the active index set.*

2 Main Results

A standard form of minimax programming problem is the problem:

$$\text{(P)} \quad \min_{x \in \mathbb{R}^n} \max_{k \in \mathbb{K}} f_k(x)$$
$$\text{s.t.} \quad g_i(x) \leqq 0, \; i = 1, \ldots, m,$$

where $f_k,\ g_i : \mathbb{R}^n \to \mathbb{R},\ k \in \mathbb{K} := \{1, \ldots, l\},\ i = 1, \ldots, m$ are convex functions.

The minimax programming problem (P) in the face of data uncertainty in the constraints can be captured by the problem

$$\text{(UP)} \quad \min_{x \in \mathbb{R}^n} \max_{k \in \mathbb{K}} \quad f_k(x)$$
$$\text{s.t.} \quad g_i(x, v_i) \leqq 0, \; i = 1, \ldots, m,$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$. The problem (UP) is to optimize convex optimization problems with data uncertainty (incomplete data), which means that input parameters of these problems are not known exactly at the time when solution has to be determined [1]. Actually there are two main approaches to deal with constrained optimization under data uncertainty, namely *robust programming approach* and *stochastic programming approach*; in stochastic programming, one works with the probabilistic distribution of uncertainty and the constraints are required to be satisfied up to prescribed level of probability [3], while robust programming approach seeks for a solution which simultaneously satisfies all possible realizations of the constraints. It seems to be more convenient to use the robust approach to study optimization problems with data uncertainty, comparing with stochastic programming approach.

In the present paper we explore optimality and duality theorems for the uncertain minimax programming problem (UP) by examining its robust (worst-case) counterpart [1]:

$$\text{(RP)} \quad \min_{x \in \mathbb{R}^n} \max_{k \in \mathbb{K}} \quad f_k(x)$$
$$\text{s.t.} \quad g_i(x, v_i) \leqq 0, \; \forall v_i \in \mathcal{V}_i, \; i = 1, \ldots, m.$$

Denote by $F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leqq 0, \; \forall v_i \in \mathcal{V}_i, \; i = 1, \ldots, m\}$ as the feasible set of (RP).

Definition 2.1 [4] We say the robust characteristic cone constraint qualification (CQ) holds if the cone

$$\bigcup_{v_i\in\mathcal{V}_i,\lambda_i\geqq 0} \operatorname{epi}\,(\sum_{i=1}^{m}\lambda_i g_i(\cdot,v_i))^*$$

is closed and convex.

Remark 2.1 One may see [4] for some properties on the cone

$$\bigcup_{v_i\in\mathcal{V}_i,\lambda_i\geqq 0} \operatorname{epi}\,(\sum_{i=1}^{m}\lambda_i g_i(\cdot,v_i))^*.$$

We establish approximate optimality theorem for (RP) under the (CQ) condition. Moreover, we formulate a Wolfe type dual problem for the primal one; and propose weak duality between the primal problem and its Wolfe type dual problem as well as strong duality which holds under the condition (CQ). We also give an example to illustrate the obtained results.

Definition 2.2 Let $\varphi(x) := \max_{k\in\mathbb{K}} f_k(x),\ x\in\mathbb{R}^n$. A point $\bar{x}\in F$ is said to be an *optimal solution* of (RP) if and only if

$$\varphi(\bar{x}) \leqq \varphi(x) \quad \forall x\in F.$$

2.1 Representation of the Normal Cone

In order to obtain Karush–Kuhn–Tucker (KKT) optimality condition in terms of the constraint functions $g_i(x,v_i)\leqq 0,\ \forall v_i\in\mathcal{V}_i,\ i=1,\ldots,m$, the *normal cone* must be explicitly expressed in their terms. Below, we present such a result under the (CQ) condition.

Lemma 2.1 *Let $\bar{x}\in C := \{x\in\mathbb{R}^n\colon g(\cdot,v)\leqq 0,\ \forall v\in\mathcal{V}\}$, where $\mathcal{V}$ is a certain convex compact uncertainty subset in $\mathbb{R}^q$. Suppose that the* (CQ) condition *holds. Then $\xi\in N_C(\bar{x})$ if and only if there exist $\bar{\lambda}\geqq 0$ and $\bar{v}\in\mathcal{V}$ such that*

$$\xi\in\bar{\lambda}\partial g(\bar{x},\bar{v}) \ \ \textit{and}\ \ \bar{\lambda}g(\bar{x},\bar{v})=0.$$

Corollary 2.1 *Let $\bar{x}\in C := \{x\in\mathbb{R}^n\colon g(\cdot,v)\leqq 0, v\in\mathcal{V}\}$, where $\mathcal{V}$ is a certain convex compact uncertainty subset in $\mathbb{R}^q$. Suppose that the Slater constraint qualification holds, that is, there exists $x_0\in\mathbb{R}^n$ such that $g(x_0,v)<0$, for all $v\in\mathcal{V}$. Then $\xi\in N_C(\bar{x})$ if and only if there exist $\bar{\lambda}\geqq 0$ and $\bar{v}\in\mathcal{V}$ such that*

$$\xi\in\bar{\lambda}\partial g(\bar{x},\bar{v}) \ \ \textit{and}\ \ \bar{\lambda}g(\bar{x},\bar{v})=0.$$

2.2 Optimality Conditions

The following theorem gives a KKT necessary condition for optimal solutions of the problem (RP).

Theorem 2.1 *Consider the problem* (RP), *assume that the* (CQ) condition *holds. If $\bar{x}$ is an optimal solution of the problem* (RP), *then there exist* $\tau := (\tau_1, \dots, \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ *and* $\lambda := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_+$, *such that*

$$
\begin{aligned}
&0 \in \sum_{k \in \mathbb{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i \in M} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}),\\
&\tau_k\big(f_k(\bar{x}) - \max_{k \in \mathbb{K}} f_k(\bar{x})\big) = 0, \quad k \in \mathbb{K},\\
&\lambda_i g_i(\bar{x}) = 0, \quad i = 1, \dots, m.
\end{aligned}
$$

Corollary 2.2 *Consider the problem* (RP), *assume that the Slater constraint qualification holds, that is, there exists* $x_0 \in \mathbb{R}^n$ *such that* $g_i(x_0, v_i) < 0$, *for all* $v_i \in \mathcal{V}_i$, $i = 1, \dots, m$. *If $\bar{x}$ is an optimal solution of the problem* (RP), *then there exist* $\tau := (\tau_1, \dots, \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ *and* $\lambda := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_+$, *such that*

$$
\begin{aligned}
&0 \in \sum_{k \in \mathbb{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i \in M} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}),\\
&\tau_k\big(f_k(\bar{x}) - \max_{k \in \mathbb{K}} f_k(\bar{x})\big) = 0, \quad k \in \mathbb{K},\\
&\lambda_i g_i(\bar{x}) = 0, \quad i = 1, \dots, m.
\end{aligned}
$$

2.3 Duality Relations

In this section we formulate a dual problem to the primal one in the sense of Wolfe [7], and explore weak and strong duality relations between them.

In connection with the robust minimax programming problem (RP), denote $\varphi(y) := \max_{k \in \mathbb{K}} f_k(y)$, we consider a dual problem in the following form:

$$
\begin{aligned}
(\mathrm{RD})_{\mathrm{W}} \qquad & \text{Maximize}_{(y, \tau v, \lambda)} \quad && \varphi(y) + \sum_{i \in M} \lambda_i g_i(y, v_i)\\
& \text{subject to} && 0 \in \sum_{k \in \mathbb{K}} \tau_k \partial f_k(y) + \sum_{i \in M} \lambda_i \partial g_i(\cdot, v_i)(y)\\
& && \tau_k(f_k(y) - \varphi(y)) = 0, \ k \in \mathbb{K}\\
& && \tau_k \geqq 0, \ \sum_{k \in \mathbb{K}} \tau_k = 1\\
& && \lambda_i \geqq 0, \ v_i \in \mathcal{V}_i, \ i \in M.
\end{aligned}
$$

Let $F_D = \{(y,\tau,v,\lambda) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \mathcal{V} \times \mathbb{R}^m_+ \colon 0 \in \sum_{k\in\mathbb{K}} \tau_k \partial f_k(y) + \sum_{i\in M} \lambda_i \partial g_i(\cdot, v_i)(y),$ $\tau_k(f_k(y) - \varphi(y)) = 0, k \in \mathbb{K}, \tau_k \geqq 0, \sum_{k\in\mathbb{K}} \tau_k = 1, \lambda_i \geqq 0, v_i \in \mathcal{V}_i, i \in M\}$ be the feasible set of $(\text{RD})_{\text{W}}$. We should note that a point $(\bar{y},\bar{\tau},\bar{v},\bar{\lambda}) \in F_D$ is called an *optimal solution* of problems $(\text{RD})_{\text{W}}$ if for all $(y,\tau,v,\lambda) \in F_D$,

$$\varphi(\bar{y}) + \sum_{i\in M} \bar{\lambda}_i g_i(\bar{y},\bar{v}_i) \geqq \varphi(y) + \sum_{i\in M} \lambda_i g_i(y,v_i).$$

The following theorem describes a weak duality relation between the primal problem (RP) and the dual problem $(\text{RD})_{\text{W}}$.

Theorem 2.2 (Weak duality) *For any feasible solution x of* (RP) *and any feasible solution (y,τ,v,λ) of* $(\text{RD})_{\text{W}}$,

$$\varphi(x) \geqq \varphi(y) + \sum_{i\in M} \lambda_i g_i(y,v_i).$$

A strong duality relation between the primal problem (RP) and the dual problem $(\text{RD})_{\text{W}}$ is given as follows.

Theorem 2.3 (Strong duality) *Let $\bar{x} \in F$ be an optimal solution of the robust problem* (RP) *such that the* (CQ) condition *is satisfied at this point. Then there exists $(\bar{\tau},\bar{v},\bar{\lambda}) \in \mathbb{R}^l_+ \times \mathbb{R}^q \times \mathbb{R}^m_+$ such that $(\bar{x},\bar{\tau},\bar{v},\bar{\lambda}) \in F_D$ is an optimal solution of problem* $(\text{RD})_{\text{W}}$.

Here comes an example to illustrate our duality results. Note that this example is modified by [5, Example 2].

Example 2.1 *Consider the following minimax optimization problem with uncertainty:*

$$(\text{RP})^1 \qquad \min_{(x_1,x_2)\in\mathbb{R}^2} \max_{k\in\{1,2\}} \{f_1(x_1,x_2), f_2(x_1,x_2)\}$$
$$s.t. \quad x_1^2 - 2v_1x_1 \leqq 0, \ v_1 \in [-1,1].$$

Let $f_1(x_1,x_2) = x_1 + x_2^2$, $f_2(x_1,x_2) = -x_1 + x_2^2$ and $g_1((x_1,x_2),v_1) = x_1^2 - 2v_1x_1$. Then the feasible set of $(\text{RP})^1$ *is $F^1 := \{(x_1,x_2) \in \mathbb{R}^2 \colon x_1^2 - 2v_1x_1 \leqq 0, \ v_1 \in [-1,1]\} = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = 0, \ x_2 \in \mathbb{R}\}$, and $\{(0,0)\}$ is the set of optimal solutions of* $(\text{RP})^1$.

Clearly, the rm Slater condition goes awry for $(\text{RP})^1$. *However*

$$\bigcup_{v_1\in[-1,1], \ \lambda_1\geqq 0} \text{epi}\,(\lambda_1 g_1(\cdot,v_1))^*$$

is closed and convex whereas the Slater condition *fails* (*one may refer to* [5]).

Now, we formulate a robust dual problem $(\mathrm{RD})^1_{\mathrm{W}}$ *for* $(\mathrm{RP})^1$ *as follows:*

$$
\begin{array}{lll}
(\mathrm{RD})^1_W & \max\limits_{(y,\tau,v,\lambda)} & \varphi(y_1,y_2)+\lambda_1 g_1\big((y_1,y_2),v_1\big) \\
& s.t. & 0\in \tau_1\partial f_1(y_1,y_2)+\tau_2\partial f_2(y_1,y_2)+\lambda_1\partial g_1(\cdot,v_1)(y_1,y_2) \\
& & \tau_1\big(f_1(y_1,y_2)-\varphi(y_1,y_2)\big)=0 \\
& & \tau_2\big(f_2(y_1,y_2)-\varphi(y_1,y_2)\big)=0 \\
& & \tau_1\geqq 0,\ \tau_2\geqq 0,\ \tau_1+\tau_2=1,\ \lambda_1\geqq 0,\ v_1\in[-1,1],
\end{array}
$$

where $\varphi(y_1,y_2)=\max\limits_{\{1,2\}}\{f_1(y_1,y_2),f_2(y_1,y_2)\}$.

By calculation, we have the set of all feasible solutions of $(\mathrm{RD})^1_{\mathrm{W}}$ *is* $F^1_D := \{((0,0),(\frac{1+2\lambda_1 v_1}{2},\frac{1-2\lambda_1 v_1}{2}),v_1,\lambda_1) : \lambda_1\in[0,\frac{1}{2}], v_1\in[-1,1]\}$. *It is not difficult to see the validness of* Theorem 2.2 (Weak duality) *and* Theorem 2.3 (Strong duality).

3 An application to robust multiobjective optimization problems

Chuong and Kim [2] employed the results of nondifferentiable minimax programming problems to study a multiobjective optimization problem, In this section, we apply the results of the robust mimimax programming problem to a robust multiobjective optimization problem. More precisely, we employ the necessary optimality conditions obtained for the robust mimimax programming problem in the previous section to derive the corresponding ones for a multiobjective optimization problem. (One can deal similarly with duality relations in this way.)

We consider the following constrained *multiobjective robust optimization problem*:

$$
\mathrm{Min}_{\mathbb{R}^l_+}\ \{f(x)\,|\,x\in F\}, \tag{RMP}
$$

where the feasible set F is same to the feasible set of (RP), and $\mathbb{R}^l_+$ denotes the nonnegative orthant of $\mathbb{R}^l$.

Note that "$\mathrm{Min}_{\mathbb{R}^l_+}$" in the above problem is understood with respect to the ordering cone $\mathbb{R}^l_+$.

Definition 3.1 A point $\bar{x}\in F$ is a *weakly Pareto solution* of problem (RMP) if and only if

$$
f(x)-f(\bar{x})\notin -\mathrm{int}\,\mathbb{R}^l_+\quad \forall x\in F,
$$

where $\mathrm{int}\,\mathbb{R}^l_+$ stands for the topological interior of $\mathbb{R}^l_+$.

The following result is a Karush–Kuhn–Tucker (KKT) necessary condition for weakly Pareto solutions of problem (RMP).

Theorem 3.1 *Let the* (CQ) condition *be satisfied at* $\bar{x} \in F$. *If* $\bar{x}$ *is a weakly Pareto solution of the problem* (RMP), *then there exist* $\tau := (\tau_1, ..., \tau_l) \in \mathbb{R}^l_+ \setminus \{0\}, \lambda := (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+$, *and* $v := (v_1, ..., v_l) \in \mathbb{R}^m_+$ *such that*

$$0 \in \sum_{k \in \mathbb{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i \in M} \lambda_i \partial g_i(\bar{x}) \text{ and } \lambda_i g_i(\bar{x}) = 0,\ i \in I.$$

References

[1] Ben-Tal, A., Nemirovski, A.: A selected topics in robust convex optimization, Math. Program., Ser B, **112**, 125–158 (2008)

[2] Chuong, T.D., Kim, D.S.: Nondifferentiable minimax programming problems with applications, Ann. Oper. Res., **251**, 73–87 (2017)

[3] Houda, M.: Comparison of approximations in stochastic and robust optimization programs, In: Hušková, M., Janžura, M. (eds.) Prague Stochastics 2006, pp. 418–425. Prague, Matfyzpress (2006)

[4] Jeyakumar V., Li, G.Y.: Strong duality in robust convex programming: complete characterizations, SIAM J. Optim., **20**, 3384–3407 (2010)

[5] Lee, J.H., Jiao, L.G.: On quasi ϵ-solution for robust convex optimization problems. Optim. Lett., **11**, 1609–1622 (2017)

[6] Rockafellar, R.T.: Convex Analysis, Princeton Univ. Press, Princeton, N. J. (1970)

[7] Wolfe, P.: A duality theorem for nonlinear programming, Quart. Appl. Math., **19**, 239–244 (1961)