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MEASUREMENT OF THE DIFFERENCE OF TWO TYPES ORTHOGONALITY IN RADON PLANES

HIROYASU MIZUGUCHI

ABSTRACT. The notion of orthogonality for vectors in inner product spaces is simple, interesting and fruitful. When moving to normed spaces, we have many possibilities to extend this notion. Recently the constants which measure the difference between these orthogonalities have been investigated. The usual orthognality in inner product spaces is symmetric. However, Birkhoff orthogonality in normed spaces is not symmetric in general. The norm plane in which Birkhoff orthogonality is symmetric is called a Radon plane. We consider the difference between Birkhoff and isosceles orthogonalities in Radon planes.

1. INTRODUCTION

In case of that X is an inner product space, an element $x \in X$ is said to be orthogonal to $y \in X$ (denoted by $x \perp y$) if the inner product $\langle x, y\rangle$ is zero. This orthogonality notion is very interesting, fruitful and have been investigated by a lot of mathematicians. In the general setting of normed spaces, many notions of orthogonality have been introduced by means of equivalent propositions to the usual orthogonality in inner product spaces: x is said to be Birkhoff orthogonal to y (denoted by $x \perp_B y$) if $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$, and said to be isosceles orthogonal to y (denoted by $x \perp_I y$) if $\|x + y\| = \|x - y\|$. These generalized orthogonality types have been studied in a lot of papers ([1] etc.).

These orthogonality relations coincide with each other in inner product spaces. However, they are different notions in general and depend

(Hiroyasu Mizuguchi) STUDENT AFFAIRS DEPARTMENT-SHINNARASHINO EDUCATION AFFAIRS SECTION, CHIBA INSTITUTE OF TECHNOLOGY

E-mail address: hiroyasu.mizuguchi@p.chibakoudai.jp.

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on the norm of the space. Recently, quantitative studies of the difference between two orthogonality types have been performed ([7, 10, 12]):

$$D(X) = \inf\left\{\inf_{\lambda\in\mathbb{R}} \|x+\lambda y\| : x, y \in S_X, x \perp_I y\right\},$$

$$D'(X) = \sup\{\|x+y\| - \|x-y\| : x, y \in S_X, x \perp_B y\},$$

$$BR(X) = \sup\left\{\frac{\|x+y\| - \|x-y\|}{\|y\|} : x, y \in X, x, y \neq 0, x \perp_B y\right\},$$

$$BI(X) = \sup\left\{\frac{\|x+y\| - \|x-y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_B y\right\},$$

$$IB(X) = \inf\left\{\frac{\inf_{\lambda\in\mathbb{R}} \|x+\lambda y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_I y\right\}.$$

Here we treat the constant $IB(X)$. The inequality $1/2 \leq IB(X) \leq 1$ holds for any normed space X and the equality $IB(X) = 1$ is equivalent to that the space X has inner product ([10]).

An orthogonality notion"$\perp$ " is called *symmetric* if $x \perp y$ implies $y \perp x$. The usual orthogonality in inner product spaces is, of course symmetric. By the definition, isosceles orthogonality in normed spaces is symmetric, too. However Birkhoff orthogonality is not symmetric in general. Birkhoff [3] proved that if Birkhoff orthogonality is symmetric in a strictly convex normed space whose dimension is at least three, then the space is an inner product space. Day [4] and James [6] showed that the assumption of strict convexity in Birkhoff's result can be released.

Theorem 1 ([4, 6])**.** *A normed space X whose dimension is at least three is an inner product space if and only if Birkhoff orthogonality is symmetric in X.*

The assumption of the dimension of the space in the above theorem cannot be omitted. A two-dimensional normed space in which Birkhoff orthogonality is symmetric is called a Radon plane.

Remark 2. *A Radon plane is made by connecting the unit spheres of a two-dimensional normed space and its dual* ([4, 8, 9])*. Thus, by absolute normalized norms and associated convex functions innumerable Radon planes can be considered.*

We consider the constant $IB(X)$ in Radon planes.

2. Result

To consider the difference between Birkhoff and isosceles orthogonalities, the results obtained by James in [5] are important.

Proposition 3 ([5]).

(i) *If* x $(\neq 0)$ *and* y *are isosceles orthogonal elements in a normed space, then* $\|x+ky\| > \frac{1}{2}\|x\|$ *for all* k.
(ii) *If* x $(\neq 0)$ *and* y *are isosceles orthogonal elements in a normed space, and* $\|y\| \leq \|x\|$, *then* $\|x+ky\| \geq 2(\sqrt{2}-1)\|x\|$ *for all* k.

From this, one can has $1/2 \leq IB(X) \leq 1$ and $2(\sqrt{2}-1) \leq D(X) \leq 1$ for any normed space.

For two elements x, y in the unit sphere in a normed space X, the sine function $s(x,y)$ is defined by

$$s(x,y) = \inf_{t\in\mathbb{R}} \|x+ty\|$$

([13]). V. Balestro, H. Martini, and R. Teixeira [2] showed the following

Proposition 4 ([2]). *A two dimensional normed space* X *is a Radon plane if and only if its associated sine function is symmetric.*

Thus for elements x, y in the unit sphere in a Radon plane X with $x \perp_I y$ we have $\inf_{\lambda\in\mathbb{R}} \|x+\lambda y\| = \inf_{\mu\in\mathbb{R}} \|y+\mu x\|$. Hence the inequality $2(\sqrt{2}-1) \leq IB(X) \leq 1$ holds for a Radon plane X.

Using Proposition 4 again, we start to consider the lower bound of $IB(X)$ in a Radon plane.

Proposition 5. *Let* X *be a Radon plane. Then*

$$IB(X) = \left\{ \|x+ky\| : \begin{array}{c} x,y \in S_X,\ \alpha \in [0,1],\ x \perp_B y, \\ k \in [0, \min\{1/2, \alpha\}],\ l \in [0,1/2] \\ \|x+ky\| = \min_{\lambda\in\mathbb{R}} \|x+\lambda y\| \\ = \min_{\mu\in\mathbb{R}} \|y+\mu x\| = \|y+lx\| \end{array} \right\}.$$

Proof. Suppose that $x \perp_I \alpha y$ for elements $x, y \in S_X$ and a number $\alpha \in \mathbb{R}$. Since $x \perp_I \alpha y$ implies $x \perp_I -\alpha y$ and $y \perp_I x/\alpha$, we can suppose $0 \leq \alpha \leq 1$. From the assumption there exists real numbers k, l satisfying $\|x+ky\| = \min_{\lambda\in\mathbb{R}} \|x+\lambda y\| = \min_{\mu\in\mathbb{R}} \|y+\mu x\| = \|y+lx\|$. In addition, the sign of k and l coincide. Thus we may assume $0 \leq k$ and $0 \leq l$. From the facts $x \perp_I \alpha y$ and $\|x+ky\| = \min_{\lambda\in\mathbb{R}} \|x+\lambda y\|$, we also have $k \leq \alpha$.

The assumption $\|x+ky\| = \min_{\lambda\in\mathbb{R}} \|x+\lambda y\|$ implies that $x+ky$ is Birkhoff orthogonal to y. From the symmetry of Birkhoff orthogonality in a Radon plane, y is Birkhoff orthogonal to $x + ky$. Using this fact, one has

$$\begin{aligned} \alpha + k \leq \|x + ky - (\alpha + k)y\| = \|x - \alpha y\| &= \|x + \alpha y\| \\ &= \|x + ky + (\alpha - k)y\| \\ &\leq \|x + ky\| + \alpha - k \end{aligned}$$

and hence $2k \leq \|x + ky\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\| \leq 1$.

In a similar way, from the fact that x is Birkhoff orthogonal to $y + lx$, we have $2l \leq \|y + lx\| \leq 1$. □

Proposition 6. *Let X be a Radon plane, an element $x \in S_X$ be isosceles orthogonality to αy for another element $y \in S_X$ and a number $\alpha \in [0, 1]$. Take numbers $k \in [0, \min\{1/2, \alpha\}]$ and $l \in [0, 1/2]$ such that $\|x + ky\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\| = \min_{\mu \in \mathbb{R}} \|y + \mu x\| = \|y + lx\|$. Then*

$$\|x + ky\| \geq \max \left\{ \frac{(\alpha + k)(1 - kl)}{(\alpha + k)(1 - kl) + k(1 - l)(\alpha - k)}, \frac{(1 + \alpha l)(1 - kl)}{(1 + \alpha l)(1 - kl) + l(1 - k)(1 - \alpha l)} \right\}.$$

Proof. It follows from $x = \{\alpha(x + ky) + k(x - \alpha y)\}/(\alpha + k)$ and $x \perp_I \alpha y$ that $\alpha + k \leq \alpha\|x + ky\| + k\|x + \alpha y\|$. For

$$c = \frac{\alpha - k}{1 + \alpha - k - \alpha l} \quad \text{and} \quad d = \frac{1 - kl}{1 + \alpha - k - \alpha l},$$

the equality $d(x + \alpha y) = (1 - c)(x + ky) + c(y + lx)$ holds, and hence one has $\|x + \alpha y\| \leq \|x + ky\|/d = (1 + \alpha - k - \alpha l)(1 - kl)^{-1}\|x + ky\|$. Thus, we obtain

$$\alpha + k \leq \frac{(\alpha + k)(1 - kl) + k(1 - l)(\alpha - k)}{1 - kl} \|x + ky\|.$$

Meanwhile, from the equality $y = \{l(-x + \alpha y) + y + lx\}/(1 + \alpha l)$, we obtain

$$1 + \alpha l \leq \frac{(1 + \alpha l)(1 - kl) + l(1 - k)(1 - \alpha l)}{1 - kl} \|x + ky\|.$$

□

Let

$$F(\alpha, k, l) = \frac{k(1 - l)(\alpha - k)}{(\alpha + k)(1 - kl)} \quad \text{and} \quad G(\alpha, k, l) = \frac{l(1 - k)(1 - \alpha l)}{(1 + \alpha l)(1 - kl)}.$$

Then from the above proposition, the inequality

$$(\|x + ky\|)^{-1} \leq 1 + \min\{F(\alpha, k, l), G(\alpha, k, l)\}$$

holds.

Let us consider the upper bound of $\min\{F(\alpha,k,l),G(\alpha,k,l)\}$.

Lemma 7. *Let* $0 \le \alpha \le 1$, $0 \le k \le \min\{\alpha, 1/2\}$ *and* $k \le l \le 1/2$. *Then*

$$\min\{F(\alpha,k,l),G(\alpha,k,l)\} = F(\alpha,k,l) \le \frac{k(1-k)}{(1+k)^2}.$$

In case of $l < k$, considering

$$\frac{(1-k)lF(\alpha,k,l) + (1-l)kG(\alpha,k,l)}{(1-k)l + (1-l)k}$$

which is greater than $\min\{F(\alpha,k,l),G(\alpha,k,l)\}$, we obtain the followings.

Lemma 8. *Let* $0 \le \alpha \le 1$, $0 \le k \le \min\{\alpha, 1/3\}$ *and* $0 \le l < k$. *Then*

$$\min\{F(\alpha,k,l),G(\alpha,k,l)\} \le \frac{k(1-k)}{(1+k)^2}.$$

Lemma 9. *Let* $0 \le \alpha \le 1$, $1/3 < k \le \min\{\alpha, 1/2\}$ *and* $0 \le l < k$. *Then*

$$\min\{F(\alpha,k,l),G(\alpha,k,l)\} \le \frac{2k(1-k)\{\sqrt{2(1-k)} - \sqrt{k}\}^2}{(1+k)\{\sqrt{2}(1-2k) + \sqrt{k(1-k)}\}^2}.$$

Both $k(1-k)/(1+k)^2$ and

$$\frac{2k(1-k)\{\sqrt{2(1-k)} - \sqrt{k}\}^2}{(1+k)\{\sqrt{2}(1-2k) + \sqrt{k(1-k)}\}^2}$$

take maximum $1/8$ at $k = 1/3$. Therefore

Theorem 10. *Let* X *be a Radon plane. Then* $8/9 \le IB(X) \le 1$.

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