THE FIXED POINT PROPERTY OF A-DIRECT SUMS OF N UNIFORMLY NON-SQUARE BANACH SPACES

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Abstract
We shall show the fixed point property of A-direct sums of N uniformly non-square Banach spaces by characterizing the nontrivialness of Dominguez-Benavides coefficient $R(1, (X_1 \oplus \cdots \oplus X_N)_A)$, that is, $R(1, (X_1 \oplus \cdots \oplus X_N)_A) < 2$.

A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called *monotone* if $\|\boldsymbol{a}\| \leq \|\boldsymbol{b}\|$ for all $\boldsymbol{a} = (a_j), \boldsymbol{b} = (b_j) \in \mathbb{R}^N$ with $|a_j| \leq |b_j|$ $(j = 1, \ldots, N)$. For $\boldsymbol{a} = (a_j)$ $|\boldsymbol{a}|$ is defined by $|\boldsymbol{a}| = (|a_j|) \in \mathbb{R}^N$. A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called *absolute* if $\|\boldsymbol{a}\| = \||\boldsymbol{a}|\|$ for all $\boldsymbol{a} \in \mathbb{R}^N$ and *normalized* if $\|e_j\| = 1$ for all $1 \leq j \leq N$, where e_j is the j-th unit vector in $\mathbb{R}^N$.

In [4] and [10] A-direct sums and AN-direct sums of N Banach spaces were introduced respectively by the following: Let $\|\cdot\|_A$ be an *arbitrary* norm on $\mathbb{R}^N$. The *A-direct sum* $(X_1 \oplus \cdots \oplus X_N)_A$ is the direct sum of $X_1, \ldots, X_N$ equipped with the norm

$$\|(x_1, \ldots, x_N)\|_A = \|(\|x_1\|, \ldots, \|x_N\|)\|_A, \ (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

and an *AN-direct sum* is an A-direct sum whose norm is defined from some absolute noramlized norm $\|\cdot\|_{AN}$ on $\mathbb{R}^N$. It is known that a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ is aboslute if and only if it is monotone ([2],[4],[14]).

In [13] Z-direct sums were introduced by the following: Let $\|\cdot\|_Z$ be an *monotone* norm on $\mathbb{R}^N_+$. The *Z-direct sum* $(X_1 \oplus \cdots \oplus X_N)_Z$ is the direct sum of $X_1, \ldots, X_N$ equipped with the norm

$$\|(x_1, \ldots, x_N)\|_Z = \|(\|x_1\|, \ldots, \|x_N\|)\|_Z, \ (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

Then we see that Z-direct sum and AN-direct sum are A-direct sum. Since an A-direct sum is isometric isomorphic to some AN-direct sum ([4]), then we have the following theorem.

Theorem 1 (cf. [4]). *Let $X_1, \ldots, X_N$ be Banach spaces. Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then the norm of $(X_1 \oplus \cdots \oplus X_N)_A$ is monotone, that is,*

$$\|(x_1, \ldots, x_N)\|_A \leq \|(y_1, \ldots, y_N)\|_A$$

for $(x_1, \dots, x_N), (y_1, \dots, y_N) \in (X_1 \oplus \dots \oplus X_N)_A$ *with* $\|x_j\| \le \|y_j\|$ $(j = 1, \dots, N)$.

As usual S_X and B_X stand for the unit sphere and the closed unit ball of X, respectively. A Banach space X is said to have the *fixed point property* (resp. *weak fixed point property*) for nonexpansive mappings if every nonexpansive self-mapping T of any nonempty bounded closed (resp. weakly compact) convex subset C of X has a fixed point (T is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$). In [6] the coefficient $R(a, X)$ called as Dominguez-Benavides coefficient(cf. [3]) was introduced by the following: For $0 \le a \le 1$ let

$$R(a, X) = \sup \left\{ \liminf_{n\to\infty} \|x_n + x\| \right\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le a$ and all weakly null sequences $\{x_n\}_n$ in the unit ball of X such that

$$\lim_{n,m\to\infty; n\ne m} \|x_n - x_m\| \le 1.$$

In this paper we shall show the fixed point property for nonexpansive mappings of A-direct sums of N uniformly non-square Banach spaces by characterizing the nontrivialness of Dominguez-Benavides coefficient R(1, $(X_1 \oplus \dots \oplus X_N)_A$), that is, R(1, $(X_1 \oplus \dots \oplus X_N)_A) < 2$.

The following theorem was proved in [6].

Theorem 2 ([6]). *Let X be a Banach space. If $R(a, X) < 1 + a$ for some $a > 0$, then X has the weak fixed point property for nonexpansive mappings.*

A Banach space X is called *uniformly non-square* ([9]) if there exists a constant $\varepsilon > 0$ such that

$$\min\{\|x + y\|, \|x - y\|\} \le 2(1 - \varepsilon) \quad \text{for all } x, y \in S_X.$$

By Theorem 2 García-Falset et. al. [8] obtained the following remarkable result.

Theorem 3 ([8]). *Let X be a uniformly non-square Banach space. Then $R(1, X) < 2$ and hence X has the fixed point property for nonexpansive mappings.*

In [7] the following notions were introduced.

Definition 4 ([7]). For $\boldsymbol{a} = (a_j) \in \mathbb{R}^N$ let supp $\boldsymbol{a} = \{j : a_j \ne 0\}$.

(i) A norm $\|\cdot\|$ on $\mathbb{R}^N$ is said to have *Property* T_1^N if

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \frac{1}{2}\|\boldsymbol{a} + \boldsymbol{b}\| = 1, \ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N \implies \text{supp } \boldsymbol{a} \cap \text{supp } \boldsymbol{b} \ne \emptyset.$$

(ii) A norm $\|\cdot\|$ on $\mathbb{R}^N$ is said to have *Property* T_∞^N if

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{a} + \boldsymbol{b}\| = 1 \implies \text{supp } \boldsymbol{a} \cap \text{supp } \boldsymbol{b} \ne \emptyset.$$

To show our key result we need the following propositions.

Proposition 5 ([11]). *Let $\{x_n^{(k)}\}_{n,k}$, $\{y_n^{(k)}\}_{n,k}$ be double sequences with nonzero terms in a Banach space X such that*

$$\lim_{k\to\infty}\lim_{n\to\infty}\|x_n^{(k)}\|>0,\ \lim_{k\to\infty}\lim_{n\to\infty}\|y_n^{(k)}\|>0.$$

Then the following are equivalent.

(i) $\lim_{k\to\infty}\liminf_{n\to\infty}\|x_n^{(k)}+y_n^{(k)}\|=\lim_{k\to\infty}\lim_{n\to\infty}(\|x_n^{(k)}\|+\|y_n^{(k)}\|)$.

(ii) $\lim_{k\to\infty}\liminf_{n\to\infty}\left\|\dfrac{x_n^{(k)}}{\|x_n^{(k)}\|}+\dfrac{y_n^{(k)}}{\|y_n^{(k)}\|}\right\|=2.$

Proposition 6 ([5]; see also [1, Chpter III, Theorem 1.5]). *Let $\{x_n\}$ be a bounded sequence in a Banach space X. Then $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that $\lim_{k,l\to\infty;k\neq l}\|x_{n_k}-x_{n_l}\|$ exists.*

Proposition 7 ([16]). *Let $\{x_n\}$ be a weakly null sequence in a Banach space X. Assume that $\lim_{n,m\to\infty;n\neq m}\|x_n-x_m\|$ exists. Then*

$$\limsup_{n\to\infty}\|x_n\|\leq\lim_{n,m\to\infty;n\neq m}\|x_n-x_m\|.$$

Proposition 8 ([12]). *Let $\boldsymbol{a}=(a_j),\boldsymbol{b}=(b_j)\in\mathbb{R}^N$ and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ be monotone. If $\|\boldsymbol{a}\|=\|\boldsymbol{b}\|$, $|a_j|\leq|b_j|$ $(j=1,\dots,N)$ and $|a_{j_0}|<|b_{j_0}|$ then $\|(\chi_{\boldsymbol{N}\setminus\{j_0\}}(j)a_j)\|=\|(b_j)\|$, where $\boldsymbol{N}=\{1,\dots,N\}$ and $\chi_{N\setminus\{j_0\}}$ is the characteristic function of $\boldsymbol{N}\setminus\{j_0\}$.*

By Theorem 1, Propositions 5, 6, 7 and 8 we can prove the following key result.

Theorem 9. *Let $X_1,\dots,X_N$ be Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property T_1^N. Then $R(1,(X_1\oplus\cdots\oplus X_N)_A)<2$ if and only if $R(1,X_j)<2$ for all $1\leq j\leq N$*

Corollary 10 (cf. [15]). *Let X and Y be Banach spaces and let a norm on $\mathbb{R}^2$ be not ℓ_1-norm. Then $R(1,(X\oplus_A Y))<2$ if and only if $R(1,X)<2$ and $R(1,Y)<2$.*

Theorem 9 combined with Theorem 3 yields nontrivialness of Dominguez-Benavides coefficient of A-direct sums of N uniformly non-square Banach spaces.

Theorem 11. *Let $X_1,\dots,X_N$ be uniformly non-square Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property T_1^N. Then $R(1,(X_1\oplus\cdots\oplus X_N)_A)<2$.*

By Theorem 11 and Theorem 2 we obtain our main results.

Theorem 12. *Let $X_1,\dots,X_N$ be uniformly non-square Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property T_1^N. Then $(X_1\oplus\cdots\oplus X_N)_A$ has the fixed point property for nonexpansive mappings.*

Theorem 13. *Let $X_1, \dots, X_N$ be uniformly non-square Banach spaces and let a norm $\|\cdot\|_A$ on $\mathbb{R}^N$ have Property T_∞^N. Then $(X_1^* \oplus \cdots \oplus X_N^*)_{A^*}$ has the fixed point property for nonexpansive mappings, where $(X_1^* \oplus \cdots \oplus X_N^*)_{A^*}$ is an A-direct sum of $X_1^*, \dots, X_N^*$ whose norm is defined by the dual norm $\|\cdot\|_A^*$.*

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