数理解析研究所講究録 第2114巻 2019年 137-143

On Ran-Reurings's fixed point theorem

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1 Introduction

Ran-Reurings's fixed point theorem [7] is a fixed point theorem in metric spaces with a partial order. In this paper, we introduce Ran-Reurings's fixed point theorem and its related results. In Section 2, we consider an asymptotic generalization of Ran-Reurings's fixed point theorem. In Sections 3 and 4, we consider applications of a fixed point theorem in metric spaces with a partial order. For fixed point theorems in metric spaces, see [1, 3, 4].

2 Asymptotic Generalization

The Banach fixed point theorem is the following: Let (X, d) be a complete metric space and T a mapping of X into itself. If T is contractive, i.e., there exists $r \in [0, 1)$ such that for any $x, y \in X$,

$$d(Tx, Ty) \leq rd(x, y), \tag{1}$$

then there exists a unique fixed point of T.

There exists a mapping which is not contractive but its iterate is contractive [1, 4]. In fact, consider $C([0, 1], \mathbb{R})$ which is the set of all continuous functions on $[0, 1]$ ($\mathbb{R}$ is the set of all real numbers). This is a Banach space with respect to the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$ for $u \in C([0, 1], \mathbb{R})$. Define a mapping of $C([0, 1], \mathbb{R})$ into itself by

$$T(u)(t) = \int_0^t u(s)ds \tag{2}$$

for $u \in C([0, 1], \mathbb{R})$ and $t \in [0, 1]$. Then we have

$$\|Tu - Tv\| \leq \|u - v\|$$

for all $u, v \in C([0,1],\mathbb{R})$. Therefore T is not contractive. Since

$$T^n(u)(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u(s)ds$$

for $u \in C([0,1],\mathbb{R})$, $t \in [0,1]$ and $n \in \mathbb{N}$ ($\mathbb{N}$ is the set of all positive integers), we have

$$\|T^n u - T^n v\| \leq \frac{1}{n!}\|u - v\|$$

for all $u, v \in C([0,1],\mathbb{R})$ and $n \in \mathbb{N}$. Hence, if we define real numbers $r_n = \frac{1}{n!}$ for $n \in \mathbb{N}$, then T_n satisfies $\|T^n u - T^n v\| \leq r_n\|u - v\|$ for all $u, v \in C([0,1],\mathbb{R})$ and $n \in \mathbb{N}$. Therefore each T^n is contractive if $n \geq 2$.

Caccioppoli's fixed point theorem is the following: Let (X, d) be a complete metric space and T a mapping of X into itself. If there exist nonnegative real numbers $\{r_n\}$ with $\sum_{n=1}^{\infty} r_n < \infty$ such that for any $x, y \in X$ and $n \in \mathbb{N}$,

$$d(T^n x, T^n y) \leq r_n d(x, y), \tag{3}$$

then there exists a unique fixed point of T.

By Caccioppoli's fixed point theorem, we obtain a unique fixed point of T defined by (2). It is noted that $\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} \frac{1}{n!} < \infty$. Moreover the Banach fixed point theorem is deduced from Caccioppoli's fixed point theorem. In fact, if T satisfies (1) for all x, y in a complete metric space X, then we have

$$d(T^n x, T^n y) \leq rd(T^{n-1}x, T^{n-1}y) \leq r^2 d(T^{n-2}x, T^{n-2}y) \cdots \leq r^n d(x, y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Moreover we have $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} < \infty$.

Recently, Ran and Reurings [7] and Nieto and López [5] consider the Banach fixed point theorem in metric spaces with a partial order. Let $(X, \leq)$ be a partially ordered set. A pair of elements $x, y \in X$ is comparable if $x \leq y$ or $y \leq x$. Let T be a mapping of X into itself. We say that T is monotone nondecreasing if for any $x, y \in X$, $x \leq y$ implies $Tx \leq Ty$.

Theorem 1 (Ran and Reurings [7], Nieto and López [5]). *Let $(X, \leq)$ be a partially ordered set with a metric d such that (X, d) is a complete metric space. Let T be a continuous and monotone nondecreasing mapping of X into itself. There exists a nonnegative real number $r \in [0, 1)$ such that for any $x, y \in X$ with $x \geq y$, (1) holds. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then there exists a fixed point of T. Moreover, if for any $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point of T is unique.*

Theorem 2 (Nieto and López [5]). *Let $(X, \leq)$ be a partially ordered set with a metric d such that (X, d) is a complete metric space. Assume that if a nondecreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$ for all $n \in \mathbb{N}$. Let T be a monotone nonincreasing mapping of X into itself. There exists a nonnegative real number $r \in [0, 1)$ such that for any $x, y \in X$ with $x \geq y$, (1) holds. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then there exists a fixed point of T. Moreover, if for any $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point of T is unique.*

In [10], we consider Caccioppoli's fixed point theorem in metric spaces with a partial order. Our result is an asymptotic generalization of theorems in [7] and [5]. In fact, Theorem 1 is deduced from Theorem 3. Theorem 2 is deduced from Theorem 4.

Theorem 3 (Toyoda and Watanabe [10]). *Let $(X, \leq)$ be a partially ordered set with a metric d such that (X, d) is a complete metric space. Let T be a continuous and monotone nondecreasing mapping of X into itself. There exist nonnegative real numbers $\{r_n\}$ with $\sum_{n=1}^{\infty} r_n < \infty$ such that for any $x, y \in X$ with $x \geq y$ and $n \in \mathbb{N}$, (3) holds. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then there exists a fixed point of T. Moreover, if for any $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point of T is unique.*

Theorem 4 (Toyoda and Watanabe [10]). *Let $(X, \leq)$ be a partially ordered set with a metric d such that (X, d) is a complete metric space. Assume that if a nondecreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$ for all $n \in \mathbb{N}$. Let T be a monotone nondecreasing mapping of X into itself. There exist nonnegative real numbers $\{r_n\}$ with $\sum_{n=1}^{\infty} r_n < \infty$ such that for any $x, y \in X$ with $x \geq y$ and $n \in \mathbb{N}$, (3) holds. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then there exists a fixed point of T. Moreover, if for any $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point of T is unique.*

Remark 1. *It is a further topic whether we can remove assumptions of monotonicity of T in Theorems 3 and 4; see [8]. Moreover, it is a further topic how to generalize Theorems 3 and 4 to metic spaces endowed with a graph; see [2].*

3 Application I

In [5], Nieto and López consider the existence of solutions for boundary value problems

$$\begin{cases} u'(t) = f(t, u(t)), \\ u(0) = u(a), \end{cases} \tag{4}$$

where $a > 0$ and f is a continuous mapping of $[0, a] \times \mathbb{R}$ into $\mathbb{R}$. A solution of (4) is a function $u \in C^1([0, a], \mathbb{R})$ satisfying (4), where $C^1([0, a], \mathbb{R})$ is the set of all continuously differentiable functions on $[0, a]$. A lower solution for (4) is a function $u \in C^1(I, \mathbb{R})$ satisfying

$$\begin{cases} u'(t) \leq f(t, u(t)), \\ u(0) \leq u(a). \end{cases}$$

Using Theorem 2, we obtain the following.

Theorem 5 ([Nieto and López [5]). *Let $a > 0$. Let f be a continuous mapping of $[0, a] \times \mathbb{R}$ into $\mathbb{R}$. Assume that there exist $\lambda > 0$, $\mu > 0$ with $\mu < \lambda$ such that for any $x, y \in \mathbb{R}$, $y \geq x$,*

$$0 \leq f(t, y) + \lambda y - (f(t, x) + \lambda x) \leq \mu(y - x).$$

Then the existence of a lower solution of (4) provides the existence of a unique solution of (4).

In the proof of Theorem 5, we use Theorem 2; see [5]. However, in Theorem 5, an assumption of the existence of a lower solution is unnecessary. In fact, using the Banach fixed point theorem, we obtain the following.

Theorem 6. *Let $a > 0$. Let f be a continuous mapping of $[0, a] \times \mathbb{R}$ into $\mathbb{R}$. Assume that there exist $\lambda > 0$, $\mu > 0$ with $\mu < \lambda$ such that for any $x, y \in \mathbb{R}$, $y \geq x$,*

$$0 \leq f(t, y) + \lambda y - (f(t, x) + \lambda x) \leq \mu(y - x).$$

Then the problem (4) has a unique solution.

Proof. Problem (4) is written as

$$\begin{cases} u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), \\ u(0) = u(a). \end{cases}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^a G(t, s) \left(f(s, u(s)) + \lambda u(s)\right) ds,$$

where

$$G(t, s) = \begin{cases} \frac{e^{\lambda(a+s-t)}}{e^{\lambda a}-1}, 0 \leq s \leq t \leq a, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda a}-1}, 0 \leq t \leq s \leq a. \end{cases}$$

Define a mapping T of $C([0, a], \mathbb{R})$ into itself by

$$(Tu)(t) = \int_0^a G(t, s) \left(f(s, u(s)) + \lambda u(s)\right) ds$$

for $u \in C([0, a], \mathbb{R})$ and $t \in [0, a]$.

The set $C([0, a], \mathbb{R})$ is a partially ordered set if we define the following order relation: $u, v \in C([0, a], \mathbb{R})$, $u \leq v$ if and only if for any $t \in [0, a]$, $u(t) \leq v(t)$. Also $C([0, a], \mathbb{R})$ is a complete metric space if we choose the metric $d(u, v) = \sup_{t\in[0,a]} |u(t) - v(t)|$ for $u, v \in C([0, a], \mathbb{R})$.

If $x, y \in \mathbb{R}$ and $t \in [0, a]$, then we have

$$|f(t, y) + \lambda y - f(t, x) - \lambda x| \leq \mu |y - x|. \tag{5}$$

In fact, if $y \geq x$, then $0 \leq f(t, y) + \lambda y - f(t, x) - \lambda x \leq \mu(y - x)$. Thus we get (5). If $x \geq y$, then $0 \leq f(t, x) + \lambda x - f(t, y) - \lambda y \leq \mu(x - y)$. Thus we get (5).

If $u, v \in C([0, a], \mathbb{R})$ and $t \in [0, a]$, then, by (5), we have

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \int_0^a G(t, s)|f(s, u(s)) + \lambda u(s) - f(s, v(s)) - \lambda v(s)|ds \\ &\leq \int_0^a G(t, s)\mu|u(s) - v(s)|ds \\ &\leq \mu d(u, v) \sup_{0\leq t\leq a} \int_0^a G(t, s)ds \\ &= \frac{\mu}{\lambda} d(u, v). \end{aligned}$$

Thus we get

$$d(Tu, Tv) \leq \frac{\mu}{\lambda} d(u, v)$$

for all $u, v \in C([0, a], \mathbb{R})$. By the Banach fixed point theorem, we obtain the existence and uniqueness of fixed points of T. □

4 Application II

In [9], we consider the existence of solutions for boundary value problems

$$\begin{cases} y''''(t) + f(t,\, y(t),\, y''(t)) = 0, \\ y(0) = y(1) = y''(0) = y''(1) = 0, \end{cases} \tag{6}$$

where f is a continuous mapping of $[0, 1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. A solution of (6) is a function $u \in C^4([0, 1], \mathbb{R})$ satisfying (6), where $C^4([0, 1], \mathbb{R})$ is the set of all fourth continuously differentiable functions on $[0, 1]$. A lower solution of (6) is a function $y \in C^4([0, 1], \mathbb{R})$ satisfying

$$\begin{cases} y''''(t) + f(t,\, y(t),\, y''(t)) \leq 0, \\ y(0) = y(1) = y''(0) = y''(1) = 0. \end{cases}$$

Using Theorem 2, we obtain the following.

Theorem 7 (Toyoda and Watanabe [9]). *Let f be a continuous mapping of $[0, 1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Assume that there exists $\mu \in (0, 8)$ such that for any $y_1, y_2, u_1, u_2 \in \mathbb{R}$ with $y_1 \leq y_2$, $u_1 \geq u_2$ and $t \in [0, 1]$,*

$$0 \leq f(t, y_1, u_1) - f(t, y_2, u_2) \leq \mu(u_1 - u_2).$$

If there exists a lower solution y such that $y'''(0) \leq \int_0^1 \int_0^t f(s, y(s), y''(s))dsdt$, then there exists a unique solution of (6).

In the proof of Theorem 7, we use Theorem 2; see [9]. However, in Theorem 7, an assumption of the existence of a lower solution is unnecessary. In fact, using the Banach fixed point theorem, we obtain the following.

Theorem 8. *Let f be a continuous mapping of $[0, 1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Assume that there exists $\mu \in (0, 8)$ such that for any $y_1, y_2, u_1, u_2 \in \mathbb{R}$ with $y_1 \leq y_2$, $u_1 \geq u_2$ and $t \in [0, 1]$,*

$$0 \leq f(t, y_1, u_1) - f(t, y_2, u_2) \leq \mu(u_1 - u_2).$$

Then there exists a unique solution of (6).

Proof. Problem (6) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)f(s,y(s),u(s))ds$$

where

$$y(t) = -\int_0^1 G(t,s)u(s)ds \tag{7}$$

and

$$G(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1. \end{cases}$$

For all $u, v \in C([0,1],\mathbb{R})$, we define $u \le v$ by $u(t) \le v(t)$ for all $t \in [0,1]$. Then $C([0,1],\mathbb{R})$ is a partially ordered set. If we define the metric d by $d(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|$ for $u, v \in C([0,1],\mathbb{R})$, then $C([0,1],\mathbb{R})$ is a complete metric space.

Let T be a mapping of $C([0,1],\mathbb{R})$ into itself by

$$Tu(t) = \int_0^1 G(t,s)f(s,y(s),u(s))ds$$

for $u \in C([0,1],\mathbb{R})$, where y is defined by (7) using u.

If $u_1, u_2 \in C([0,1],\mathbb{R})$ and $t \in [0,1]$, then we have

$$|f(t,y_1(t),u_1(t)) - f(t,y_2(t),u_2(t))| \le \mu|u_1(t) - u_2(t)|, \tag{8}$$

where y_1, y_2 are defined by (7) using u_1, u_2. In fact, let $u_1, u_2 \in C([0,1],\mathbb{R})$ and $t \in [0,1]$. If $u_1(t) \ge u_2(t)$, then we have $y_1(t) = -\int_0^1 G(t,s)u_1(s)ds \le -\int_0^1 G(t,s)u_2(s)ds = y_2(t)$. Note that $G(t,s) \ge 0$ for all $(t,s) \in [0,1] \times [0,1]$. Then we have $0 \le f(t,y_1(t),u_1(t)) - f(t,y_2(t),u_2(t)) \le \mu(u_1(t) - u_2(t))$. Thus we get (8). If $u_1(t) \le u_2(t)$, then we have $y_2(t) = -\int_0^1 G(t,s)u_2(s)ds \le -\int_0^1 G(t,s)u_1(s)ds = y_1(t)$. Then we have $0 \le f(t,y_2(t),u_2(t)) - f(t,y_1(t),u_1(t)) \le \mu(u_2(t) - u_1(t))$. Thus we get (8).

Therefore, for $u_1, u_2 \in C([0,1],\mathbb{R})$ and $t \in [0,1]$, by (8), we have

$$\begin{aligned} |Tu_1(t) - Tu_2(t)| &\le \int_0^1 G(t,s)|f(s,y_1(s),u_1(s)) - f(s,y_2(s),u_2(s))|ds \\ &\le \mu \int_0^1 G(t,s)|u_1(s) - u_2(s)|ds \\ &\le \mu d(u_1,u_2) \int_0^1 G(t,s)ds \\ &\le \frac{\mu}{8} d(u_1,u_2). \end{aligned}$$

Note that $\int_0^1 G(t,s)ds = \frac{1}{2}t(1-t)$. Thus we get

$$d(Tu_2, Tu_2) \le \frac{\mu}{8} d(u_1,u_2)$$

for all $u_1, u_2 \in C([0,1],\mathbb{R})$. By the Banach fixed point theorem, we obtain the existence and uniqueness of fixed points of T. □

Remark 2. *It is a further topic whether, as well as Applications I and II, we can remove conditions of theorems which are applications of fixed point theorems in metric spaces with a partial order; see [6], [8] and [11]*

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