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順序距離空間における不動点定理と非線形境界値問題への適用

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1. Introduction

A coupled fixed point theorem is a combination between fixed point results for contractive type mappings and the monotone iterative method proposed by Bhaskar and Lakshmikantham [1]. Several authors [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] investigated it. It is a strong tool to study a existence and uniqueness solution of boundary value problems for several ordinary differential equations, see [1, 4, 11, 12]. Recently in [12], Jleli et.al extend and generalize several existing results in the literature. They also show the existence and uniqueness of solutions of the following fourth-order two-point boundary value problem for elastic beam equations:

$$\begin{cases} u''''(t) = f(t, u(t), u(t)), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where f is a continuous mapping of $[0,1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$.

We are also concerned about higher order boundary value problems. In particular, for the existence of a solution the use of a fixed point theorem is a very popular method. So, for instance, we consider the following problem,

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

or, for example, the next one (see [12]):

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{2}$$

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where f is a continuous mapping of $[0,1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. We will show that some coupled fixed point theorems are very useful in order to get a solution of these boundary value problems.

For the existence and uniqueness of solutions for the fourth-order two-point boundary value problem for (1), many researchers have studied, see [13, 14, 15] The proof is carried out using the Leray-Schauder fixed point theorem, etc.

In this article, using the method of coupled fixed point theorem in [1, 4, 5, 7, 12], we show the existence of solutions for (1) and (2).

2. Fixed point theorem

First of all, we cited the following definitions and preliminary results will be useful later. Let (X,d) be a metric space endowed with a partial order $\preceq$. We say that a mapping $F: X \to X$ is nondecreasing if for any $x, y \in X$,

$$x \preceq y \Rightarrow Fx \preceq Fy.$$

Let Φ denote the set of all functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying

(a) φ is continuous and nondecreasing;
(b) $\varphi^{-1}(\{0\}) = \{0\}$.

Let Ψ denote the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying

(c) $\lim_{t\to r+} \psi(t) > 0$ (and finite) for all $r > 0$;
(d) $\lim_{t\to 0+} \psi(t) = 0$.

Let Θ denote the set of all functions $\theta : [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying

(e) θ is continuous;
(f) $\theta(s1, s2, s3, s4) = 0$ if and only if $s1s2s3s4 = 0$.

Examples of functions ψ of Ψ are given in [7]; see also [4, 16]. Examples of functions θ in Θ are given in [12].

In [12, Theorem 3.1, 3.2], the following fixed point theorem is obtained. We require an additional assumption to the metric space X with a partial order $\preceq$: We say that $(X, d, \preceq)$ is regular if $\{a_n\}$ is a nondecreasing sequence in X with respect to $\preceq$ such that $a_n \to a \in X$ as $n \to \infty$, then $a_n \preceq a$ for all n.

Theorem 1. *Let (X,d) be a complete metric space endowed with a partial order $\preceq$ and $F : X \to X$ a nondecreasing mapping such that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any $x, y \in X$ with*

$x \succeq y$,

$$\varphi(d(Fx, Fy)) \leq \varphi(d(x,y)) - \psi(d(x,y)) + \theta(d(x,Fx), d(y,Fy), d(x,Fy), d(y,Fx)).$$

Suppose also that the following (i) or (ii) hold.

(i) F *is continuous*

(ii) $(X, d, \leq)$ *is regular.*

Also supose that there exists $x_0 \in X$ *such that* $x_0 \preceq Fx_0$ *(or* $x_0 \succeq Fx_0$*). Then* F *admits a fixed point, that is, there exists* $\overline{x} \in X$ *such that* $\overline{x} = F\overline{x}$.

3. Fixed point theorem for monotone mapping

In this section, for mappings F of $X \times X$ into X, we introduce a monotone property. Moreover we consider fixed point theorems for monotone mappings which have this monotone property. We say that a mapping F of $X \times X$ into X is mixed monotone if F is nondecreasing in its first variable and nonincreasing in its second, that is, for $x, y, u, v \in X$,

$$x \succeq u, y \preceq v \Rightarrow F(u,v) \preceq F(x,y),$$

and a mapping $\tilde{F}$ of $X \times X$ into X is reverse mixed monotone if $\tilde{F}$ is nonincreasing in its first variable and nondecreasing in its second, that is, for $x, y, u, v \in X$,

$$x \succeq u, y \preceq v \Rightarrow \tilde{F}(u,v) \succeq \tilde{F}(x,y).$$

Let (X, d) be a metric space, Let F and $\tilde{F}$ be mappings of $X \times X$ into X. We also consider the mapping A of $X \times X$ into $[0, \infty)$ and the mapping B of $X \times X \times X \times X$ into $[0, \infty)$ defined by

$$A(x,y) = \frac{d(x, F(x,y)) + d(y, \tilde{F}(x,y))}{2}, (x,y) \in X \times X,$$

$$B(x,y,u,v) = \frac{d(x, F(u,v)) + d(y, \tilde{F}(u,v))}{2}, (x,y,u,v) \in X \times X \times X \times X.$$

Definition 2. *Mappings* F *and* $\tilde{F}$ *admit a pre-coupled fixed point, if there exists* $(a,b) \in X \times X$ *such that* $a = F(a,b)$ *and* $b = \tilde{F}(a,b)$.

We require additional assumptions to the metric space X with a partial order $\preceq$:

Definition 3. *Let* (X, d) *be a complete metric space endowed with a partial order* $\preceq$. *We say that*

(i) $(X, d, \preceq)$ *is nondecreasing-regular ($\uparrow$-regular) if a nondecreasing sequence* $\{x_n\} \subset X$ *converges to* x, *then* $x_n \preceq x$ *for all* n;
(ii) $(X, d, \preceq)$ *is nonincreasing-regular ($\downarrow$-regular) if a nonincreasing sequence* $\{x_n\} \subset X$ *converges to* x, *then* $x_n \succeq x$ *for all* n.

Motivated by [12, Theorem 3.4, 3.5], we have the following fixed point theorem.

Theorem 4. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$, $F : X \times X \to X$ a mixed monotone mapping and $\tilde{F} : X \times X \to X$ a reverse mixed monotone mapping. We assume that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any $x, y, u, v \in X$ with $x \succeq u$, $y \preceq v$, the following inequality holds:*

$$\varphi\left(\frac{d(F(x,y),F(u,v)) + d(\tilde{F}(x,y),\tilde{F}(u,v))}{2}\right)$$
$$\leq \varphi\left(\frac{d(x,u)+d(y,v)}{2}\right) - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right)$$
$$+ \theta\left(A(x,y), A(u,v), B(x,y,u,v), B(u,v,x,y)\right).$$

Suppose also that the following (i) or (ii) hold.
(i) *F and $\tilde{F}$ are continuous*
(ii) *$(X, d, \preceq)$ is nondecreasing-regular and nonincreasing-regular.*

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0), y_0 \succeq \tilde{F}(x_0, y_0), \quad \textit{or}$$

$$x_0 \succeq F(x_0, y_0), y_0 \preceq \tilde{F}(x_0, y_0),$$

then F and $\tilde{F}$ admit a pre-coupled fixed point, that is, there exists $(a, b) \in X \times X$ such that $a = F(a, b)$ and $b = \tilde{F}(a, b)$.

Proof. See, [17]. □

4. Applications

In this section, we study the existence of solutions of two types fourth-order two-point boundary value problems for elastic beam equations. As another applications, we can consider two types third-order two-point boundary value problems, see [17]. In particular, the following result is an extension of the result in [12].

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = A, u'(0) = B, u''(1) = C, u'''(1) = D, \end{cases} \tag{3}$$

with $I = [0,1]$ and $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, where $C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a set of continuous mappings of $I \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Let Ω be a set of functions ω of $[0,\infty)$ into $[0,\infty)$ satisfying

(i) ω is nondecreasing;

(ii) there exists $\psi \in \Psi$ such that $\omega(r) = \frac{r}{2} - \psi(\frac{r}{2})$ for all $r \in [0,\infty)$.

For examples of such functions, see [7].

Next we consider the following assumptions $(A1)$ and $(A2)$.

(A1) There exists $\omega \in \Omega$ such that for all $t \in I$ and for all $a, b, c, e \in \mathbb{R}$, with $a \geq c$ and $b \leq e$,

$$0 \leq f(t,a,b) - f(t,c,e) \leq \omega(a-c) + \omega(e-b).$$

(A2) There exist $\alpha, \beta \in C(I, \mathbb{R})$ which are solutions of

$$\alpha(t) \leq \int_0^1 G(t,s) f(s, \alpha(s), \beta(s)) ds, t \in I,$$

$$\beta(t) \geq \int_0^1 H_1(t,s) f(s, \alpha(s), \beta(s)) ds, t \in I,$$

where the Green functions G and H_1 are defined by

$$G(t,s) = \begin{cases} \frac{1}{6}s^2(3t-s), & (0 \leq s \leq t \leq 1), \\ \frac{1}{6}t^2(3s-t), & (0 \leq t \leq s \leq 1), \end{cases}$$

$$H_1(t,s) = \begin{cases} 0, & (0 \leq s \leq t \leq 1), \\ s-t, & (0 \leq t \leq s \leq 1). \end{cases}$$

Note that

$$\int_0^1 G(t,s) f(s, u(s), v(s)) ds$$
$$= \int_0^1 H_2(t,s) \int_0^1 H_1(s,r) f(r, u(r), v(r)) dr ds,$$

where the green function H_2 is defined by

$$H_2(t,s) = \begin{cases} t-s, & (0 \leq s \leq t \leq 1), \\ 0, & (0 \leq t \leq s \leq 1). \end{cases}$$

It is easy to show that

$$0 \leq G(t,s) \leq \frac{1}{2}t^2 s \text{ for all } t, s \in I,$$

and

$$0 \leq H_1(t,s) \leq \min\{s,t\} \text{ for all } t, s \in I.$$

Now we have the following theorem.

Theorem 5. *Under the assumptions* $(A1)$ *and* $(A2)$, *the fourth-order two-point boundary value problem (3) has a solution.*

Proof. See, [17]. □

As an application of our results, we also prove the existence of solutions of the following fourth-order two-point boundary value problem, see [13, 14, 15]:

$$\begin{cases} u''''(t) = f(t, u(t), u''(t)), \\ u(0) = A, u(1) = B, u''(0) = C, u''(1) = D, \end{cases} \tag{4}$$

with $I = [0,1]$ and $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. We take the set of functions Ω same way as in former result, and the assumptions $(A1)$ and $(A2)$ are same as those of former result with respect to the following Green functions G and H.

$$G(t,s) = \begin{cases} \frac{1}{6}s(1-t)(2t - s^2 - t^2), & (0 \le s \le t \le 1), \\ \frac{1}{6}t(1-s)(2s - t^2 - s^2), & (0 \le t \le s \le 1), \end{cases}$$

and

$$H(t,s) = \begin{cases} s(1-t) & (0 \le s \le t \le 1), \\ t(1-s) & (0 \le t \le s \le 1). \end{cases}$$

Note that

$$\begin{aligned} &\int_0^1 G(t,s) f(s, u(s), v(s)) ds \\ &= \int_0^1 H(t,s) \int_0^1 H(s,r) f(r, u(r), v(r)) dr ds, t \in I. \end{aligned}$$

It is easy to show that

$$0 \le G(t,s) \le \frac{1}{3} st \text{ for all } t, s \in I,$$

and

$$0 \le H(t,s) \le \min\{s,t\} \text{ for all } t, s \in I.$$

Theorem 6. *Under the assumptions* $(A1)$ *and* $(A2)$, *the fourth-order two-point boundary value problem (4) has a solution.*

Proof. See, [17] □

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