2-local isometries and the reflexivity property of certain spaces of continuous maps

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1 Introduction

The studies about local maps were started by Larson, Kadison and Sourour. In 1988, Larson[9] studied local automorphisms of Banach algebra and obtained the first results concerning to local maps. In 1990, Kadison[8] exhibited the results concerning to local derivations on von neumann algebras. Larson and Sourour[10] got the results of local derivations of $B(X)$ for a Banach space $X$.

The studies of 2-local maps were initiated by Šemrl[13]. He got the results about 2-local automorphisms and 2-local derivations in 1997. Inspired by his results, Molnár[12] started the studies about 2-local isometries in 2002. He considered the group of all surjective complex linear isometries. If $X$ is locally compact Hausdorff space, Györy[3] studies that 2-local isometries are complex linear isometries on the set of all continuous functions vanishing at infinity $C_0(X)$. Hatori, Miura, Oka and Takagi[4] got the results in the case of the uniform algebras in 2007. $C^{(n)}[0,1]$ denotes the set of all $n$-times continuously differentiable functions on $[0,1]$ with $\|f\|_C = \sup_{t\in[0,1]} \sum_{k=0}^{n} |f^{(k)}(t)|/k!$. In 2018, Kawamura, Koshimizu and Miura[7] studied about $C^{(n)}[0,1]$. They got the results that 2-local isometries are surjective complex linear isometries on each space. In recent years, the case
of surjective real linear isometries are studied. Hosseini[5] studied $C^{(n)}[0, 1]$ with $\|f\|_n = \max\{|f(0)|, |f'(0)|, |f^{(2)}(0)|, \ldots, |f^{(n-1)}(0)|, \|f^{(n)}\|_{\infty}\}$ in 2017. The results about 2-local isometries in the case of real linear isometries is fewer than the case of complex linear isometries. I get the result about surjective real linear isometries. I will prove it.

2 Fundamental definitions

In this paper, $\mathbb{R}$ stands for the set of all real numbers. The symbol $\mathbb{C}$ stands for all complex numbers.

**Definition 2.1 (isometry).** Let $(X, d_X), (Y, d_Y)$ be metric spaces. Let $T$ be a map $X$ into $Y$. If $d_X(x_1, x_2) = d_Y(T(x_1), T(x_2))$ for all points $x_1, x_2 \in X$, then $T$ is called an isometry.

Note that $T$ is injective if $T$ is an isometry.

**Definition 2.2.** Let $X$ be a Banach space. The set of all surjective complex linear isometries on $X$ is denoted by $Iso_{\mathbb{C}}(X)$. The set of all surjective real linear isometries on $X$ is denoted by $Iso_{\mathbb{R}}(X)$.

**Definition 2.3 (2-local isometry).** Let $X$ be a Banach space. Let $T$ be a map on $X$. If for each pair of elements $f, g \in X$ there exists $T_{f,g} \in Iso_{\mathbb{C}}(X)$ (or $\in Iso_{\mathbb{R}}(X)$) such that $T_{f,g}(f) = T(f)$ and $T_{f,g}(g) = T(g)$ depending on $f$ and $g$, then $T$ is called a 2-local isometry.

We note that no continuity, surjectivity nor linearity are assumed for $T$.

**Definition 2.4.** Let $C[0, 1]$ denote the set of all complex-valued functions $f$ on the closed interval endowed with the supremum norm

$$\|f\|_{\infty} = \sup\{|f(t)| : t \in [0, 1]\}.$$  

Then $(C[0, 1], \|\cdot\|_{\infty})$ is a Banach algebra.

**Definition 2.5 (Choquet boundary).** Let $X$ be a locally compact Hausdorff space. Let $A$ be a uniform algebra on $X$. Define a subset $E$ of $X$ by $E = \{t \in X : f(x) = 1\}$ for some $f \in A$. Then $E$ is called a peak set for $A$. For every $x \in X$, $E_x$
is a peak set for \( A \). If \( \{x\} = \bigcap_\alpha E_\alpha \), \( x \) is called a weak peak point of \( A \). Define \( Ch(A) \) by \( Ch(A) = \{x \in X : x \text{ is a weak peak point for } A\} \). Then \( Ch(A) \) is called the Choquet boundary of \( A \).

**Definition 2.6 (reflexivity).** Let \( X \) be a Banach space. We say that \( Iso_\mathbb{R}(X) \) is 2-local reflexive if every 2-local isometry is in \( Iso_\mathbb{R}(X) \).

### 3 Surjective real linear isometries on \( C[0, 1] \)

In this section, we consider the form of surjective real linear isometries (Theorem 3.1). This theorem was essentially proved by Ellis[2] or Miura[11]. We note that the Choquet boundary and the Shilov boundary of \( C[0, 1] \) corresponds to the closed interval \([0, 1]\).

**Theorem 3.1.** A map \( T \) is a surjective real linear isometry on \( C[0, 1] \) if and only if there exist a continuous function \( T(1) : [0, 1] \rightarrow \{z \in \mathbb{C} : |z| = 1\} \) and a homeomorphism \( \varphi : [0, 1] \rightarrow [0, 1] \) such that one of the following equalities

\[
\begin{cases}
T(f)(t) = T(1)f \circ \varphi(t) & (f \in C[0, 1], t \in [0, 1]) \\
T(f)(t) = T(1)\overline{f \circ \varphi(t)} & (f \in C[0, 1], t \in [0, 1]).
\end{cases}
\]

**Proof.** First, we assume that a map \( T : C[0, 1] \rightarrow C[0, 1] \) is a surjective real linear isometry on \( C[0, 1] \). The Choquet boundary of \( C[0, 1] \) coincides with the closed interval \([0, 1]\). By a theorem of Miura[11] and the connectivity of \([0, 1]\), one of the following equalities

\[
\begin{cases}
Tf(t) = T(1)f \circ \varphi(t) & (f \in C[0, 1], t \in [0, 1]) \\
Tf(t) = T(1)\overline{f \circ \varphi(t)} & (f \in C[0, 1], t \in [0, 1]).
\end{cases}
\]

Next, we assume that there exist a continuous function \( T(1) : [0, 1] \rightarrow \{z \in \mathbb{C} : |z| = 1\} \) and a homeomorphism \( \varphi : [0, 1] \rightarrow [0, 1] \) such that one of the following equalities

\[
\begin{cases}
T(f)(t) = T(1)f \circ \varphi(t) & (f \in C[0, 1], t \in [0, 1]) \\
T(f)(t) = T(1)\overline{f \circ \varphi(t)} & (f \in C[0, 1], t \in [0, 1]).
\end{cases}
\]

We infer that \( T \) is a surjective real linear isometry on \( C[0, 1] \).
4 2-local isometries in $C[0, 1]$

The studies about 2-local isometries were started by Molnár[12]. If there exists $T_{f,g} \in \text{Iso}_\mathbb{R}(C[0,1])$ such that $Tf = T_{f,g}f$ and $Tg = T_{f,g}g$ for every pair of elements $f, g \in C[0,1]$, then $T$ is called 2-local isometry.

The following is the main result in this paper.

**Theorem 4.1.** Let $T$ be a 2-local isometry on $C[0,1]$. Then $T$ is a 2-local isometry. Thus $\text{Iso}_\mathbb{R}(C[0,1])$ is 2-local reflexive.

To prove Theorem 4.1, we can reduce the case of $T(1) = 1$ (Proposition 4.1). When we assume that $T(1) = 1$, for every element $f \in C[0,1]$ there exists an isometry $T_{1,f}$ such that $T(1) = T_{1,f}(1)$. Since $T(1) = 1$, we get $T_{1,f}(1) = 1$. By Theorem 3.1, $T$ satisfies one of the following equalities

\[
\begin{cases}
Tf(t) = T_{1,f}(1) \circ \varphi_{1,f}(t) = f(\varphi_{1,f}(t)) & (f \in C[0,1], t \in [0,1]) \\
Tf(t) = T_{1,f}(1) \circ \varphi_{1,f}(t) = f(\varphi_{1,f}(t)) & (f \in C[0,1], t \in [0,1]),
\end{cases}
\]

where $\varphi_{1,f}$ is a homeomorphism. When we put $t_0$ such that $\varphi_{1,f}(t) = t_0$, one of the following equalities

\[
\begin{cases}
Tf(t) = f(t_0) \\
Tf(t) = f(t_0).
\end{cases}
\]

**Proposition 4.1.** Let $T$ be a 2-local isometry on $C[0,1]$. When $T(1) = 1$, $T$ is a 2-local isometry.

**Proof.** Let $Id$ be the identity map of $C[0,1]$. Since $T$ is a 2-local isometry, for every $f \in C[0,1]$ there exists $T_{f,Id} \in \text{Iso}_\mathbb{R}(C[0,1])$ such that $T(f) = T_{f,Id}(f)$ and $T(Id) = T_{f,Id}(Id)$, also there exists $T_{1,Id} \in \text{Iso}_\mathbb{R}(C[0,1])$ such that $T(1) = T_{1,Id}(1)$ and $T(Id) = T_{1,Id}(Id)$. By Theorem 3.1, $T_{f,Id}$ and $T_{1,Id}$ are represented by

\[
\begin{cases}
T_{f,Id}(t) = T_{f,Id}(1) \circ \varphi_{f,Id}(t) & (g \in C[0,1], t \in [0,1]) \\
\text{or} \\
T_{f,Id}(t) = T_{f,Id}(1) \circ \varphi_{f,Id}(t) & (g \in C[0,1], t \in [0,1]),
\end{cases}
\]
\[
\begin{cases}
T_{1,Idg}(t) = T_{1,Id}(1)g \circ \varphi_{1,Id}(t) & (g \in C[0,1], t \in [0,1]) \\
\text{or} & \\
T_{1,Idg}(t) = T_{1,Id}(1)g \circ \varphi_{1,Id}(t) & (g \in C[0,1], t \in [0,1]),
\end{cases}
\]

where \(\varphi_{f,Id}\) and \(\varphi_{1,Id}\) are homeomorphisms on \([0,1]\) respectively. Since \(T_{1,Id}(1) = T(1) = 1\), \(T_{1,Id}\) is represented by

\[
\begin{cases}
T_{1,Idg}(t) = g \circ \varphi_{1,Id}(t) & (g \in C[0,1], t \in [0,1]) \\
\text{or} & \\
T_{1,Idg}(t) = g \circ \varphi_{1,Id}(t) & (g \in C[0,1], t \in [0,1]).
\end{cases}
\] (2)

We define a set \(E_{t_{0}f}\) by

\[E_{t_{0}f} = \{t \in [0,1] : T_{I}(t) = f(t_{0})\} \]

for every \(f \in C[0,1], t_{0} \in [0,1]\). Now, \(E_{t_{0}f}\) is a subset of \([0,1]\). By the definition of \(E_{t_{0}f}\), \(E_{t_{0}Id}\) is represented by \(E_{t_{0}Id} = \{t \in [0,1] : T(Id)(t) = Id(t_{0})\}\). Since \(TI(Id) = T_{1,Id}\) and (2), we get

\[TI(Id) = T_{1,Id}Id = Id \circ \varphi_{1,Id} = \varphi_{1,Id}.\] (3)

We get \(E_{t_{0}Id} = \{t \in [0,1] : \varphi_{1,Id}(t) = t_{0}\}\) since (3) and \(Id(t_{0}) = t_{0}\). Since \(\varphi_{1,Id}\) is a homeomorphism, \(E_{t_{0}Id}\) is a singleton.

We take \(b_{t_{0}} \in [0,1]\) such that \(\{b_{t_{0}}\} = E_{t_{0}Id}\). We have \(TI(Id)(b_{t_{0}}) = Id(t_{0}) = t_{0}\) by \(b_{t_{0}} \in E_{t_{0}Id}\) and the definition of \(E_{t_{0}Id}\). Therefore we obtain

\[\varphi_{1,Id}(b_{t_{0}}) = t_{0}\] (4)

by (3). Furthermore we have

\[TI(Id)(b_{t_{0}}) = Tf,Id(Id)(b_{t_{0}}) = T_{f,Id}(1)Id \circ \varphi_{f,Id}(b_{t_{0}}) = T_{f,Id}(1)\varphi_{f,Id}(b_{t_{0}})\]

by \(TI(Id) = T_{f,Id}Id\) and (1). By (5) and \(T(Id)(b_{t_{0}}) = t_{0}\), we have \(T_{f,Id}(1)\varphi_{f,Id}(b_{t_{0}}) = t_{0}\). Since \(\varphi_{f,Id}(b_{t_{0}})\) is in \([0,1]\) and \(t_{0}\) is in \([0,1]\), \(T(Id)(b_{t_{0}})\) is a real number which is a scalar of modulars 1. we get

\[T_{f,Id}(1)(b_{t_{0}}) = 1.\] (6)

Therefore we obtain

\[\varphi_{f,Id}(b_{t_{0}}) = t_{0}.\] (7)
We consider $E_{t_{of}} = \left\{ t \in [0,1] : \frac{Tf(t)}{f(t)} = \frac{f(t_0)}{f(t_0)} \right\}$ for every $f \in C[0,1]$. Since $Tf = T_{f,Id}$ and (1), we get

$$\begin{align*}
&Tf(b_{t_0}) = T_{f,Id}(1)(b_{t_0})f \circ \varphi_{f,Id}(b_{t_0}) \\
&Tf(b_{t_0}) = T_{f,Id}(1)(b_{t_0})f \circ \varphi_{f,Id}(b_{t_0}).
\end{align*}$$

By (6), we have

$$\begin{align*}
&Tf(b_{t_0}) = f \circ \varphi_{f,Id}(b_{t_0}) \\
&Tf(b_{t_0}) = f \circ \varphi_{f,Id}(b_{t_0}).
\end{align*}$$

By (7), we have

$$\begin{align*}
&Tf(b_{t_0}) = f(t_0) \\
&Tf(b_{t_0}) = f(t_0).
\end{align*}$$

Therefore $b_{t_0}$ is an element of $E_{t_{of}}$. Since $f$ is an arbitrary element of $C[0,1]$, we get $E_{t_0,Id} = \{b_{t_0}\} = \bigcap_{f \in C[0,1]} E_{t_{of}}$.

Let $\psi$ be a map $[0,1]$ into $[0,1]$ such that $\{\psi(t_0)\} = \bigcap_{f \in C[0,1]} E_{t_{of}}$. Since $\{b_{t_0}\} = \bigcap_{f \in C[0,1]} E_{t_{of}}$, we get $\psi(t_0) = b_{t_0}$. By $TId = T_{1,Id}Id$ and (2), we have

$$\begin{align*}
&TId(\psi(t_0)) = T_{1,Id}Id(\psi(t_0)) \\
&= Id \varphi_{1,Id}(\psi(t_0)) \\
&= \varphi_{1,Id}(\psi(t_0)) \\
&= \varphi_{1,Id}(b_{t_0}).
\end{align*}$$

By (4), we get

$$TId(\psi(t_0)) = t_0. \tag{8}$$

We will prove that a map $\psi$ is bijective. Let $x \in [0,1]$ be $x = \varphi_{1,Id}(y)$ for every $y \in [0,1]$. We obtain $b_{\varphi_{1,Id}(y)} = \psi(\varphi_{1,Id}(y)) \in E_{\varphi_{1,Id}(y),Id}$. We get $TId = \varphi_{1,Id}$ by (3). By $TId = \varphi_{1,Id}$ and (8), we get $\varphi_{1,Id}(\psi(\varphi_{1,Id}(y))) = TId(\psi(\varphi_{1,Id}(y))) = \varphi_{1,Id}(y)$. Since $\varphi_{1,Id}$ is a homeomorphism, we get $\psi(\varphi_{1,Id}(y)) = y$. By $x = \varphi_{1,Id}(y)$, $y$ is represented by $\psi(x) = y$. Therefore $\psi$ is surjective.

We take $t_1$, $t_2 \in [0,1]$ and assume that $t_1 \neq t_2$. We notice $\psi(t_1) = b_{t_1} \in E_{t_{1,f}}$ and $\psi(t_2) = b_{t_2} \in E_{t_{2,f}}$ ($f \in C[0,1]$). We get $TId(\psi(t_1)) = \varphi_{1,Id}(\psi(t_1))$ by (3). Since we have $TId(\psi(t_1)) = t_1$ by (8), we get $\varphi_{1,Id}(\psi(t_1)) = t_1$. In the same way, we get $\varphi_{1,Id}(\psi(t_2)) = t_2$. By the assumption $t_1 \neq t_2$, we get $\varphi_{1,Id}(\psi(t_1)) \neq \varphi_{1,Id}(\psi(t_2))$. We obtain $\psi(\psi(t_1)) = \psi(t_2)$. Therefore $\psi$ is injective.
By (4) and (7), we get $\varphi_{1,Id}(b_{t_{0}}) = \varphi_{f,Id}(b_{t_{0}})$. Since $b_{t_{0}} = \psi(t_{0})$ ($t_{0} \in [0,1]$), we have $\varphi_{1,Id}(\psi(t_{0})) = \varphi_{f,Id}(\psi(t_{0}))$. Since $\psi$ is a bijection, for every $t \in [0,1]$ we represent $\varphi_{1,Id}(t) = \varphi_{f,Id}(t)$. We get

$$
\varphi_{1,Id} = \varphi_{f,Id}.
$$

(9)

Let $i$ be a constant function : $[0,1] \rightarrow i$. A map $T$ is represented by

$$
\begin{cases}
T_i(\psi(t_0)) = i(t_0) = i \\
\text{or} \\
T_i(\psi(t_0)) = \overline{i(t_0)} = -i
\end{cases}
$$

for every $t_0 \in [0,1]$. Since $\psi$ is bijective and $[0,1]$ is connected, $T$ satisfies either of the cases

(a) $T$ satisfies $T_i = i$ for every $t \in [0,1]$

or

(b) $T$ satisfies $T_i = -i$ for every $t \in [0,1]$.

First, we consider the case (a). We get

$$
TId = T_{f,Id}(1)Id \circ \varphi_{f,Id} = T_{f,Id}(1)\varphi_{f,Id}
$$

for the identity map $Id$ of $C[0,1]$. By the above equation and (3), we get $\varphi_{1,Id} = T_{f,Id}(1)\varphi_{f,Id}$. By (9), we get $T_{f,Id}(1) = 1$. Since (9) and $T_{f,Id}(1) = 1$, and we get

$$
Tf = T_{f,Id}(1)f \circ \varphi_{f,Id} = f \circ \varphi_{f,Id} = f \circ \varphi_{1,Id}.
$$

Consequently, in the case (a), $T$ is represented by $Tf = f \circ \varphi_{1,Id}$ for every $f \in C[0,1]$. Next, we consider the case (b). Let $U$ be a map : $C[0,1] \rightarrow C[0,1]$ such that $U = \overline{T}$. We notice $U$ is a 2-local isometry. For the constant functions $1, i \in C[0,1]$ we have $U(1) = \overline{T(1)} = 1$ and $U(i) = \overline{T(i)} = \overline{-i} = i$. we apply the case (a) to $U$, we get $\overline{Tf} = Uf = f \circ \varphi_{1,Id}$. So we get $Tf = f \circ \varphi_{1,Id}$. Therefore when $T(1) = 1$, one of the following equalities

$$
\begin{cases}
Tf(t) = f\varphi_{1,Id}(t) & (f \in C[0,1], t \in [0,1]) \\
Tf(t) = f\varphi_{1,Id}(t) & (f \in C[0,1], t \in [0,1])
\end{cases}
$$

By Theorem 3.1, $T$ is a surjective real linear isometry on $C[0,1]$. □
Proposition 4.2. Let $T$ be a 2-local isometry on $C[0,1]$. Then $T$ satisfies $|T(1)(t)| = 1$ $(t \in [0,1])$.

Proof. Since $T$ is a 2-local isometry, for every $f \in C[0,1]$ there exists $T_{f,1} \in Iso_{\mathbb{R}}(C[0,1])$ such that $T_{f,1}(f) = T(f)$ and $T_{f,1}(1) = T(1)$. Since $T_{f,1}$ is an element of $Iso_{\mathbb{R}}(C[0,1])$, there exists $T_{f,1}(1)$ such that $|T_{f,1}(1)| = 1$. By $T_{f,1}(1) = T(1)$, there exists $T(1)$ such that $|T(1)(t)| = 1$ $(t \in [0,1])$. \qed

Proposition 4.3. Let $T$ be a 2-local isometry on $C[0,1]$. Define a map $S$ by $S = \overline{T(1)}T$. Then $S$ is a 2-local isometry on $C[0,1]$ such that $S(1) = 1$.

Proof. Since $T$ is a 2-local isometry, for every pair of elements $f, g \in C[0,1]$ there exist $T_{f,g} \in Iso_{\mathbb{R}}(C[0,1])$ such that $T_{f,g}f = T f$ and $T_{f,g}g = T g$. Define a map $S_{f,g}$ by $S_{f,g} = \overline{T(1)}T_{f,g}$. Since $T_{f,g}$ is a real linear isometry, we get that for every $\alpha, \beta \in \mathbb{R}, u, v \in C[0,1]$ 

$$S_{f,g}(\alpha u + \beta v) = \overline{T(1)}T_{f,g}(\alpha u + \beta v)$$

$$= \overline{T(1)}(\alpha T_{f,g}(u) + \beta T_{f,g}(v))$$

$$= \alpha \overline{T(1)}T_{f,g}(u) + \beta \overline{T(1)}T_{f,g}(v))$$

$$= \alpha S_{f,g}(u) + \beta S_{f,g}(v).$$

Consequently, $S_{f,g}$ is a real linear map. We get that for every $u \in C[0,1]$ 

$$\|S_{f,g}(u)\|_{\infty} = \|\overline{T(1)}T_{f,g}(u)\|_{\infty}$$

$$= \|T_{f,g}(u)\|_{\infty}$$

$$= \|u\|_{\infty}.$$

So $S_{f,g}$ is an isometry. Since $T_{f,g}$ is a surjective real linear isometry on $C[0,1]$, $T_{f,g}$ is bijective. There exists a map $T_{f,g}^{-1}$ which is an inverse of $T_{f,g}$. Define a map $v$ by $v = T_{f,g}^{-1}T(1)u$ for every $u \in C[0,1]$, then $v$ is an element of $C[0,1]$. We get $S_{f,g}(v) = \overline{T(1)}T_{f,g}T_{f,g}^{-1}T(1)u = u$. We notice $S_{f,g}$ is surjective. Therefore $S_{f,g}$ is a surjective real linear isometry on $C[0,1]$. By the assumption, $S_{f,g} = \overline{T(1)}T_{f,g}$.

We have 

$$S_{f,g}f = \overline{T(1)}T_{f,g}f$$

$$= \overline{T(1)}T f$$

$$= S f.$$
By the same way, we get $S_{f,g} g = S g$. Therefore $S$ is a 2-local isometry. For the constant function $1 \in C[0,1]$ we get $S(1) = \overline{T(1)} T(1) = 1$. \hfill \Box

**Proof of Theorem 4.1.** Let $S$ be a map $S = \overline{T(1)} T$. By Proposition 4.3, $S$ is a 2-local isometry of $C[0,1]$ such that $S(1) = 1$. We apply Proposition 4.1 to $S$, $S$ satisfies that one of the following equalities

$$
\begin{cases}
S f(t) = f \circ \varphi(t) & (t \in [0,1]) \\
S f(t) = \frac{f \circ \varphi(t)}{f \circ \varphi(t)} & (t \in [0,1]),
\end{cases}
$$

where $\varphi$ is a homeomorphism on $[0,1]$. Since $S = \overline{T(1)} T$, we get $T(1) S = T(1) \overline{T(1)} T = T$. Therefore $T$ satisfies that one of the following equalities

$$
\begin{cases}
T f(t) = T(1) f \circ \varphi(t) & (f \in C[0,1], t \in [0,1]) \\
T f(t) = T(1) \frac{f \circ \varphi(t)}{f \circ \varphi(t)} & (f \in C[0,1], t \in [0,1]).
\end{cases}
$$

By Theorem 3.1, $T$ is a surjective real linear isometry. Therefore $\text{Iso}_\mathbb{R}(C[0,1])$ is 2-local reflexive. \hfill \Box

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**References**


