On Spaces of Lipschitz Maps
with Values in a Uniform Algebra

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In this paper, linear always means complex linear, especially Banach algebra always
means complex Banach algebra. Isometries with respect to the \( s \)-norm between vector
valued Lipschitz spaces were studied by Hatori and Oi \[2\]. We prove a version of their
results (Main Theorem A). There are literatures which study isometries with respect
to the \( \max \)-norm between vector valued Lipschitz spaces \[1, 4\]. In this paper, we
exhibit the form of isometries with respect to the \( \max \)-norm under an additional
condition (Main Theorem B) (cf. \[7\]).

1 Definitions

In this section, we introduce some basic definitions.

**Definition 1.1.** Let \( X \) be a compact metric space and \( E \) a normed space. A map
\( f : X \rightarrow E \) is called a Lipschitz map if

\[
(L(f) := \sup_{x, y \in X, x \neq y} \frac{\|f(x) - f(y)\|_E}{d(x, y)} < \infty.
\]

The number \( L(f) \) is called the Lipschitz constant of \( f \). We shall denote by \( \text{Lip} (X, E) \)
the space of all Lipschitz maps from \( X \) into \( E \). We write the space \( \text{Lip} (X, \mathbb{C}) \) just by

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Lip (X) for simplification. There are several norms on the Lipschitz space Lip (X, E):
s-norm \( \| \cdot \|_s \) is defined by \( \| \cdot \|_s = \| \cdot \|_\infty + L(\cdot) \), and max-norm \( \| \cdot \|_{\max} \) by \( \| \cdot \|_{\max} = \max \{ \| \cdot \|_\infty, L(\cdot) \} \). If B is a Banach algebra, the space \( (\text{Lip} (X, B), \| \cdot \|_s) \) is a Banach algebra, and the space \( (\text{Lip} (X, B), \| \cdot \|_{\max}) \) is a Banach space (In general, submultiplicativity needs not hold.).

2 Main Theorem A

The next is the theorem of isometries with respect to the s-norm between Lipschitz spaces. In this section, we give an outline of the proof of this theorem.

**Theorem 2.1** (Main Theorem A). For \( j = 1, 2 \), let \( X_j \) be a compact metric space, \( Y_j \) a compact Hausdorff space, and \( A_j \) a uniform algebra. If \( U : (\text{Lip} (X_1, A_1), \| \cdot \|_s) \to (\text{Lip} (X_2, A_2), \| \cdot \|_s) \) is a unital surjective linear isometry, then there exist

- a continuous map \( \psi : X_2 \times \text{Ch} (A_2) \to X_1 \) such that for every \( y' \in \text{Ch} (A_2) \)
  \[ \psi (\cdot, y') : X_2 \to X_1 \] is a surjective isometry,

and

- a homeomorphism \( \tau : \text{Ch} (A_2) \to \text{Ch} (A_1) \)

such that \( (U(F)(x'))(y') = (F'(\psi (x', y')))(\tau (y')) \) for every \( x' \in X_2, y' \in \text{Ch} (A_2) \), and \( F \in \text{Lip} (X_2, A_2) \).

**Remark 2.2.** A map \( U \) being unital means \( U(1) = 1 \). The space \( \text{Ch} (A) \) denotes a Choquet boundary of \( A \). (If \( Y \) is a compact metric space and \( A \) is a subspace of \( (C(Y), \| \cdot \|_\infty) \), \( \text{Ch} (A) = \{ y \in Y \mid \tau_y \in \text{ext} \{ \varphi \in A^* \mid \| \varphi \| = \varphi (1) = 1 \} \} \) where \( \tau_y \) is the evaluation map at \( y \in Y \).)

**Outline of the proof of the Main Theorem A**

First, we regard \( \text{Lip} (X_j, A_j) \) as a subspace of \( C (X_j \times Y_j) \). We apply a theorem of Jarosz [3], then we find that \( U \) is an isometry also with respect to the supremum norm. Using partition of unity, we find that the uniform closure of \( \text{Lip} (X_j, A_j) \) coincides with \( C (X_j, A_j) \). So we can extend \( U \) from \( C (X_1, A_1) \) onto \( C (X_2, A_2) \) which is a unital surjective linear isometry with respect to the supremum norm. We denote this
map by $\tilde{U}^\infty$. We define maps

\[ S : \{ \varphi' \in C(X_2, A_2)^* | \| \varphi' \| = \varphi'(1) = 1 \} \rightarrow \{ \varphi \in C(X_1, A_1)^* | \| \varphi \| = \varphi(1) = 1 \} \]

by $S(\varphi') := \varphi' \circ \tilde{U}^\infty$ and

\[ S' : \{ \varphi \in C(X_1, A_1)^* | \| \varphi \| = \varphi(1) = 1 \} \rightarrow \{ \varphi' \in C(X_2, A_2)^* | \| \varphi' \| = \varphi'(1) = 1 \} \]

by $S'(\varphi) := \varphi \circ (\tilde{U}^\infty)^{-1}$. Then, $S$ and $S'$ are well-defined, $S'$ is an inverse map of $S$, and $S$ is a $w^*$-homeomorphism.

For $j = 1, 2$, we define a set

\[ K_j := \text{ext} \{ \varphi \in C(X_j, A_j)^* | \| \varphi \| = \varphi(1) = 1 \} . \]

Then we find that $S(K_2) = K_1$ by some easy argument of extreme points. We note that the Choquet boundary of $C(X_j, A_j)$ coincides with $X_j \times \text{Ch}(A_j)$. If we define a homeomorphism $\Phi_j : X_j \times \text{Ch}(A_j) \rightarrow K_j$ by $\Phi_j(x, y) = \varphi(x, y)$ where $\varphi(x, y)$ is the evaluation at $(x, y)$ for $j = 1, 2$, then the map $\Phi_j^{-1} \circ S \circ \Phi_j$ is a homeomorphism between $X_2 \times \text{Ch}(A_2)$ and $X_1 \times \text{Ch}(A_1)$. So we can define continuous maps $\psi_1 : X_2 \times \text{Ch}(A_2) \rightarrow X_1$, $\psi_2 : X_2 \times \text{Ch}(A_2) \rightarrow \text{Ch}(A_1)$ by $(\psi_1, \psi_2) = \Phi_j^{-1} \circ S \circ \Phi_j$. By the similar way, we consider the homeomorphism $\Phi_j^{-1} \circ S_j^{-1} \circ \Phi_j$ between $X_1 \times \text{Ch}(A_1)$ and $X_2 \times \text{Ch}(A_2)$, and define continuous maps $\psi'_1 : X_1 \times \text{Ch}(A_1) \rightarrow X_2$, $\psi'_2 : X_1 \times \text{Ch}(A_1) \rightarrow \text{Ch}(A_2)$ by $(\psi'_1, \psi'_2) = \Phi_j^{-1} \circ S_j^{-1} \circ \Phi_j$. Then for every $x' \in X_2$ and $y' \in \text{Ch}(A_2)$, $((U(F))(x'))(y') = S\left(\varphi'(x', y')\right)(F) = (F(\psi_1(x', y')))(\psi_2(x', y'))$. We shall observe the maps $\psi_1, \psi_2$.

At the first, We show that the map $\psi_2$ needs not depend on the first variable $x' \in X_2$, that is, the equality $\psi_2(x'_1, y') = \psi_2(x'_2, y')$ holds for any $x'_1, x'_2 \in X_2$ and $y' \in \text{Ch}(A_2)$. Suppose that there are $x'_1 \neq x'_2 \in X_2$ and $y' \in \text{Ch}(A_2)$ such that $\psi_2(x'_1, y') \neq \psi_2(x'_2, y')$. Then there is $h \in A_1$ such that $h(\psi_2(x'_1, y')) \neq h(\psi_2(x'_2, y'))$ since $A_1$ is a uniform algebra. By the direct computation, we assert that $L(1 \otimes h) = 0$ and $L(U(1 \otimes h)) \neq 0$. On the other hand $U$ preserves the Lipschitz constant because $U$ preserves the s-norm and the supremum norm. This is a contradiction. Hence the map $\psi_2$ needs not depend on the first variable.

Then we define continuous maps $\tau : \text{Ch}(A_2) \rightarrow \text{Ch}(A_1)$ by $\tau(y') = \psi_2(x', y')$ ($y' \in \text{Ch}(A_2)$) for some $x' \in X_2$, and $\tau' : \text{Ch}(A_1) \rightarrow \text{Ch}(A_2)$ by $\tau'(y) = \psi'_2(x, y)$.
(y \in \text{Ch}(A_1)) \text{ for some } x \in X_1. \text{ We can check that the map } \tau \text{ is a homeomorphism between } \text{Ch}(A_2) \text{ and } \text{Ch}(A_1). \text{ Moreover, } \tau' \text{ is an inverse map of } \tau. \text{ On the other hand, the maps } \psi_1(\cdot, y') : X_2 \to X_1 \text{ and } \psi_1' : X_1 \to X_2 \text{ are bijective for each } y' \in \text{Ch}(A_2) \text{ and } y \in \text{Ch}(A_1) \text{ respectively. Moreover, } \psi_1(\cdot, y') = \psi_1'(\cdot, \tau(y'))^{-1} \text{ and } \\
\psi_1'(\cdot, \tau^{-1}(y))^{-1} \text{ hold for each } y' \in \text{Ch}(A_2) \text{ and } y \in \text{Ch}(A_1) \text{ respectively. }

These indicate that it is sufficient to show that \psi_1(\cdot, y'_0) : X_2 \to X_1 \text{ is a contractive map for each } y'_0 \in \text{Ch}(A_2) \text{ which proves the Main Theorem A. Take } y'_0 \in \text{Ch}(A_2) \text{ arbitrarily. We prove that } d(\psi_1(x'_1, y'_0), \psi_1(x'_2, y'_0)) \leq d(x'_1, x'_2) \text{ for every distinct } x'_1, x'_2 \in X_2. \text{ We define a function } f_{\psi_1(x'_2, y'_0)} : X_1 \to \mathbb{C} \text{ by } f_{\psi_1(x'_2, y'_0)}(x) = d(x, \psi_1(x'_2, y'_0)) \text{ for } x \in X_1. \text{ Then } f_{\psi_1(x'_2, y'_0)} \text{ is in Lip } (X_1) \text{ and } L \left(f_{\psi_1(x'_2, y'_0)}\right) = 1. \text{ Therefore, }

\begin{align*}
&d(\psi_1(x'_1, y'_0), \psi_1(x'_2, y'_0)) \\
&= d(\psi_1(x'_1, y'_0), \psi_1(x'_2, y'_0)) - d(\psi_1(x'_1, y'_0), \psi_1(x'_2, y'_0)) \\
&= |f_{\psi_1(x'_2, y'_0)}(\psi_1(x'_1, y'_0)) - f_{\psi_1(x'_2, y'_0)}(\psi_1(x'_2, y'_0))| \\
&= \left|\left(\left(f_{\psi_1(x'_2, y'_0)} \otimes 1\right)(\psi_1(x'_1, y'_0))\right)(\tau(y'_0)) - \left(\left(f_{\psi_1(x'_2, y'_0)} \otimes 1\right)(\psi_1(x'_2, y'_0))\right)(\tau(y'_0))\right| \\
&\leq \left\|\left(U \left(f_{\psi_1(x'_2, y'_0)} \otimes 1\right)\right)(x'_1) - \left(U \left(f_{\psi_1(x'_2, y'_0)} \otimes 1\right)\right)(x'_2)\right\|_{\infty(Y_2)} \\
&\leq d(x'_1, x'_2) L \left(U \left(f_{\psi_1(x'_2, y'_0)} \otimes 1\right)\right).
\end{align*}

Since \( U \) preserves the Lipschitz constant, we have

\[ L \left(U \left(f_{\psi_1(x'_2, y'_0)} \otimes 1\right)\right) = L \left(f_{\psi_1(x'_2, y'_0)} \otimes 1\right) = L \left(f_{\psi_1(x'_2, y'_0)}\right) = 1. \]

Hence we have

\[ d(\psi_1(x'_1, y'_0), \psi_1(x'_2, y'_0)) \leq d(x'_1, x'_2) L \left(U \left(f_{\psi_1(x'_2, y'_0)} \otimes 1\right)\right) = d(x'_1, x'_2) \]

and \( \psi_1(\cdot, y'_0) \) is a contractive map. We complete the outline of the proof of the Main Theorem A.
3 Main Theorem B

In this section, we consider isometries with respect to the max-norm between Lipschitz spaces. We exhibit the Main Theorem B and give an outline of the proof of this theorem.

Definition 3.1 (K pair). Let $X_1$ and $X_2$ be compact metric spaces. We say that the ordered pair $(X_1, X_2)$ of these two sets is a $K$ pair if the following two conditions are satisfied.

- (K1) For $j = 1, 2$, if we take any $x_1, x_2 \in X_j$, there are finitely many $x_1^\circ, \ldots, x_n^\circ \in X_j$ such that $d(x_1, x_1^\circ) < 1$, $d(x_i^\circ, x_{i+1}^\circ) < 1$ ($i = 1, \ldots, n - 1$), $d(x_n^\circ, x_2) < 1$.
- (K2) For any bijection $\psi : X_2 \to X_1$ and positive $\varepsilon$, the following statement holds; if $x'_1, x'_2 \in X_2$ and $d(x'_1, x'_2) < \varepsilon$ implies that $d(\psi(x'_1), \psi(x'_2)) = d(x'_1, x'_2)$, then $\psi$ is an isometry.

Theorem 3.2 (Main Theorem B). For $j = 1, 2$, let $X_j$ be a compact metric space, $Y_j$ a compact Hausdorff space, and $A_j$ a uniform algebra. We assume that $(X_1, X_2)$ is a $K$ pair. If $U : (\text{Lip}(X_1, A_1), \| \cdot \|_{\text{max}}) \to (\text{Lip}(X_2, A_2), \| \cdot \|_{\text{max}})$ is a unital surjective linear isometry, then there exist

- a continuous map $\psi : X_2 \times \text{Ch}(A_2) \to X_1$ such that for every $y' \in \text{Ch}(A_2)$, $\psi(\cdot, y') : X_2 \to X_1$ is a surjective isometry,

and

- a homeomorphism $\tau : \text{Ch}(A_2) \to \text{Ch}(A_1)$ such that $(U(F)(x'))(y') = (F(\psi(x', y')))(\tau(y'))$ for every $x' \in X_2$, $y' \in \text{Ch}(A_2)$, and $F \in \text{Lip}(X_2, A_2)$.

Outline of the proof of the Main Theorem B

We can prove by the same way as the outline of the proof of Theorem 2.1 that there are continuous maps $\psi_1 : X_2 \times \text{Ch}(A_2) \to X_1$, $\psi_2 : X_2 \times \text{Ch}(A_2) \to \text{Ch}(A_1)$ such that for every $x' \in X_2$, $y' \in \text{Ch}(A_2)$, $((U(F))(x'))(y') = (F(\psi_1(x', y')))(\psi_2(x', y'))$ holds.
At the first, we show that the map $\psi_2$ needs not depend on the first variable $x' \in X_2$, that is, the equality $\psi_2(x_1', y') = \psi_2(x_2', y')$ holds for any $x_1', x_2' \in X_2$ and $y' \in \text{Ch}(A_2)$. By the condition (K 1), it suffices to show that the equality $\psi_2(x_1', y') = \psi_2(x_2', y')$ holds for every $x_1', x_2' \in X_2$ with $d(x_1', x_2') < 1$ and $y' \in \text{Ch}(A_2)$. If not, there exist $x_1^0, x_2^0 \in X_2$ with $d(x_1^0, x_2^0) < 1$ and $y^0 \in \text{Ch}(A_2)$ such that $\psi_2(x_1^0, y^0) \neq \psi_2(x_2^0, y^0)$. Let $\varepsilon_0 = \frac{1 - d(x_1^0, x_2^0)}{2}$. We take an open neighborhood $V \subset Y_1$ of $\psi_2(x_1^0, y^0)$ which doesn't contain $\psi_2(x_2^0, y^0)$. Then there is a peaking function $h \in A_1$ such that $h(\psi_2(x_1^0, y^0)) = 1$, and $|h(y)| < \varepsilon_0$ for every $y \in Y_1 \setminus V$. Especially $|h(\psi_2(x_2^0, y^0))| < \varepsilon_0$. It is clear that $\|1 \otimes h\|_{\text{max}} = 1$. On the other hand,

$$L(U(1 \otimes h)) \geq \frac{|U(1 \otimes h)(x_1^0, y^0) - U(1 \otimes h)(x_2^0, y^0)|}{d(x_1^0, x_2^0)} = \frac{|h(\psi_2(x_1^0, y^0)) - h(\psi_2(x_2^0, y^0))|}{d(x_1^0, x_2^0)} > \frac{1 - 2\varepsilon_0}{d(x_1^0, x_2^0)} = 1$$

holds, hence we get $\|U(1 \otimes h)\|_{\text{max}} > 1$. This contradicts to the fact that $U$ preserves the max-norm. Thus $\psi_2$ needs not depend on the first variable.

By the Theorem of Jarosz [3], $U$ is also an isometry with respect to the supremum norm. We can extend $U$ from the uniform closure of $\text{Lip}(X_1, A_1)$, which is $C(X_1, A_1)$, onto the uniform closure of $\text{Lip}(X_2, A_2)$, which is $C(X_2, A_2)$, that is a unital surjective linear isometry with respect to the supremum norm. Since $C(X_j, A_j)$ is a uniform algebra, a theorem of Nagasawa [5] yields that $U$ is multiplicative. For each $y' \in \text{Ch}(A_2)$, we define a map $U_{y'} : \text{Lip}(X_1) \longrightarrow \text{Lip}(X_2)$ by

$$U_{y'}(f) = ((U(f \otimes 1))((\cdot)))(y')$$

for $f \in \text{Lip}(X_1)$. $U_{y'}$ is a unital homomorphism. So by [6, Theorem 5-1] , there is a Lipschitz map $\phi_{y'} : X_2 \longrightarrow X_1$ such that $U_{y'}(f) = f \circ \phi_{y'}$ for every $f \in \text{Lip}(X_1)$. It is easy to check the equality $\phi_{y'} = \psi_1(\cdot, y')$. Hence $\psi_1(\cdot, y')$ is a Lipschitz map.

Next we prove that $\psi_1(\cdot, y') : X_2 \longrightarrow X_1$ is a surjective isometry for each $y' \in \text{Ch}(A_2)$. Let $\varepsilon_0 = \frac{1}{\max(1, L(\psi_1(\cdot, y')))}$. By (K 2) and the descriptions in the outline of the proof of Theorem 2.1, it suffices to show that for every $x_1', x_2' \in X_2$ with $d(x_1', x_2') < \varepsilon_0$, the equality $d(x_1', x_2') \geq d(\psi_1(x_1', y'), \psi_1(x_2', y'))$ holds. We define $f_{\psi_1(x_2', y')} \in \text{Lip}(X_1)$ by $f_{\psi_1(x_2', y')}(x) = \min \{d(x, \psi_1(x_2', y')), 1\}$ for
$x \in X_1$, then we have $L(\psi_1(x'_2, y')) \leq 1$, $\|f_{\psi_1}(x'_2, y')\|_{\infty} \leq 1$. By the definition of $\varepsilon_0$, we get $d(\psi_1(x'_1, y'), \psi_1(x'_2, y')) \leq 1$. Hence $f_{\psi_1}(x'_2, y')(\psi_1(x'_1, y')) = d(\psi_1(x'_1, y'), \psi_1(x'_2, y'))$, and
\[
d(\psi_1(x'_1, y'), \psi_1(x'_2, y'))
= |f_{\psi_1}(x'_1, y')(\psi_1(x'_1, y')) - f_{\psi_1}(x'_2, y')(\psi_1(x'_2, y'))|
\leq \|U(f_{\psi_1}(x'_2, y') \otimes 1)(x'_2)\|_{\infty(Y_{2})}
\leq L(U(f_{\psi_1}(x'_2, y') \otimes 1))d(x'_1, x'_2).
\]

Thus we get the desired inequality. Now we complete the outline of the proof of the Main Theorem B.

\[\square\]

In the next section, we observe some examples of K pairs, and Main Theorem B without the condition, K pair.

4 K pairs

In the Main Theorem B, we assume that $(X_1, X_2)$ is a K pair. We give some examples of K pairs.

Example 4.1.

1. If $a < b$, the pair of closed intervals $[a, b], [a, b]$ is a K pair.
2. Let $\overline{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with the usual distance, then $(\overline{D}, \overline{D})$ is a K pair.
3. Let $K = \{(0) \times [-1, 1]\} \cup \{(t, \frac{1}{2}t) \mid 0 \leq t \leq 2\} \cup \{(t, -\frac{1}{2}t) \mid 0 \leq t \leq 2\} \subset \mathbb{R}^2$ with the usual distance, then $(K, K)$ is a K pair.

It is not difficult to check that these three pairs above are K pairs.

Example 4.2. Let $X_1 = X_2 = Y_1 = Y_2 = \{a, b\}$ where the distance of $a$ and $b$ is 2, then $(X_1, X_2)$ is not a K pair because it doesn’t satisfy (K1). We define a map $\phi : X_2 \times Y_2 \rightarrow X_1 \times Y_1$ by
\[
\phi((a, a)) = (a, a), \ \phi((a, b)) = (b, a)
\]
\[ \phi((b,a)) = (a,b), \ \phi((b,b)) = (b,b) \]
and maps \( \psi_1 : X_2 \times Y_2 \to X_1, \ \psi_2 : X_2 \times Y_2 \to Y_1 \) by \( \phi = (\psi_1, \psi_2) \). Let \( U : \text{Lip}(X_1, C(Y_1)) \to \text{Lip}(X_2, C(Y_2)) \) be
\[
((U(F))(x'))(y') = (F(\psi_1(x', y')))(\psi_2(x', y'))
\]
for \( x' \in X_2, \ y' \in Y_2 \), and \( F \in \text{Lip}(X_1, C(Y_1)) \). Then \( U \) is a unital surjective linear isometry with respect to the max-norm. Actually for every \( F \in \text{Lip}(X_j, C(Y_j)) \),
\[
L(F) = \frac{\|F(a) - F(b)\|_{\infty}}{2} \leq \|F\|_{\infty}.
\]
Hence the max-norm coincides with the supremum norm in this case. The map \( U \) is clearly an isometry with respect to the supremum norm. But \( U \) cannot be represented as the form in Theorem 3.2.

**Example 4.3.** Let \( H = \{(0) \times [-1,1] \} \cup \{[0,3] \times \{0\}\} \cup \{[3] \times [-1,1]\} \subset \mathbb{R}^2 \) with the usual distance. Then \( (H, H) \) is not a K pair. To prove this, we define a bijection \( \psi : H \to H \) by
\[
\psi((x, y)) = \begin{cases} 
(x, y) & ((x, y) \in \{0\} \times [-1,1]) \\
(x, y) & ((x, y) \in \{3\} \times [-1,1]) \\
(x, -y) & ((x, y) \in \{0\} \times [0,3])
\end{cases}
\]
Then \( d((x_1, y_1), (x_2, y_2)) < 2 \) implies that \( d((x_1, y_1), (x_2, y_2)) = d(\psi((x_1, y_1)), \psi((x_2, y_2))) \) but \( \psi \) is not an isometry. Let \( Y \) be any compact Hausdorff space. We define a map \( U : \text{Lip}(H, C(Y)) \to \text{Lip}(H, C(Y)) \) by
\[
((U(F))(x'))(y') = (F(\psi(x')))(y')
\]
for \( x' \in H, \ y' \in Y \), and \( F \in \text{Lip}(H, C(Y)) \). This \( U \) is not represented by the form in Theorem 3.2, but \( U \) is a unital surjective linear isometry with respect to the max-norm.

**References**


