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Constructing order relations on semigroups*

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Abstract

We present an approach to construct order relations on semigroups via inverses along elements and survey recent results on this topic.

Contents

1 Introduction

Let S be a semigroup. By S^1 we denote the monoid generated by S. We denote by $E(S)$ the set of idempotents of S and by 2^S the set of subsets of S.

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The detailed and self contained information on semigroup theory can be found in [22].

An inverse of an element $a \in S$ is an element $a^{-1} \in S$ such that $a^{-1}a = aa^{-1} = 1$.

Generalized inverse of a given element $a \in S$ is an element $b \in S$ associated with a in the following way:

1. This element exists for a class of elements larger than the class of invertible elements.

2. This element has some of the properties of the usual inverse.

3. This element equals to the usual inverse of a if it exists.

The concept of a generalized inverse seems to have been first mentioned in print in 1903 by Fredholm [12], where a particular generalized inverse (called by him pseudoinverse) of an integral operator was given. The class of all pseudoinverses was characterized in 1912 by Hurwitz [23], who used the finite dimensionality of the null spaces of the Fredholm operators to give a simple algebraic construction. The detailed and self-contained exposition on this subject can be found for example in [2].

Definition 1.1. Let $a \in S$.

1. We say that a is *(von Neumann) regular* if $a \in aSa$.

2. A particular solution to $axa = a$ is called an *inner inverse* of a and is denoted by a^-.

3. A solution of the equation $xax = x$ is called an *outer inverse* of a and is denoted by $a^=$.

4. An inner inverse of a that is also an outer inverse is called a *reflexive inverse* and is denoted by a^+.

The set of all inner (resp. outer, resp. reflexive) inverses of a is denoted by $a\{1\}$ (resp. $a\{2\}$, resp. $a\{1,2\}$).

Remark 1.2. Let $a \in S$ and $a^- \in a\{1\}$. Then $aa^-, a^-a \in E(S)$. Furthermore, $a^-aa^- \in a\{1,2\}$.

Definition 1.3. A semigroup S is *regular* if all its elements are regular.

The definitions of group, Moore-Penrose and Drazin inverses are standard and can be found in the literature (see, for example, [2], [16], [8]). We provide them here for the completeness.

Definition 1.4. Let $a \in S$.

1. The element a is group invertible if there is $a^\# \in a\{1,2\}$ that commutes with a.

2. The element a is Drazin invertible if there is $a^D \in a\{2\}$ that commutes with a and $a^{k+1}a^D = a^k$.

3. If $*$ is an involution in S, then a is Moore-Penrose invertible if there is $a^\dagger \in a\{1,2\}$ such that $aa^\dagger$ and $a^\dagger a$ are symmetric with respect to $*$.

Each of these inverses is unique if exists.

Theorem 1.5. *[38] The element a has a Drazin inverse a^D if a positive power a^n of a is group invertible and $a^D = (a^{n+1})^{\#} a^n$.*

Remark 1.6. If $e \in E(S)$ then $e^\# = e^D = e$. If, in addition, $e^* = e$ then $e^\dagger = e$.

Example 1.7. Let $S = \mathbb{M}_3(\mathbb{Z}_2)$, and pose $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $o = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$. Then

1. $a^\# = a^\dagger = a^D = a$,

2. $b^\dagger = b^t$,

3. $b^D = o$,

4. There is no $b^\#$.

Proof. 1. By Lemma above.

2. Easy to verify since $bb^t = a$.

3. Since $b^2 = o$, o is a Drazin inverse of b with $k = 2$.

4. Suppose that there is $b^\# = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Then $bb^\# b = \begin{pmatrix} 0 & x_{21} \\ 0 & 0 \end{pmatrix} \Rightarrow$ $b^\# = \begin{pmatrix} x_{11} & x_{12} \\ 1 & x_{22} \end{pmatrix}$. But then $bb^\# = \begin{pmatrix} x_{21} & x_{22} \\ 0 & 0 \end{pmatrix}$ and $b^\# b = \begin{pmatrix} 0 & x_{11} \\ 0 & 1 \end{pmatrix}$, a contradiction.

$\square$

The inverse along an element was introduced in [26]. We recall the definition and properties of this inverse. Note that in our paper, this new inverse is denoted by a^{-d} instead of $a^{\|d}$, extending the case $d = 1$. In this survey we follow the outline of the works [14, 15].

Definition 1.8. [26, Definition 4] Given elements $a, d \in S$, we say that a is invertible along d if there exists $b \in S$ such that $bad = d = dab$ and $b \in dS^1 \cap S^1 d$. If such an element b exists then it is unique (see [26, Theorem 6]) and is denoted by a^{-d}.

Another characterization is the following:

Lemma 1.9. *[26, Lemma 3] An element $a \in S$ is invertible along $d \in S$ if and only if there exists $b \in S$ such that $bab = b$, $bS^1 = dS^1$ and $S^1 b = S^1 d$. In this case $a^{-d} = b$.*

The inverse along an element is an outer inverse. It satisfies

$$a^{-d} = d(ad)^{\#} = (da)^{\#} d$$

and belongs to the double centralizer (double commutant) of $\{a, d\}$, see [26, Theorem 10]. It exists if and only if $d \in dadS^1 \cap S^1 dad$, for the details see [27].

Remark 1.10. For any $a^{=} \in a\{2\}$ $a^{=} = a^{-a^{=}}$. It follows that any outer inverse of a is an inverse of a along some element.

For specific choices of d, we recover the classical generalized inverses:

Lemma 1.11. *[26, Theorem 11]*
1. $a^{\#} = a^{-a}$,
2. $a^{\dagger} = a^{-a^*}$,
3. $a^D = a^{-a^n}$ *for a certain integer n.*

There is also another concept by Drazin [11]

Definition 1.12. Let S be any semigroup and let $a, b, c, y \in S$. We call y a (b, c)-inverse of a if the following two conditions hold:

1. $y \in (bSy) \cap (ySc)$;
2. $yab = b$, $cay = c$.

The main difference is that this notion uses two parameters instead of one. For example, (see [11, p. 2]), it can be useful when considering come parametric inverses, such as Chipman's "weighted inverse" ([3, pp. 114-176] and also [2, pp. 119-120]) and Cline and Greville's "W-weighted pseudo-inverse" ([7]). We prove that it is the only significant difference.

Lemma 1.13. *Suppose that $b = yab$ and $c = cay$. The following are equivalent:*

- $y \in (bSy) \cap (ySc)$;
- $y \in bS^1 \cap S^1 c$.

Proof. If $y = bny = ymc$ for some $n, m \in S$ then obviously $y \in bS^1 \cap S^1 c$.

Now suppose that $y \in bS^1 \cap S^1 c$. It follows that $y = bn = mc$ for some $m, n \in S \cup \{1\}$. So $y = bn = yabn = yay = b(na)y = y(am)c$ where $am, na \in S$ as $a \in S$. ☐

Corollary 1.14. *The (d, d) inverse of a is exactly the same as a^{-d}.*

Our survey paper basically follows the recent papers [14, 15] by Mary and the authors. It is divided to the parts described below. Section 2 is devoted to the study of the classical partial orders on semigroups via outer inverses. In particular, we give a new characterization of the Hartwig–Nambooripad order. In Section 3 we define new extensions of the Hartwig–Nambooripad partial order on arbitrary semigroups, and show that these extensions can be defined by outer inverses in the case of epigroups. In Section 4, we discuss partial orders based on outer inverses and inverses along an element and their connection to classical orders. In Section 4.2 the most general definition is given, and Section 4.3 proves the transitivity of the introduced relations. In Section 4.4 some of the introduced orders are compared with the sharp and Drazin partial orders. Section 4.5 contains the discussion of some further properties of the introduced relations. Finally in Section 5 we consider connection between introduced relations and Mitra unified theory.

2 Partial orders on semigroups and outer inverses

The first partial order on semigroups was defined on inverse semigroups by Vagner in 1952 [43], as the abtract counterpart of the inclusion of partial transformations in the case of the symmetric inverse semigroup.

Definition 2.1. Semigroup S is an inverse semigroup if for every $a \in S$ there exists $a^+ \in a\{1,2\}$ and such a^+ is unique.

Let S be an inverse semigroup. For $a, b \in S$ Vagner defined the partial order ω by $a\omega b$ if and only if (iff) $a^+a = a^+b$. Its restriction to the commutative subsemigroup of idempotents leads to the identification of commutative bands with semilattices. More generally, the set $E(S)$ of idempotents of a semigroup S can be partially ordered by the rule: for all $e, f \in E(S)$, $e\omega f$ if and only if $e = ef = fe$ (see Clifford and Preston [5] or Lyapin [25] for semigroups and semilattices, see also Kaplansky [24] for the ring case).

Then in 1980 Hartwig [17] and Nambooripad [39] discovered independently an extension of the previous partial orders for regular semigoups. Finally, in 1986, Mitsch [34] defined a partial order on arbitrary semigroups, that coincide with the previous ones on regular semigroups.

Since we do not restrict our attention to regular semigroups, the classical relations may fail to be reflexive. Therefore, we adopt in this paper the following convention: by a partial order, we always mean an antisymmetric and transitive relation only, no reflexivity is required. If it is needed, one can use the reflexive closure $\overline{\mathcal{K}} = \mathcal{K} \bigcup \{(a,a) | a \in S\}$ of the relation $\mathcal{K}$.

2.1 The natural partial order

The natural partial order on regular semigroups was defined in 1980 independently by Hartwig [17] and Nambooripad [39]. At this time, regular semigroups occupied already a prominent place within semigroup theory. This order was later extended by Mitsch to non-regular semigroups in [34]. If S is a regular semigroup then the relation ω can be defined by $a\omega b$ iff there exist $e, f \in E(S)$ such that $a = eb = bf$. For idempotents, the defined relation reduces to $e\omega f \Leftrightarrow ef = fe = e$. In [34], one finds many different ways to express the natural partial order (on regular semigroups). Some of them use reflexive inverses.

Lemma 2.2. *[34, Lemma 1] For a regular semigroup S, the following conditions are equivalent:*

1. *$a = eb = bf$ for some $e, f \in E(S)$;*
2. *$a = aa'b = ba''a$ for some $a', a'' \in a\{1,2\}$;*
3. *$a = aa'b = ba'a$ for some $a' \in a\{1,2\}$;*
4. *$a'a = a'b$ and $aa' = ba'$ for some $a' \in a\{1,2\}$, see also [17];*
5. *$a = ab'b = bb'a, a = ab'a$ for some $b' \in b\{1,2\}$;*
6. *$a = axb = bxa, a = axa, b = bxb$ for some $x \in S$;*
7. *$a = eb$ and $aS \subseteq bS$ for some idempotent e such that $aS^1 = eS^1$, see also [39];*
8. *$a = xb = by$, $xa = a$ for some $x, y \in S$.*

We also add the following equivalence due to Hartwig and Luh, see [28].

Lemma 2.3. *Let $a, b \in S$ and b be regular. Then the following two conditions are equivalent:*

1. *$a = xb = by$, $xa = a$ for some $x, y \in S$;*
2. *$a = bzb$ for some $z \in S$ and $b\{1\} \subseteq a\{1\}$.*

Let S be an arbitrary semigroup, $a, b \in S$. Following Drazin [9] and Petrich [40] we introduce here the notations:

- $a\mathcal{J}b$ iff $a = eb = bf$ for some $e, f \in E(S)$;
- $a <^- b$ iff $a^-a = a^-b$ and $aa^- = ba^-$ for some $a^- \in a\{1\}$;
- $a\mathcal{N}b$ iff $a = axa = axb = bxa$ for some $x \in S$;
- $a\mathcal{M}b$ iff $a = xb = by, xa = a$ for some $x, y \in S^1$;
- $a\mathcal{P}b$ iff $a = pa = pb = bp = ap$ for some $p \in S^1$;
- $a\mathcal{H}b$ iff $a = bxb$ for some $x \in S^1$ and $b\{1\} \subseteq a\{1\}$.

Here the relations $\mathcal{J}$, $\mathcal{N}$, $\mathcal{M}$, $\mathcal{P}$ and $\mathcal{H}$ are named after Jones, Nambooripad, Mitsch, Petrich and Hartwig correspondingly. For any semigroup S, it holds that $\mathcal{N} \subseteq \mathcal{J} \subseteq \mathcal{M}$, ($\mathcal{N}$ is stronger than $\mathcal{J}$ which is stronger than $\mathcal{M}$). In addition, $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{P}$ are partial orders, see Lemma 3.1 and Lemma 9.1 in [40]. Relations $\mathcal{M}$ and $\mathcal{P}$ are always reflexive, but $\mathcal{N}$ is reflexive only on regular semigroups. Actually, it holds that on regular semigroups, all the previous relations coincide. Relation $<^-$ is due to Hartwig [17] and called the minus partial order.

Lemma 2.4. *(Folklore) For any semigroup S it holds that $<^- = \mathcal{N}$.*

Also this proves that $\mathcal{N}$ is a partial order (Lemma 3.1 in [40]).

The detailed account for properties of minus order on semigroups, monoids, and rings can be found in [17, 6, 39]. An obvious advantage of this partial order to compare with the others [1, 6, 10, 17, 19] is that its properties are strongly improved if one improves the ring properties of the base algebraic structure, namely, changes the semigroup S to a monoid or to a ring satisfying certain conditions. Finally, in the case of matrix ring over the field of complex numbers the minus-order has a lot of different equivalent characterizations. In particular, Hartwig in [17] has considered the following binary relation called the rank-subtractivity. Let $M_n(\mathbb{F})$ denote the set of all $n \times n$ matrices with the coefficients from an arbitrary field $\mathbb{F}$.

Definition 2.5. A pair of matrices $A, B \in M_n(\mathbb{F})$ is called rank-subtractive if

$$\operatorname{rk}(B - A) = \operatorname{rk} B - \operatorname{rk} A \ . \tag{1}$$

It worth to note that the equality 1 is equivalent to the inequality $\operatorname{rk}(B-A) \leq \operatorname{rk} B - \operatorname{rk} A$ since the inequality $\operatorname{rk}(B - A) \geq \operatorname{rk} B - \operatorname{rk} A$ holds for arbitrary matrices A and B.

This relation defines a partial order on matrices. Indeed, it is clearly reflexive and anti-symmetric, it is also transitive since if pairs (A, B) and (B, C) are rank-subtractive then one has

$$\operatorname{rk}(C - A) = \operatorname{rk}((B - A) + (C - B)) \leq \operatorname{rk}(B - A) + \operatorname{rk}(C - B) =$$

$$= \operatorname{rk} B - \operatorname{rk} A + \operatorname{rk} C - \operatorname{rk} B = \operatorname{rk} C - \operatorname{rk} A,$$

i.e., the pair (A, C) is rank-subtractive.

In [17, Theorem 3.2] it is proved that rank-subtractivity partial order is equivalent to minus-order on $M_n(\mathbb{F})$.

It is known that for matrices over a field, $a <^- b \Leftrightarrow b\{1\} \subseteq a\{1\}$ ([28, Theorem 2.2]). Since its appearance, Mitsch's partial order has been extensively studied, see for example [34], [35], [21], [40], [36], [37]. Whereas relation $\mathcal{H}$ appears implicitly in Hartwig's work, it has not been studied as a relation on arbitrary semigroups.

2.2 Characterization of the relation $\mathcal{N}$ by means of outer inverses

The known characterizations of the natural partial order on a regular semigroup involve reflexive inverses (inner inverses do the same job since the minus order coincide with the Nambooripad order). However, characterizations involving outer inverses were usually not studied until recently, see [31], [4], [41]. The following proposition gives equivalent characterizations of relation $\mathcal{N}$ in terms of outer inverses (The equivalence 1. $\Leftrightarrow$ 4. is stated in [31], where in fact the implication 1. $\Rightarrow$ 4. is proved, see [31, Lemma 4]).

Proposition 2.6. *[14, Proposition 1] Let $a, b \in S$. Then the following statements are equivalent:*

1. *$a = bb^{=}b$ for some $b^{=} \in b\{2\}$;*
2. *$a = ab^{=}a = ab^{=}b = bb^{=}a$ for some $b^{=} \in b\{2\}$;*
3. *$a = ab_l^{=}a = ab_l^{=}b = bb_r^{=}a$ for some $b_l^{=}, b_r^{=} \in b\{2\}$;*
4. *$a = axa = axb = bya$ for some $x, y \in S$;*
5. *$a = axa = axb = bxa$ for some $x \in S$.*

3 Extending orders on arbitrary semigroups

The proofs of the results provided in this section can be found in [14].

3.1 Definitions

By definition, for $a, b \in S$, $a\mathcal{N}b$ implies that a is regular. To compare non-regular elements, we define the relation Γ as follows:

Definition 3.1. Let $a, b \in S$. Then $a\Gamma b$ if there exist $x, y \in S^1$ such that $a = axb = bya$ and $b\{1\} \subseteq a\{1\}$.

Obviously, $\Gamma \subseteq \mathcal{H}$ as $a\Gamma b$ implies $a = byaxb$ for $a, b \in S$.

To compare relation Γ with Nambooripad's relation $\mathcal{N}$, we define the left, right and the symmetric version of Γ, in an apparently different form.

Definition 3.2. Let $a, b \in S$. We define relations Γ_l, Γ_r, and $\Gamma_{\mathcal{P}}$ on $S \times S$ as follows:

1. If b is not regular, then $a\Gamma_l b$ (resp. $a\Gamma_r b$, $a\Gamma_{\mathcal{P}} b$) iff there exists $x \in S^1$ such that $a = axb$ (resp. there exists $y \in S^1$ such that $a = bya$, there exist $x \in S^1$ such that $a = axb = bxa$);
2. If b is regular, then $a\Gamma_l b$ (resp. $a\Gamma_r b$, $a\Gamma_{\mathcal{P}} b$) iff there exist $x, y \in S^1$, such that $a = axa = axb = bya$ (resp. there exist $x, y \in S^1$, such that $a = aya = axb = bya$, there exists $x \in S^1$, such that $a = axa = axb = bxa$).

It happens that Γ is the intersection of Γ_l and Γ_r.

Lemma 3.3. *Let $a, b \in S$. The following statements are equivalent:*

1. $a\Gamma b$;
2. $a\Gamma_l b$ *and* $a\Gamma_r b$.

The derivation of $\Gamma_{\mathcal{P}}$ from Γ is then similar to Petrich's definition of partial order $\mathcal{P}$ from Mitsch's $\mathcal{M}$.

Corollary 3.4. *Let $a, b \in S$. The following statements are equivalent:*

1. $a\Gamma_{\mathcal{P}} b$;
2. *There exists $x \in S^1$ such that $a = axb = bxa$, and $b\{1\} \subseteq a\{1\}$.*

By Proposition 2.6, if b is regular, then $a\Gamma_l b$ (resp. $a\Gamma_r b$, $a\Gamma b$, $a\Gamma_{\mathcal{P}} b$) if and only if $a\mathcal{N}b$ (Lemma 3.5 below). It then holds trivially that $\mathcal{N} \subseteq \Gamma_{\mathcal{P}} \subseteq \Gamma$ by Proposition 2.6, therefore Γ may be seen as an extension of the Hartwig-Nambooripad order.

Lemma 3.5. *Let $a, b \in S$ such that b is regular. Then the following statements are equivalent;*

1. $a\Gamma_l b$;
2. $a\Gamma_r b$;
3. $a\mathcal{N}b$;
4. $a\Gamma_{\mathcal{P}} b$;
5. *There exists $b^+ \in b\{1,2\}$, $a = ab^+b = bb^+a = ab^+a$.*

Corollary 3.6. *Let S be a regular semigroup. Then $\Gamma = \Gamma_l = \Gamma_r = \Gamma_{\mathcal{P}}$ is the natural partial order.*

3.2 The orders $\mathcal{H}$, $\Gamma_l, \Gamma_r, \Gamma$ and $\Gamma_{\mathcal{P}}$

Let S be an arbitrary semigroup. Recall that partial orders are not assumed reflexive in the paper. Actually, relations $\mathcal{H}$, $\Gamma_l, \Gamma_r, \Gamma$ and $\Gamma_{\mathcal{P}}$ are reflexive iff S is regular.

Lemma 3.7. *Relations $\mathcal{H}, \Gamma_l\Gamma_r, \Gamma$ and $\Gamma_{\mathcal{P}}$ are partial orders.*

The following example shows that the relations Γ_l and Γ_r are different in general.

Example 3.8. Let $M = \langle a, b | ab = a \rangle$ be the free monoid generated by a and b quotiented by the relation $ab = a$. Elements of M are of the form $b^q a^p$ ($p \geq 0, q \geq 0$) and $b^q a^p b^n a^m = b^q a^{p+m}$ if $p > 0$ and $b^{q+n} a^m$ when $p = 0$. The identity is the only regular element in M. As $a = axb$ for $x = 1$ then $a\Gamma_l b$. But a is not below b for Γ_r.

Relations Γ and $\mathcal{M}$ are not comparable in general. The first examples below show that each of Γ, Γ_l, Γ_r, $\Gamma_{\mathcal{P}}$ does not imply relation $\mathcal{M}$.

Example 3.9. Let $S = \langle a, b | ab = ba = a\rangle$ be the free semigroup generated by a and b quotiented by the relation $ab = ba = a$. Elements of S are of the form a^p, b^q $(p > 0, q > 0)$ and $a^p b^q = b^q a^p = a^p$. There are no regular elements in S. As $a = axb = bxa$ for $x = 1$ then $a\Gamma b$ (also $a\Gamma_{\mathcal{P}} b$). But a is not below b for Mitsch's partial order. Indeed, if $a = xb$ then $x = a$ and $xa = a^2 \neq a$.

Let us remind that a semigroup is *nilpotent* of degree k if it has a zero, every product of k elements equals the zero, and some product of $(k-1)$ elements is non-zero.

Example 3.10. Let $S = N \times G$ be the direct product of a commutative nilpotent semigroup N by a group G, and let $a = (a_1, a_2), b = (b_1, b_2) \in S$. The only regular element of N is 0_N.

1. We first consider Mitsch's order. Assume that $a\mathcal{M}b$. Then there exist $x = (x_1, x_2), y = (y_1, y_2) \in S^1$ such that $a = xa = xb = by$. It follows that $x_1^p a_1 = a_1$ for all $p \geq 0$ and $a_1 = 0_N$, or $x_1 = 1$ and $a = b$. Also as $a = xa$ then $a_2 = x_2 a_2$ and $x_2 = 1_G$. Then $a_2 = x_2 b_2 = b_2$. As conversely $(0_N, a_2)\mathcal{M}(n, a_2)$ for any $n \in N$ then $a\mathcal{M}b$ if and only if $a_1 = 0_N$ and $a_2 = b_2$, or $a = b$.

2. We consider now the relation Γ_l. Assume that $a\Gamma_l b$. If b is regular then $b_1 = 0_N$. There exist $x = (x_1, x_2), y = (y_1, y_2) \in S^1$ such that $a = axa = axb = bya$ and therefore $a_1 = 0$ and $a_2 = b_2$. But in this latter case $a = b = (0, a_2)$, thus $a = b$. If $b_1 \neq 0_N$ then b is not regular. There exists $x = (x_1, x_2) \in S^1$ such that $a = axb$. In this latter case $a_1 = 0$. But conversely, for any b such that $b_1 \neq 0_N$ then $(0, a_2) = (0, a_2)(0, b_2^{-1})(b_1, b_2)$ and $a\Gamma_l b$. As it also holds that $(0, a_2) = (b_1, b_2)(0, b_2^{-1})(0, a_2)$ then $\Gamma_l = \Gamma_r = \Gamma = \Gamma_{\mathcal{P}}$. Note that $(0, b_2^{-1})$ is an outer inverse of b.

Next example shows that the relations $\mathcal{P}$, $\mathcal{M}$, and $\mathcal{J}$ do not imply the order Γ.

Example 3.11. Let $S = \langle a, e, b | be = eb = a, e^2 = e\rangle$. As $a = eb$ and $e^2 = e$ then $ea = a$ and it holds that $a = ea = eb = be$, that is $a\mathcal{P}b$ (and in particular $a\mathcal{M}b$, also $a\mathcal{J}b$). But a may not be written as a product involving a and b and a is not below b for the partial order Γ. Indeed elements of S are of the form e, a^p, b^q $(p > 0, q > 0)$ with the product $a^p b^q = b^q a^p = a^{p+q}$, $a^p e = e a^p = a^p$ and $b^q e = e b^q = a^q$ $(p > 0, q > 0)$.

Below we show that the relation $\mathcal{H}$ implies neither Γ_l, nor $\mathcal{M}$.

Example 3.12. Let S be the set of all nonnegative integers with an operation of addition. The only regular element in S is 0. As $5 = 2 + 1 + 2$ then $5\mathcal{H}2$ but $5 \not\Gamma_l 2$ as $5 + x + 2 \neq 5$ for all $x \in \mathbb{N}$. Also $5 \not\mathcal{M} 2$ as $x + 2 = 2$ implies $x = 0$.

3.3 Characterization based on outer inverses

As we have seen in the case of a regular semigroup, the partial order $\Gamma(=\Gamma_l = \Gamma_r = \Gamma_{\mathcal{P}})$ coincides with the natural partial order which admits characterizations by means of outer inverses (Proposition 2.6).

More generally, for a regular element a and an arbitrary element b, the relations $a\Gamma_l b$ (resp. $a\Gamma_r b, a\Gamma b, a\Gamma_{\mathcal{P}} b$) can be defined by means of outer inverses of b.

Proposition 3.13. *Let $a, b \in S$ such that a is regular. Then the following statements are equivalent.*

1. $a = axb$ *for some* $x \in S^1$;
2. $a = ab_l^= b$ *for some* $b_l^= \in b\{2\}$.

If moreover $a = axa = axb = bya$ for some $x, y \in S^1$, then it actually holds that $a = ab^= a = ab^= b = bb^= a$ for some $b^= \in b\{2\}$.

Dual arguments give us the following corollary:

Corollary 3.14. *Let $a, b \in S$ such that a is regular. Then the following statements are equivalent.*

1. $a = bya$ *for some* $y \in S^1$;
2. $a = bb_r^= a$ *for some* $b_r^= \in b\{2\}$.

Additional assumptions $a = axb = bya = aya$ for some $x, y \in S^1$ leads to $a = ab^= a = ab^= b = bb^= a$ for some $b^= \in b\{2\}$.

Also the following statements are equivalent.

1. $a = axb = bya$ *for some* $x, y \in S^1$;
2. $a = ab_l^= b = bb_r^= a$ *for some* $b_l^=, b_r^= \in b\{2\}$.

Additional assumption $axa = a$ leads to $a = ab^= a = ab^= b = bb^= a$ for some $b^= \in b\{2\}$.

Finally we consider the relation $\mathcal{H}$.

Lemma 3.15. *Let $a, b \in S$ such that a is regular. Then the following statements are equivalent.*

1. $a = bxb$ *for some* $x \in S^1$;
2. $a = ab_l^= b = bb_r^= a$ *for some* $b_l^=, b_r^= \in b\{2\}$.

We deduce from the previous results the following corollary.

Corollary 3.16. *Let $a, b \in S$ such that a is regular. Then the following statements are equivalent.*

1. $a\mathcal{H}b$;

2. $a = ab_l^= b = bb_r^= a$ for some $b_l^=, b_r^= \in b\{2\}$, and moreover, if b is regular, then $a = ab^= a = ab^= b = bb^= a$ for some $b^= \in b\{2\}$;

3. $a\Gamma b$.

In epigroups (in particular finite or periodic semigroups), the previous characterizations by outer inverses remain valid for a non-regular element a of S. Recall that for epigroups the Jones's partial order $\mathcal{J}$ and the Mitsch's partial order $\mathcal{M}$ coincide ([21, Proposition 2.8]).

Proposition 3.17. *Let S be an epigroup, and $a, b \in S$. Then the following statements are equivalent.*

1. $a\Gamma_l b$;

2. $a = ab_l^= b$ *for some* $b_l^= \in b\{2\}$ *and if b is regular, then*

$$a = ab^= a = ab^= b = bb^= a$$

for some $b^= \in b\{2\}$.

Proof. Let $a, b \in S$ such that $a\Gamma_l b$. Then there exists $x \in S^1$, such that $a = axb$. As $a = axb$ then $a = a(xb)^n$ for all $n \geq 0$. Let m be the Drazin index of xb. It holds that $(xb)^{n+1}(xb)^D = (xb)^n$ for all $n \geq m$, and $(xb)^D \in (xb)\{2\}$ commutes with xb. It follows that $[(xb)^D x]b[(xb)^D x] = [(xb)^D x]$ and $(xb)^D x \in b\{2\}$. But also $a[(xb)^D x]b = axb[(xb)^D] = a(xb)^{m+1}(xb)^D = a(xb)^m = a$. If b is regular and $a\Gamma_l b$, then a is regular and by Proposition 3.13 $a = ab^= a = ab^= b = bb^= a$ for some $b^= \in b\{2\}$. The converse is straightforward. ☐

By dual arguments, the same statement holds for Γ_r, Γ and $\Gamma_{\mathcal{P}}$.

Remark 3.18. If in Example 3.10 we take N and G finite, then semigroup $S = N \times G$ is finite whence S is an epigroup, and still we can find elements $a, b \in S$ such that $a\Gamma b$ but $a \not\mathcal{M} b$. Also, assume that N is of nilpotency index more than 4, and let $b_1 \in N$, $b_1^3 \neq 0_N$. Then set $b = (b_1, 1_G)$ and $a = b^3$. As a and b are not regular, then $a\mathcal{H}b$, but as $a_1 = b_1^3 \neq 0_N$, then $a \not\Gamma b$.

4 Partial orders based on inverses and parametric functions

There are many other partial orders on semigroups which can be defined via the generalized inverse elements, so-called G-based orders. Detailed and self-contained information on the topic can be found in the monograph [33]. Recently unified approach to different generalized inverses via so-called inverses along elements was developed in the works [26, 11]. The nice feature of inverses along elements is that this concept recovers classical generalized inverses by specification of the chosen elements. In particular, the group inverse, Drazin inverse, Moore-Penrose, etc. appear as particular cases.

The aim of the present section is to introduce a new family of partial order relations on semigroups on the base of generalized inverses along the elements and to investigate their algebraic properties. It turns out that this leads to general unified theory of partial orders on semigroups and well-known semigroup order relations can be obtained using our approach. We recall that in order to simplify definitions, by a partial order we always mean an antisymmetric and transitive relation only.

This sections is based on results from [15]. Since original proofs can be found there, most of them are omitted here.

4.1 Partial orders based on outer inverses and inverses along an element

The orders which are finer than the minus partial order and which are based on specific inner inverses have been extensively studied, see, for example [29], [30], [31], [32], [20], [42], [41]. On the other hand, the study of partial orders based on specific outer inverses (in the sense of Definition 1.1) is relatively new, and goes back to Mitra and Hartwig [31]. In this paper, $\Theta : S \to 2^S$ denotes a (multi-valued) function. We will specifically study the case investigated in [31] where Θ sends an element to a (possibly empty) subset of its outer inverses: $\Theta(x) \subseteq x\{2\}$ for any $x \in S$. Mitra and Hartwig defined a relation $<^\Theta$ as follows.

Definition 4.1. [31] For $a, b \in S$ $a <^\Theta b$ means that there exists an outer inverse $b^=$ of b, $b^= \in \Theta(b)$, such that $a = bb^=b$.

It is proved in [31, Lemma 6] that on regular semigroups, any partial order finer than the minus partial order is of this form for a specific choice of function Θ. The two drawbacks of this definition are:

- if $a <^\Theta b$ then a is regular, hence the relation is not suitable for comparing non-regular elements;
- $<^\Theta$ is not a partial order in general, as next example shows.

Example 4.2. Let $S = \mathbb{M}_3(\mathbb{Z}_2)$, and pose $a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\delta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let Θ be a map such that $\Theta(c) = \{\delta\}$ and $\Theta(b) = \{d\}$. It is easy to verify that $\delta c \delta = \delta$ and $dbd = d$. As $b = c\delta c$ and $a = bdb$ then $a <^\Theta b$ and $b <^\Theta c$. But $a \neq c\delta c$ hence $a \not<^\Theta c$.

4.2 Definitions and general properties

Following the same lines as [31], given a (multi-valued) function $\Theta : S \to 2^S$ with values in the set of of outer inverses, $\Theta(a) \subseteq a\{2\} (\forall a \in S)$, we define the relation $\mathcal{N}^\Theta$ by replacing any instance of x in the definition by elements of $\Theta(b)$.

Definition 4.3. Let $a, b \in S$.

1. $a\mathcal{N}^\Theta b$ if there exists $b^= \in \Theta(b)$, such that $a = ab^=a = ab^=b = bb^=a$.
2. $a\Gamma^\Theta b$ if there exist $b_l^=, b_r^= \in \Theta(b)$, such that $a = ab_l^=b = bb_r^=a$ and $b\{1\} \subseteq a\{1\}$.
3. If b is not regular, then $a\Gamma_l^\Theta b$ if there exists $b_r^= \in \Theta(b)$, such that $a = ab_r^=b$.
4. If b is regular, then $a\Gamma_l^\Theta b$ if there exist $b_l^=, b_r^= \in \Theta(b)$, such that $a = ab_l^=a = ab_l^=b = bb_r^=a$.
5. If b is not regular, then $a\Gamma_r^\Theta b$ if there exists $b_r^= \in \Theta(b)$, such that $a = bb_r^=a$.
6. If b is regular, then $a\Gamma_r^\Theta b$ if there exist $b_l^=, b_r^= \in \Theta(b)$, such that $a = ab_r^=a = ab_l^=b = bb_r^=a$.
7. If b is not regular, then $a\Gamma_\mathcal{P}^\Theta b$ if there exists $b^= \in \Theta(b)$, such that $a = ab^=b = bb^=a$.
8. If b is regular, then $a\Gamma_\mathcal{P}^\Theta b$ if there exist $b^= \in \Theta(b)$,such that $a = ab^=a = ab^=b = bb^=a$.

It happens that Γ^Θ is the intersection of Γ_l^Θ and Γ_r^Θ.

Lemma 4.4. $\Gamma^\Theta = \Gamma_l^\Theta \cap \Gamma_r^\Theta$.

As a consequence, it always holds that

$$\mathcal{N}^\Theta \subseteq \Gamma_\mathcal{P}^\Theta \subseteq \Gamma_l^\Theta \cap \Gamma_r^\Theta = \Gamma^\Theta.$$

Of special interest will be the following functions Θ:

- $\Theta : b \mapsto b\{2\}$. In this case we have $\mathcal{N}^\Theta = \mathcal{N} =<^\Theta$.
- $\Theta^\# : b \mapsto \{b^\#\}$, the group inverse of b, or $\Theta^D : b \mapsto \{b^D\}$, the Drazin inverse of b.
- Let $\Delta : S \to 2^S$. We pose $\Theta_\Delta : b \mapsto \{b^{-d} | d \in \Delta(b)\}$. Here, for $b \in S$, $\Delta(b)$ is not included in $b\{2\}$ in general, but $\Theta(b)$ is. To simplify the notations, we denote the relation $<^{-\Delta}$ (resp. $\mathcal{N}^{-\Delta}$, $\Gamma^{-\Delta}$, $\Gamma_l^{-\Delta}$, $\Gamma_r^{-\Delta}$, $\Gamma_\mathcal{P}^{-\Delta}$) instead of $<^{\Theta_\Delta}$ (resp. $\mathcal{N}^{\Theta_\Delta}$, Γ^{Θ_Δ}, $\Gamma_l^{\Theta_\Delta}$, $\Gamma_r^{\Theta_\Delta}$, $\Gamma_\mathcal{P}^{\Theta_\Delta}$). For instance, if $\Delta^\#$ is such that $\Delta^\#(b) = b$ for each $b \in S$, then $\Theta_{\Delta^\#} = \Theta^\#$.

Let us illustrate these notions.

Example 4.5. Let $S = T_3$ be the semigroup, which consists of all maps from the set $\{1, 2, 3\}$ to itself with the composition operation. We write (mnk) for the function which sends 1 to m, 2 to n, and 3 to k.

Let $a = (333), b = (131)$. Then there exists $b' = (222)$ such that $a = ab'b = bb'a = ab'a$ or equally $a\Gamma_{\mathcal{P}}b$. Let $\Delta : S \to 2^S$ such that $\Delta(b) = \{x = (111), y = (333)\}$. Obviously x and y are outer inverses of b. Hence $b^{-x} = x$ and $b^{-y} = y$. But $bb^{-x}a = (131)(111)(333) = (131)(111) = (111) \neq a$ and $bb^{-y}a = (131)(333)(333) = (131)(333) = (111)$. Finally, there is no any $d \in \Delta(b)$ such that $a = bb^{-d}a$. Hence $a \cancel{\Gamma_l^{-\Delta}} b$.

Lemma 4.6. *Let $a, b \in S$ such that $a <^{\Theta} b$ for some Θ. Then $b\{1\} \subseteq a\{1\}$.*

Lemma 4.7. *Let $\Delta = \Theta$. Then $<^{-\Delta} = <^{\Theta}$ (resp. $\mathcal{N}^{-\Delta} = \mathcal{N}^{\Theta}$, $\Gamma^{-\Delta} = \Gamma^{\Theta}$, $\Gamma_l^{-\Delta} = \Gamma_l^{\Theta}$, $\Gamma_r^{-\Delta} = \Gamma_r^{\Theta}$ and $\Gamma_{\mathcal{P}}^{-\Delta} = \Gamma_{\mathcal{P}}^{\Theta}$).*

The following lemma is the direct consequence of the definitions:

Lemma 4.8. *Let $a, b \in S$ and $b^{=} \in b\{2\}$ such that $a = bb^{=}b$. Then*

1. *$a = ab^{=}a = ab^{=}b = bb^{=}a$;*
2. *if, in addition, b is invertible along d and $a = bb^{-d}b$ then a is invertible along d with $a^{-d} = b^{-d}$ and a^{-d} is a reflexive inverse of a.*

Example 4.9. Let us show, that the expressions $ab^{-d}b$, $bb^{-d}a$, $ab^{-d}a$ and $bb^{-d}b$ might be different elements of S. Let $\mathbb{F}$ be an arbitrary field, $S = M_3(\mathbb{F})$,

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; d = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in S.$$

Then $dbd = d$ and therefore b is invertible along d. In this case $ab^{-d}b = c$, $bb^{-d}a = a$, $ab^{-d}a = d$ and $bb^{-d}b = b$.

Corollary 4.10. $<^{\Theta} \subseteq \mathcal{N}^{\Theta}$.

In general $a\mathcal{N}^{\Theta}b$ does not imply $a <^{\Theta} b$ as the following example shows.

Example 4.11. Let $S = \mathbb{M}_3(\mathbb{Z}_2)$. We consider $a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and the map Δ such that $\Delta(b) = \{d\}$. Then it is easy to verify that:

1. b is invertible along d with $b^{-d} = d$.
2. $a = ab^{-d}b = bb^{-d}a = ab^{-d}a$.
3. $a \neq bb^{-d}b$.

Finally, $a\mathcal{N}^{-\Delta}b$ and $a \not<^{-\Delta} b$.

Theorem 4.12. *The relations $\mathcal{N}^\Theta$, Γ_l^Θ, Γ_r^Θ, $\Gamma_\mathcal{P}^\Theta$ and Γ^Θ are partial orders.*

Next theorem gives an equivalent characterization of relation $<^\Theta$ in so-called G-based form (see [32]) and as the relation $\mathcal{N}^\Theta$ with an additional condition.

Theorem 4.13. *The following are equivalent:*

1. $a <^\Theta b$;
2. *There exists* $b^= \in \Theta(b) \cap a\{2\}$ *such that* $a = ab^=b = bb^=a = ab^=a$;
3. *There exists* $a^+ \in \Theta(b) \cap a\{1, 2\}$ *such that:*
   - $aa^+ = ba^+$;
   - $a^+a = a^+b$.

Finally we consider compatibility with multiplication by invertible elements.

Lemma 4.14. *Let* $b^= = b^=bb^=$. *Then for any invertible* $p \in S$ *it holds that:*

1. $(p^{-1}b^=) = (p^{-1}b^=)(bp)(p^{-1}b^=)$ *or equally* $p^{-1}b^= \in (bp)\{2\}$;
2. $(b^=p^{-1}) = (b^=p^{-1})(pb)(b^=p^{-1})$ *or equally* $b^=p^{-1} \in (pb)\{2\}$.

Lemma 4.15. *Let* $a, b, p \in S$ *and* p *be invertible.*
If $p^{-1}\Theta(b) \subseteq \Theta(bp)$, *then* $a <^\Theta b \Rightarrow ap <^\Theta bp$.
In particular, if $\Theta(x) = x\{2\}\ \forall x \in S$, *the following are equivalent.*

1. $a <^\Theta b$;
2. $ap <^\Theta bp$;
3. $pa <^\Theta pb$.

Corollary 4.16. *By dual agruments, if* $\Theta(b)p^{-1} \subseteq \Theta(pb)$, *then* $a <^\Theta b \Rightarrow pa <^\Theta pb$.

Corollary 4.17. *The same statements hold for* $\mathcal{N}^\Theta$.

4.3 Transitivity of $<^{-\Delta}$

In [31], the question of transitivity of $<^\Theta$ was investigated. We specialize here to the case where $\Theta(b) = \Theta_\Delta(b) = \{b^{-d} | d \in \Delta(b)\}$.

Proposition 4.18. *Assume that for all* $b, c, d, \delta \in S$ *satisfying the conditions*
1. *b invertible along* $d \in \Delta(b)$,
2. *c invertible along* $\delta \in \Delta(c)$,
3. $b < c$ *with* $b = cc^{-\delta}c$,

there exists $t \in \Delta(c)$ *such that* $(c^{-\delta}cb^{-d}cc^{-\delta})S^1 = tS^1$ *and* $S^1(c^{-\delta}cb^{-d}cc^{-\delta}) = S^1t$.

Then $<^{-\Delta}$ *is transitive.*

Proof. Let $a, b, c \in S$ be such that $a <^{-\Delta} b$ and $b <^{-\Delta} c$. Then there exists $d \in \Delta(b)$ such that b is invertible along d and $a = bb^{-d}b$, suppose that there exists $\delta \in \Delta(c)$, such that c is invertible along δ and $b = cc^{-\delta}c$.

Consider $x = c^{-\delta}cb^{-d}cc^{-\delta}$, then $cxc = bb^{-d}b = a$ and

$$xcx = c^{-\delta}cb^{-d}cc^{-\delta}cb^{-d}cc^{-\delta} = c^{-\delta}cb^{-d}bb^{-d}cc^{-\delta} = x.$$

It follows that x is an outer inverse of c, and in particular $x = c^{-x}$. Since $xS^1 = tS^1$ and $S^1t = S^1x$, $c^{-x} = c^{-t}$ by Lemma 1.9. It follows that there exists $t \in \Delta(c)$ such that $a = cc^{-t}c$. Finally, $a <^{-\Delta} c$.

□

Corollary 4.19. *Assume that $\Delta : S \to 2^S$ is a constant function ($\Delta(x) = \Delta_0$ for some fixed $\Delta_0 \subseteq S$) such that either*

- *$\Delta_0 = \delta_0 S^1, S^1\delta_0, S^1\delta_0 S^1$ or $\delta_0 S^1 \cap S^1\delta_0$ for an element $\delta_0 \in S$ (Δ_0 is a right, left, two-sided principal ideal, or the intersection of the left and right principal ideals generated by δ_0);*
- *if $c \in \Delta_0$ then $d \in \Delta_0 \Leftrightarrow cS^1 = dS^1$ (Δ_0 is an R-class, for Green's relation R, see [13], [22, Chapter 2]);*
- *if $c \in \Delta_0$ then $d \in \Delta_0 \Leftrightarrow S^1c = S^1d$ (Δ_0 is an L-class, for Green's relation L);*
- *if $c \in \Delta_0$ then $d \in \Delta_0 \Leftrightarrow cS^1 = dS^1$ and $S^1c = S^1d$ (Δ_0 is an H-class, for Green's relation H).*

Then $<^{-\Delta}$ is a partial order.

Note that in the case of a single-valued constant function, transitivity holds in a trivial way.

Lemma 4.20. *Let $\delta \in S$ and define $\delta : S \to 2^S$ by $\delta(x) = \{\delta\}$. Assume also that $a, b, c \in S$ are such that $a <^{-\delta} b$ and $b <^{-\delta} c$. Then $a = b$.*

But it also holds trivially in other cases.

Proposition 4.21. *Define $\Theta^D : S \to 2^S$ by $\Theta^D(x) = \{x^D\}$, Drazin inverse of x, and let $a, b, c \in S$ such that $a <^{\Theta^D} b$ and $b <^{\Theta^D} c$. Then $a = b$.*

The conclusion then also holds for $\Theta^\# : x \mapsto \{x^\#\}$, as the group inverse is a special Drazin inverse. More generally, if $\Theta(b) \subseteq b\{1\}$, then $a <^{\Theta} b$ implies $a = b$.

4.4 Comparison with the sharp partial order and the Drazin partial order

We now consider the case of special non-constant functions. Recall that $C(x)$ denotes the centralizer of x and $CC(x) = C(C(x))$ its double centralizer.

Proposition 4.22. *Let $\Delta : S \to 2^S$ sends each x to a subset of its centralizer namely, $\Delta(x) \subset C(x)$. Then $a <^{-\Delta} b$ implies that $ab = ba$.*

The sharp partial order is defined in [29] by $a <^{\#} b$ iff $aa^{\#} = ba^{\#} = a^{\#}b$. It actually coincides with the partial order on completely regular semigroups (semigroups with all elements group invertible) defined by Drazin in [9] by $a\mathcal{S}b$ iff $a^2 = ba = ab$. (just multiply on the left and on the right either by a^2 or by $(a^{\#})^2$).

Corollary 4.23. *Let $C : b \mapsto C(b)$. Then $<^{-C}$ is the sharp partial order.*

Proof. Let $a <^{-C} b$. Then there exists $d \in C(b)$ such that $a = bb^{-d}b$. By Lemma 4.8, b^{-d} is an inner and outer inverse of a. As b and d commute, they belong to $C(\{b,d\})$ hence they commute with b^{-d}. It follows that $a = bb^{-d}b$ commutes with b^{-d}, that is a is group invertible with $a^{\#} = b^{-d}$. Then $aa^{\#} = bb^{-d}bb^{-d} = bb^{-d} = ba^{\#}$ and $a^{\#}a = b^{-d}bb^{-d}b = b^{-d}b = a^{\#}b$, hence $a <^{\#} b$.
Conversely, assume $a <^{\#} b$, that is $aa^{\#} = ba^{\#}$ and $a^{\#}a = a^{\#}b$. As $aa^{\#} = a^{\#}a$, then $a^{\#}$ commute with b. As also $a^{\#} = b^{-a^{\#}}$, then $bb^{-a^{\#}}b = ba^{\#}b = ba^{\#}a = aa^{\#}a = a$. Finally $a <^{-C} b$. □

Corollary 4.24. *Let $\Theta_C : b \mapsto C(b) \cap b\{2\}$. Then $<^{\Theta_C}$ is the sharp partial order.*

Proposition 4.25. *For all $b \in S$, $\Theta_C(b) = \{b^{-d} | d \in C(b)\}$.*
As a consequence, all of the defined relations remain unchanged under the substitution Θ_C for $-C$.

Next, we give a third characterization of the sharp partial order based on centralizers.

Proposition 4.26. *Let $a, b \in S$. Then the following statements are equivalent:*

1. *$a <^{\Theta_C} b$;*
2. *$a <^{\#} b$;*
3. *$a <^{-} b$ in the semigroup $C(b)$.*

Since $<^{-C} \subseteq \mathcal{N}^{-C}$, the sharp partial order is finer than $\mathcal{N}^{-C}$. Next example shows that in general it is strictly finer.

Example 4.27. Let $S = T_3$ and pose $a = (112)$, $b = (132)$. Then a is not group invertible. Indeed, suppose that there exists $a^{\#}$. Then $a = aa^{\#}a = aaa^{\#} = (111)a^{\#} = (111) \neq a$, contradiction. As a result $a \not<^{\#} b$. But it is easy to see that $b^{-1} = b$. Let $b^{=} = b \in C(b)$. Then $a = ab^{=}b = bb^{=}a$ and $ab^{=}a = (112)(132)(112) = (112) = a$.

Next we study relation $<^{-CC}$ based on the double centralizer. We start with a useful lemma.

Lemma 4.28. *Let $a \in S$ be invertible along $d \in CC(a)$. Then $a^{-d} \in CC(a)$.*

Corollary 4.29. *Relation $<^{-CC}$ is a partial order.*

As $CC(x) \subseteq C(x)$ for all $x \in S$, then $<^{-CC} \subseteq <^{-C}$ and $<^{-CC}$ is finer than the sharp partial order. Next example shows that it is strictly finer in general.

Example 4.30. Let $S = T_3$ and $b = (122)$.
Then obviously $(111), (122), (123), (222) \in C(b)$ but $(121) \notin C(b)$ and as a consequence $(121) \notin CC(b)$. Indeed $(121)(122) = (122) \neq (121) = (122)(121)$.Now let $d = (mnk) \in CC(b)$. Then d commutes with (111) and (222) or equally $m = 1$, $n = 2$. It follows that $CC(b) = \{(123), (122)\}$.

As b is not invertible, $b^{-(123)}$ doesn't exists. But $b^{-b} = b$. Then $bb^{-b}b = b$. It follows that if $a <^{-\Delta_{CC}} b$ then $a = b$. But for example $(111) <^{\#} b$.

Proposition 4.31. *Let $\Theta_{CC} : b \mapsto CC(b) \cap b\{2\}$. Then for all $b \in S$, $\Theta_{CC}(b) = \{b^{-d} | d \in CC(b)\}$.*
As a consequence, all of the defined relations remain unchanged under the substitution Θ_{CC} for $-CC$.

Proposition 4.32. *Let $a, b \in S$. Then the following statements are equivalent:*

1. *$a <^{-CC} b$ in S;*
2. *$a <^{\#} b$ in $CC(b)$;*
3. *$a <^{-} b$ in $CC(b)$;*
4. *$a <^{\Theta_{CC}} b$ in S.*

We finally consider the star and Drazin partial orders. Let S be a semigroup with a proper involution. The star partial order is defined by $a <^* b$ iff $aa^* = ba^*$ and $a^*a = a^*b$, and the Drazin partial order by $a <^{\dagger} b$ iff $aa^\dagger = ba^\dagger$ and $a^\dagger a = a^\dagger b$. It is stated in [10] that the two partial orders coincide on the set of $*-$regular (Moore-Penrose invertible) elements.

Theorem 4.33. *Let S be a semigroup with a proper involution, and let*

$$\Theta^* : b \mapsto \Theta^*(b) = \{b^= \in b\{2\} | b^= b = (b^= b)^* \text{ and } bb^= = (bb^=)^*\}.$$

Then $<^{\Theta^*}$ is the Drazin partial order.

Proof. Let $a, b \in S$ such that $a <^{\Theta^*} b$. Then by Theorem 4.13 there exists $a^+ \in \Theta^*(b)$ such that $aa^+ = ba^+$ and $a^+a = a^+b$. By Lemma 4.8 $a^+ \in a\{1, 2\}$. Since $a^+ \in \Theta^*(b)$, $aa^+ = ba^+ = (ba^+)^* = (aa^+)^*$ and $a^+a = a^+b = (a^+b)^* = (a^+a)^*$. It follows that $a^+ = a^\dagger$ and as a consequence $a <^{\dagger} b$.

Now suppose that $a <^{\dagger} b$. It follows that $aa^\dagger = ba^\dagger$ and $a^\dagger a = a^\dagger b$. Then $a^\dagger ba^\dagger = a^\dagger aa^\dagger = a^\dagger$, $(aa^\dagger)^* = aa^\dagger = ba^\dagger = (ba^\dagger)^*$ and dually, $a^\dagger b = (a^\dagger)^*$. It follows that $a^\dagger \in \Theta^*(b)$ and as a consequence $a <^{\Theta^*} b$. □

Proposition 4.34. *Consider $\Delta^* : b \mapsto \Delta^*(b) = \{x \in S \mid xb = (xb)^*$ and $bx = (bx)^*\}$. Then for all $b \in S$, $\Theta^*(b) = \{b^{-d} | d \in \Delta^*(b)\}$.*
As a consequence, all of the defined relations remain unchanged under the substitution Θ^* for $-\Delta^*$.

Since $<^{\Theta^*} \subseteq \mathcal{N}^{\Theta^*}$, the Drazin partial order is finer than $\mathcal{N}^{\Theta^*}$. Next example shows that it is strictly finer in general.

Example 4.35. Let $S = \mathbb{M}_3(\mathbb{Z}_2)$ with a transposition as an involution. We consider $a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is easy to verify that $bbb = b$, $a = bba = abb = aba$. Since $bb = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is symmetric, $b \in \Theta^*(b)$ and finally $a\mathcal{N}^{\Theta^*}b$. Note that $a = aa = a^* = a^\dagger$. Then $a^\dagger a = a \neq a^\dagger b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It follows that $a \not<^* b$.

4.5 Other properties of the partial order $\mathcal{N}^\Theta$

Proposition 4.36. *Let $a, b \in S$ and $a\, \mathcal{N}^\Theta b$ for some Θ. Then either $a = b$ or a^{-1} doesn't exist.*

Corollary 4.37. *Any invertible $a \in S$ is a maximal element with respect to $\mathcal{N}^\Theta$.*

Let us consider $\Delta^\#$ such that $\Delta^\#(b) = b$ for each $b \in S$. It is evident that $a <^{-\Delta^\#} b$ implies $a = b$, so that $<^{-\Delta^\#}$ $(=<^{\Theta^\#})$ is the diagonal relation. The next example shows that it doesn't hold for $\mathcal{N}^{-\Delta^\#}$.

Example 4.38. Let $S = T_3$ be the semigroup, which consists of all maps from the set $\{1, 2, 3\}$ to itself with the composition operation. We write (mnk) for the function which sends 1 to m, 2 to n, and 3 to k.

Let $a = (323), b = (321)$. Then $b^\# = (321)$ and $a = ab^\#b = bb^\#a = ab^\#a$ or equally $a\mathcal{N}^{-\Delta^\#}b$.

Remark 4.39. In the Example above $a^\# = a^\#bb^\# = b^\#ba^\# = a^\#ba^\#$ that is $a^\#\mathcal{N}^{-\Delta^\#}b^\#$. But the equivalence $a^\# = a^\#ba^\#$ may not holds in general as the following example shows.

Example 4.40. Let $S = T_3$, $a = (211)$ and $b = (231)$. Then $a^\# = a$ and $b^\# = (312)$. In this case we obtain the following equations: $a = ab^\#b = bb^\#a = ab^\#a$. But $a^\#ba^\# = (111) \neq a^\#$

Proposition 4.41. *Let $a, b \in S$ such that a is group invertible, and assume that function $\Delta : S \to 2^S$ satisfies $\Delta(b) \subseteq C(b)$. If $a\mathcal{N}^{-\Delta}b$ with $a = ab^{-d}b = bb^{-d}a = ab^{-d}a$ for some $d \in \Delta(b)$, then $a^\# = a^\#bb^{-d} = b^{-d}ba^\#$.*

Remark 4.42. We note that $\Delta^\#$ satisfies the hypothesis of Proposition 4.41.

Lemma 4.43. *Let $a, b \in S$ and $a\mathcal{N}^\Theta b$ with $a = ab^= b = bb^= a = ab^= a$. Then $ab^=, b^= a \in E(S)$.*

$ab^= = (ab^=a)b^=$, $b^=a = b^=(ab^=a)$.

Lemma 4.44. *Let $a, b \in S$. If $a\mathcal{N}^\Theta b$ and b is an idempotent in S then a is idempotent and $a = ab = ba = a^2$.*

Proposition 4.45. *Let $S = T_n$. Then the n-tuple $(i \ldots i)$ for each $i \in \{1, \ldots, n\}$ are minimal elements with respect to $\mathcal{N}^\Theta$.*

In [18] different cases of maximal elements under the minus partial order were studied. Since $<^\Theta \subseteq \mathcal{N}^\Theta \subseteq <^-$, all results obtained for the minus order hold for $<^\Theta$ and $\mathcal{N}^\Theta$, and all the results obtained in this section for $\mathcal{N}^\Theta$ hold for $<^\Theta$.

5 Connection to Mitra's unified theory

It was Mitra who suggested a unified approach to matrix partial orderings via generalized inverses in [30].

Let $G : S \to 2^S$ denote a (multi-valued) function. Mitra defined a relation $<^G$ as follows.

Definition 5.1. [30] For $a, b \in S$ $a <^G b$ means that there exists an inner inverse a^- of a, $a^- \in G(a)$, such that $aa^- = ba^-$ and $a^-a = a^-b$.

There is a strong connection between $<^G$ and $<^\Theta$, see Theorem 4.13. In this section we investigate it further.

Theorem 5.2. *Suppose that $<$ is some order, possibly, non-transitive, such that $a < b \Rightarrow a <^- b$. Then $<$ can be represented in the form $<^\Theta$ for some $\Theta : S \to 2^S$.*

Proof. Let $a, b \in S$ such that $a < b$. By conditions $a <^- b$ that is $aa^- = ba^-$ and $a^-a = a^-b$ for some $a^- \in a\{1\}$. Then $a^-aa^-ba^-aa^- = a^-aa^-aa^-aa^- = a^-aa^-$. It follows that $a^-aa^- \in b\{2\}$. Also $ba^-aa^-b = aa^-aa^-a = a$.

Define Θ in the following way: $\Theta(b) = \{a^-aa^- \mid \text{for every } a \in S \text{ with } a < b\}$, where a^- is chosen from $a\{1\}$ as above, that is $aa^- = ba^-$ and $a^-a = a^-b$.

Thus we obtain that Θ satisfies the condition $x < y \Rightarrow x <^\Theta y$ for every $x, y \in S$. Suppose that there exist $x, y \in S$ such that $x <^\Theta y$ for $x \not< y$. Then there exists some $y^= \in \Theta(y)$ such that $x = yy^=y$. By the definition of Θ we have that $y^= = a^-aa^-$ for some $a < y$ and a^- is such that $a^-a = a^-y$ and $aa^- = ya^-$. Then $x = yy^=y = ya^-aa^-y = a < y$, a contradiction. It follows that $<^\Theta$ is equivalent to $<$. □

Note that the definition of $a <^G b$ uses a function of a as far as $a <^\Theta b$ uses a function of b. As a consequence of the theorem above we obtain:

Corollary 5.3. *Let $G : S \to 2^S$. Then we may find $\Theta : S \to 2^S$ such that $a <^G b \Leftrightarrow a <^\Theta b$.*

Remark 5.4. The above theorem is not valid for $<^G$ as the following example shows.

Example 5.5. Let $S = T_3$, $a = (111), b = (122), c = (133)$. Then $aa = a = ab = ba = ac = ca$. Thus q so $a <^- b$ and $a <^- c$. Suppose that we want to find G such that $a <^G b$ but $a \not<^G c$. Let us find all $a^- \in a\{1\}$ with $aa^- = ba^-$. It follows that $aa^- = (111) = ba^- = (122)a^- \Rightarrow a^- = (111) = a$. Then $a \in G(a)$. But $aa = ca = ac \Rightarrow a <^G c$, a contradiction.

Corollary 5.6. *It follows that there exists $\Theta : S \to 2^S$ such that there is no $<^G$ equivalent to $<^\Theta$. Thus $<^\Theta$ is more general relation.*

We now compare $<^G$ and $\mathcal{N}^\Theta$. Recall that relation $\mathcal{N}^\Theta$ is transitive.

Remark 5.7. The following example shows that the relation $a <^G b$ is not transitive in general:

Example 5.8. [30, Example 1]

Let $S = M_3(\mathbb{R})$ and $a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Put $G(x) = \{x^\dagger\}$ if $rank(x) = 1$ and $G(x) = \{x\{1\}\}$ otherwise. Here $rank(a) = 1$, $a^\dagger = a$ and $aa = ab = ba$. It follows that $a <^G b$.

Let $b^- = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to see that $bb^-b = b$ and as a consequence $b^- \in b\{1\}$ and $b^- \in G(b)$ since $rank(b) = 2$. Also $bb^- = b^- = cb^-$ and $b^-b = b = b^-c$. It follows that $b <^G c$.

Finally, $a <^G b <^G c$. However, one can check that $a = a^\dagger a \neq a^\dagger c = ac = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus $a \not<^G c$ and $<^G$ is not transitive.

We conclude our analysis by the following:

Corollary 5.9.
1. There exists G such that there is no Θ with $<^G = \mathcal{N}^\Theta$.
2. There exists Θ such that there is no G with $<^G = \mathcal{N}^\Theta$.

Proof. 1. In Example 5.8 relation $<^G$ is not transitive and as a consequence there is no Θ with $<^G = \mathcal{N}^\Theta$.

2. In Example 5.5 $a <^G b \Rightarrow a <^G c$ for any G. Let Θ be such that $\Theta(b) = \{a\}$ and $\Theta(c) = \emptyset$. Then $a\mathcal{N}^\Theta b$ but $a \not\mathcal{N}^\Theta c$ and as a consequence there is no G with $<^G = \mathcal{N}^\Theta$. $\square$

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