The topology of the space of rational curves on a toric variety and related problems

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Abstract

We report about the recent joint work with A. Kozlowski [16] (cf. [15], [18]) concerning to the topology of spaces of rational curves on a toric variety and related problems.

1 Introduction

Spaces of maps. For connected spaces X and Y, let $\mathrm{Map}^*(X,Y)$ denote the space consisting of all base point preserving continuous maps from X to Y with the compact-open topology, and for each class $D \in \pi_0(\mathrm{Map}^*(X,Y))$, let $\mathrm{Map}^*_D(X,Y)$ denote the path-component of $\mathrm{Map}^*(X,Y)$ corresponding to the homotopy class D. When $X = \mathbb{C}\mathrm{P}^1$ and Y is a complex manifolds, we denote by $\mathrm{Hol}^*_D(S^2,Y)$ the space of all base point preserving holomorphic maps $f \in \mathrm{Map}^*_D(S^2,Y) = \Omega^2_D Y$.

Convex rational polyhedral cones. *A convex rational polyhedral cone* is the subset of $\mathbb{R}^m$ of the form

$$\sigma = \mathrm{Cone}(S) = \mathrm{Cone}(\boldsymbol{m}_1,\cdots,\boldsymbol{m}_s) = \{\sum_{k=1}^{s} \lambda_k \boldsymbol{m}_k : \lambda_k \geq 0\} \tag{1.1}$$

for some finite set $S = \{\boldsymbol{m}_k : 1 \leq k \leq s\} \subset \mathbb{Z}^m$ and it is called *strongly convex* if $\sigma \cap (-\sigma) = \{\mathbf{0}_m\}$, where we set $\mathbf{0}_m = (0,0,\cdots,0) \in \mathbb{R}^m$. When S is the emptyset $\emptyset$, we set $\mathrm{Cone}(\emptyset) = \{\mathbf{0}_m\}$ and we may also regard it as one of convex rational polyhedral cones.

Fans and toric varieties. Let X be an m dimensional irreducible normal algebraic variety over $\mathbb{C}$. One says that X is *a toric variety* if it has an algebraic action of of an algebraic torus $\mathbb{T}^m_{\mathbb{C}} = (\mathbb{C}^*)^m$, such that the orbit $\mathbb{T}^m_{\mathbb{C}} \cdot *$ of some point $* \in X$ is dense in X and isomorphic to $\mathbb{T}^m_{\mathbb{C}}$. A toric variety X is characterized up to isomorphism by its *fan* Σ, which is a finite collection of strongly convex rational polyhedral cones in $\mathbb{R}^m$ such that every face τ of $\sigma \in \Sigma$ belongs to Σ and the intersection $\sigma_1 \cap \sigma_2$ of any two elements $\sigma_1, \sigma_2 \in \Sigma$ is a face of each σ_k $(k=1,2)$. We denote by $X_{\boldsymbol{\Sigma}}$ the toric variety associated to the fan Σ.

Polyhedral products and homogenous coordinates. Let K be a simplicial complex on the index set $[r]=\{1,2,\cdots,r\}$,[1] and let (X,A) be pair of spaces such that $A\subset X$. Then define *the polyhedral product* $\mathcal{Z}_K(X,A)$ with respect to K by the union $\mathcal{Z}_K(X,A)=\bigcup_{\sigma\in K}(X,A)^\sigma$, where $(X,A)^\sigma=\{(x_1,\cdots,x_r)\in X^r : x_k\in A \text{ if } k\notin\sigma\}$.

Note that the space $\mathcal{Z}_K(D^2,S^1)$ is usually called *the moment-angle complex* of K.

Definition 1.1. Let Σ be a fan in $\mathbb{R}^m$ such that $\{\mathbf{0}_m\}\subsetneqq\Sigma$, and let $\Sigma(1)=\{\rho_1,\cdots,\rho_r\}$ denote the set of all one dimensional cones in Σ.

(i) For each integer $1\leq k\leq r$, we denote by $\boldsymbol{n}_k\in\mathbb{Z}^m$ *the primitive generator* of ρ_k, such that $\rho_k\cap\mathbb{Z}^m=\mathbb{Z}_{\geq 0}\cdot\boldsymbol{n}_k$. Note that $\rho_k=\mathrm{Cone}(\boldsymbol{n}_k)$.

(ii) Let $\mathcal{K}_\Sigma$ denote *the underlying simplicial complex of* Σ defined by

$$\mathcal{K}_\Sigma=\Big\{\{i_1,\cdots,i_s\}\subset[r] : \mathrm{Cone}(\boldsymbol{n}_{i_1},\cdots,\boldsymbol{n}_{i_s})\in\Sigma\Big\}. \tag{1.2}$$

It is easy to see that $\mathcal{K}_\Sigma$ is a simplicial complex on the index set $[r]$.

(iii) Next, define the subgroup $G_\Sigma\subset\mathbb{T}^r_{\mathbb{C}}$ by

$$G_\Sigma=\{(\mu_1,\cdots,\mu_r)\in\mathbb{T}^r_{\mathbb{C}} : \prod_{k=1}^r(\mu_k)^{\langle\boldsymbol{n}_k,\boldsymbol{m}\rangle}=1 \text{ for all } \boldsymbol{m}\in\mathbb{Z}^m\}, \tag{1.3}$$

where $\langle\boldsymbol{u},\boldsymbol{v}\rangle=\sum_{k=1}^m u_kv_k$ for $\boldsymbol{u}=(u_1,\cdots,u_m)$ and $\boldsymbol{v}=(v_1,\cdots,v_m)\in\mathbb{R}^m$.

(iv) Consider the natural G_Σ-action on $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*)$ given by coordinate-wise multiplication, i.e.

$$(\mu_1,\cdots,\mu_r)\cdot(x_1,\cdots,x_r)=(\mu_1x_1,\cdots,\mu_rx_r)$$

for $((\mu_1,\cdots,\mu_r),(x_1,\cdots,x_r))\in G_\Sigma\times\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*)$. □

Let $I(\mathcal{K}_\Sigma)=\{\sigma\subset[r] : \sigma\notin\mathcal{K}_\Sigma\}$ and consider the orbit space $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*)/G_\Sigma$. Now recall the following result due to D. Cox.

Theorem 1.2 (D. Cox; [4], [5]). *Let Σ be a fan in $\mathbb{R}^m$ as in Definition 1.1 and suppose that the set $\{\boldsymbol{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{R}^m$.*

(i) *Then there is a natural isomorphism*

$$X_{\boldsymbol{\Sigma}}\cong\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*)/G_\Sigma. \tag{1.4}$$

(ii) *If $f:\mathbb{C}\mathrm{P}^s\to X_{\boldsymbol{\Sigma}}$ is a holomorphic map, then there exists an r-tuple $D=(d_1,\cdots,d_r)\in(\mathbb{Z}_{\geq 0})^r$ of non-negative integers satisfying the condition $\sum_{k=1}^r d_k\boldsymbol{n}_k=\mathbf{0}_m$, and homogenous polynomials $f_i\in\mathbb{C}[z_0,\cdots,z_s]$ of degree d_i $(i=1,2,\cdots,r)$ such that the polynomials $\{f_i\}_{i\in\sigma}$ have no common root except $\mathbf{0}_{s+1}\in\mathbb{C}^{s+1}$ for each $\sigma\in I(\mathcal{K}_\Sigma)$ and that the diagram*

$$\begin{array}{ccc}\mathbb{C}^{s+1}\setminus\{\mathbf{0}_{s+1}\} & \xrightarrow{(f_1,\cdots,f_r)} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*) \\ {\scriptstyle\gamma_s}\downarrow & & \downarrow{\scriptstyle q_\Sigma} \\ \mathbb{C}\mathrm{P}^s & \xrightarrow{f} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*)/G_\Sigma=X_{\boldsymbol{\Sigma}}\end{array} \tag{1.5}$$

[1] In this paper by a simplicial complex K we always mean an *an abstract simplicial complex*, and we always assume that a simplicial complex K contains the empty set $\emptyset$.

is commutative, where two map $\gamma_s : \mathbb{C}^{s+1} \setminus \{\mathbf{0}_{s+1}\} \to \mathbb{C}\mathrm{P}^s$ *and* $q_\Sigma : \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \to X_{\mathbf{\Sigma}}$ *denote the canonical Hopf fibering and the canonical projection, respectively. In this case, we call this holomorphic map* f *a holomorphic map of degree* $D = (d_1, \cdots, d_r)$ *and we represent it as* $f = [f_1, \cdots, f_r]$.

(iii) *If* $g_i \in \mathbb{C}[z_0, \cdots, z_s]$ *is a homogenous polynomial of degree* d_i $(1 \leq i \leq r)$ *such that* $f = [f_1, \cdots, f_r] = [g_1, \cdots, g_r]$, *there exists some element* $(\mu_1, \cdots, \mu_r) \in G_\Sigma$ *such that* $f_i = \mu_i \cdot g_i$ *for each* $1 \leq i \leq r$. □

Assumptions. From now on, let Σ be a fan in $\mathbb{R}^m$ as in Definition 1.1, and assume that $X_{\mathbf{\Sigma}}$ is simply connected.[2] Thus, we can identify $X_{\mathbf{\Sigma}} = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma$, and we shall assume that the following condition holds.

(1.5.1) There is an r-tuple $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$.

2 Spaces of rational curves on a toric variety

Let P^d denote the space of all monic polynomials $f(z) = z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d \in \mathbb{C}[z]$ of degree d, and we set

$$\mathrm{P}^D = \mathrm{P}^{d_1} \times \mathrm{P}^{d_2} \times \cdots \times \mathrm{P}^{d_r}. \tag{2.1}$$

Now let $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ be an r-tuple of positive integers satisfying (1.5.1) and consider a base point preserving holomorphic map $f = [f_1, \cdots, f_r] : \mathbb{C}\mathrm{P}^s \to X_{\mathbf{\Sigma}}$ of the degree D for the case $s = 1$.

In this situation, we make the identification $\mathbb{C}\mathrm{P}^1 = S^2 = \mathbb{C} \cup \infty$ and choose the points ∞ and $[1, 1, \cdots, 1]$ as the base points of $\mathbb{C}\mathrm{P}^1$ and $X_{\mathbf{\Sigma}}$ respectively. Then, by setting $z = \frac{z_0}{z_1}$, for each $1 \leq k \leq r$ we can view f_k as a monic polynomial $f_k(z) \in \mathrm{P}^{d_k}$ in the complex variable z. Thus we can identify the space $\mathrm{Hol}_D^*(S^2, X_{\mathbf{\Sigma}})$ of all base point preserving holomorphic maps $f : S^2 \to X_{\mathbf{\Sigma}}$ of the degree D with some spaces of r-tuples $(f_1(z), \cdots, f_r(z)) \in \mathrm{P}^D$ as follows.

Definition 2.1. (1) First consider the case that the r-tuple $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ satisfies the condition (1.5.1).

In this case, for any r-tuple $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ satisfying the condition (1.5.1), one can identify the space $\mathrm{Hol}_D^*(S^2, X_{\mathbf{\Sigma}})$ with the space all r-tuples $(f_1(z), \cdots, f_r(z)) \in \mathrm{P}^D$ satisfying the condition

(†) For any $\sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma)$, the polynomials $f_{i_1}(z), \cdots, f_{i_s}(z)$ have no common root i.e. $(f_{i_1}(\alpha), \cdots, f_{i_s}(\alpha)) \neq (0, \cdots, 0)$ for any $\alpha \in \mathbb{C}$.

[2] One can show that the set $S = \{\boldsymbol{n}_k : 1 \leq k \leq r\}$ of primitive generators spans $\mathbb{Z}^m$ over $\mathbb{Z}$ if $X_{\mathbf{\Sigma}}$ is simply connected. Thus the set S also spans $\mathbb{R}^m$ and the assumption of Theorem 1.2 is satisfied.

Then define the natural inclusion map

$$i_D : \mathrm{Hol}^*_D(S^2, X_{\mathbf{\Sigma}}) \to \mathrm{Map}^*(S^2, X_{\mathbf{\Sigma}}) = \Omega^2 X_{\mathbf{\Sigma}} \tag{2.2}$$

by

$$i_D(f_1(z), \cdots, f_r(z))(\alpha) = \begin{cases} [f_1(\alpha), \cdots, f_r(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1, 1, \cdots, 1] & \text{if } \alpha = \infty \end{cases} \tag{2.3}$$

where we choose the points ∞ and $[1, 1, \cdots, 1]$ as the base points of S^2 and $X_{\mathbf{\Sigma}}$.

Since $\mathrm{Hol}^*_D(S^2, X_{\mathbf{\Sigma}})$ is path-connected, the image of i_D is contained in a certain path-component of $\Omega^2 X_{\mathbf{\Sigma}}$, which is denoted by $\Omega^2_D X_{\mathbf{\Sigma}}$. Thus we have a natural inclusion

$$i_D : \mathrm{Hol}^*_D(S^2, X_{\mathbf{\Sigma}}) \to \mathrm{Map}^*_D(S^2, X_{\mathbf{\Sigma}}) = \Omega^2_D X_{\mathbf{\Sigma}}. \tag{2.4}$$

(2) Next we shall consider the general case.

Indeed, for each r-tuple $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ of positive integers, let H^{Σ}_D denote the space of r-tuples $(f_1(z), \cdots, f_r(z)) \in \mathrm{P}^D$ satisfying the condition (†).

Note that $H^{\Sigma}_D = \mathrm{Hol}^*_D(S^2, X_{\mathbf{\Sigma}})$ when the condition $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$ is satisfied. □

Definition 2.2. (i) A map $f : X \to Y$ is called *a homology equivalence through dimension* N (resp. *a homotopy equivalence through dimension* N) if the induced homomorphism $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$ (resp. $f_* : \pi_k(X) \to \pi_k(Y)$) is an isomorphism for any $k \leq N$.

(ii) We say that a set $S = \{\boldsymbol{n}_{i_1}, \cdots, \boldsymbol{n}_{i_s}\}$ is a *a primitive collection* if it does not span a cone in Σ but any proper subset of it does. Then define integers $r_{\min}(\Sigma)$ and $d(D, \Sigma)$ by

$$r_{\min}(\Sigma) = \min\{s = \mathrm{card}(S) \in \mathbb{Z}_{\geq 1} : S = \{\boldsymbol{n}_{i_1}, \cdots, \boldsymbol{n}_{i_s}\} \text{ is a primitive collection}\}, \tag{2.5}$$

$$d(D; \Sigma) = (2r_{\min}(\Sigma) - 3)d_{min} - 2, \text{ where } d_{\min} = \min\{d_1, \cdots, d_r\}. \tag{2.6}$$

Atiyah-Jones-Segal type Theorem. The main purpose of this note is to report the following result.

Theorem 2.3 ([16])**.** *Let $X_{\mathbf{\Sigma}}$ be a simply connected non-singular toric variety, and let $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ be an r-tuple of positive integers such that $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$.*

Then the inclusion map

$$i_D : \mathrm{Hol}^*_D(S^2, X_{\mathbf{\Sigma}}) \to \Omega^2_D X_{\mathbf{\Sigma}} \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^2, S^1)$$

is a homotopy equivalence through dimension $d(D; \Sigma)$ if $r_{\min}(\Sigma) \geq 3$, and it is a homology equivalence through dimension $d(D; \Sigma) = d_{min} - 2$ if $r_{\min}(\Sigma) = 2$. □

Corollary 2.4. *Let $X_{\mathbf{\Sigma}}$ be a simply connected non-singular toric variety, and suppose that there is an r-tuple $D_* = (m_1, \cdots, m_r) \in (\mathbb{Z}_{\geq 1})^r$ of positive integers such that $\sum_{k=1}^r m_k \boldsymbol{n}_k = \mathbf{0}_m$. Then for each $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ of positive integers, there is a map*

$$j_D : H^{\Sigma}_D \to \Omega^2_D X_{\mathbf{\Sigma}} \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^2, S^1)$$

which is a homotopy equivalence through dimension $d(D; \Sigma)$ if $r_{\min}(\Sigma) \geq 3$, and a homology equivalence through dimension $d(D; \Sigma) = d_{min} - 2$ if $r_{\min}(\Sigma) = 2$. □

Related topics and problems. Finally we shall comment about the related topics.

Remark 2.5. (i) When $X_{\mathbf{\Sigma}} = \mathbb{C}\mathrm{P}^s$ with $s \geq 2$, we can obtain the more sharper result (see the detail in [13] and [14]).

(ii) If $X_{\mathbf{\Sigma}}$ is compact, M. Guest [7] proved that the map i_D is a homotopy equivalence through dimension $d_{min} - 1$ and this result is stronger than that of Theorem 2.3 when $r_{\min}(\Sigma) = 2$. In this reason the author and A. Kozlowski are wondering whether the map i_D might be *a homotopy equivalence* through dimension $d(D, \Sigma)$ even if $r_{\min}(\Sigma) = 2$.

(iii) More generally one can consider the similar problem for the inclusion map i_D : $\mathrm{Hol}^*_D(\mathbb{C}\mathrm{P}^s, X_{\mathbf{\Sigma}}) \to \mathrm{Map}^*_D(\mathbb{C}\mathrm{P}^s, X_{\mathbf{\Sigma}})$ when $s \geq 2$, and this problem was really investigated by J. Mostovoy and E. Munguia-Villanueva in [21].

(iv) One can consider the space of resultants related to the space $\mathrm{Hol}^*_D(S^2, X_{\mathbf{\Sigma}})$ of rational curves on toric varieties and it seems very interesting to investigate whether a similar Atiyah-Jones-Segal conjecture holds for this space. Indeed, when $X_{\mathbf{\Sigma}} = \mathbb{C}\mathrm{P}^s$, this problem was solved very nicely in [15]. In the subsequent paper [18], Kozlowski and the author will discuss about the homotopy types of the space of resultants related to toric varieties. $\square$

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