Liftings of vector valued Siegel modular forms

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Abstract

Chenevier–Lannes [3] gave a general description of the space of vector valued Siegel modular cusp forms by using Arthur's multiplicity formula. In particular, this description tells us the existence of many liftings of cusp forms. In this survey article, we will explain the description of Chenevier–Lannes, and give several examples of liftings, which contain Ibukiyama's conjecture [6].

1 Siegel modular forms

Let $\mathrm{Sp}_n(\mathbb{R})$ be the symplectic group of rank n defined by

$$\mathrm{Sp}_n(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} \right\}.$$

This group acts on the Siegel upper half space

$$\mathfrak{H}_n = \{ Z \in \mathrm{Sym}_n(\mathbb{C}) \mid \mathrm{Im}(Z) > 0 \}$$

by

$$g\langle Z \rangle = (AZ+B)(CZ+D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R}),\ Z \in \mathfrak{H}_n.$$

For $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ with $k_1 \geq \cdots \geq k_n$, we denote by $(\rho_{\mathbf{k}}, V_{\mathbf{k}})$ the irreducible representation of $\mathrm{U}(n)$ of highest weight $\mathbf{k}$. A $V_{\mathbf{k}}$-valued holomorphic function $F\colon \mathfrak{H}_n \to V_{\mathbf{k}}$ is called a *Siegel modular cusp form of vector weight* $\rho_{\mathbf{k}}$ if

1. $F(\gamma\langle Z \rangle) = \rho_{\mathbf{k}}(CZ+D)F(Z)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z})$ and $Z \in \mathfrak{H}_n$;
2. F has a Fourier expansion of the form
$$F(Z) = \sum_{T \in \mathrm{Sym}_n(\mathbb{Q}), T > 0} A_F(T) e^{2\pi\sqrt{-1}\mathrm{tr}(TZ)}, \quad A_F(T) \in V_{\mathbf{k}}.$$

The space of Siegel modular cusp forms of vector weight $\rho_{\mathbf{k}}$ is denoted by $S_{\rho_{\mathbf{k}}}(\mathrm{Sp}_n(\mathbb{Z}))$. Also write

- $S_k(\mathrm{Sp}_n(\mathbb{Z})) = S_{\rho_{(k,\dots,k)}}(\mathrm{Sp}_n(\mathbb{Z}))$ for the space of scalar weight $\rho_{(k,\dots,k)} = \det^k$;
- $S_{k,j}(\mathrm{Sp}_2(\mathbb{Z})) = S_{\rho_{(k+j,k)}}(\mathrm{Sp}_2(\mathbb{Z}))$ for the space of degree 2 and of vector weight $\rho_{(k+j,k)} = \det^k \mathrm{Sym}(j)$.

There is a Hecke theory for $S_{\rho_{\mathbf{k}}}(\mathrm{Sp}_n(\mathbb{Z}))$. Through this article, we always assume that $F \in S_{\rho_{\mathbf{k}}}(\mathrm{Sp}_n(\mathbb{Z}))$ is a Hecke eigenform.

2 Langlands conjecture

It is strongly desired that there is a locally compact group $\mathcal{L}_{\mathbb{Q}}$ such that the irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ are parametrized by the n-dimensional irreducible representations of $\mathcal{L}_{\mathbb{Q}}$. The group $\mathcal{L}_{\mathbb{Q}}$ is called the *hypothetical Langlands group*. Using a description of Mœglin–Waldspurger [10], each n-dimensional (semisimple) representation of $\mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C})$ should correspond to an irreducible isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$.

Let G be a split reductive algebraic group over $\mathbb{Q}$, and $\widehat{G}$ be the complex Langlands dual group of G. Roughly speaking, the **Langlands conjecture** asserts that one would associate a homomorphism

$$\psi\colon \mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$$

with each irreducible cuspidal automorphic representation π of $G(\mathbb{A}_{\mathbb{Q}})$. In particular, for a rational representation $r\colon \widehat{G} \to \mathrm{GL}_n(\mathbb{C})$, one should obtain an L-function

$$L(s, \pi, r) = L(s, r \circ \psi),$$

where the right hand side is the Godement–Jacquet L-function [4] of the automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ associated to $r \circ \psi$, which satisfies a functional equation with respect to $s \leftrightarrow 1 - s$.

A Hecke eigenform $F \in S_{\rho_{\mathbf{k}}}(\mathrm{Sp}_n(\mathbb{Z}))$ gives an irreducible cuspidal automorphic representation π_F of the symplectic group $\mathrm{Sp}_n(\mathbb{A}_{\mathbb{Q}})$ or the general symplectic group $\mathrm{GSp}_n(\mathbb{A}_{\mathbb{Q}})$. The dual groups of Sp_n and GSp_n are $\mathrm{SO}_{2n+1}(\mathbb{C})$ and $\mathrm{GSpin}_{2n+1}(\mathbb{C})$, respectively. They have rational representations

$$\mathrm{std}\colon \mathrm{SO}_{2n+1}(\mathbb{C}) \to \mathrm{GL}_{2n+1}(\mathbb{C}) \quad \text{and} \quad \mathrm{spin}\colon \mathrm{GSpin}_{2n+1}(\mathbb{C}) \to \mathrm{GL}_{2^n}(\mathbb{C})$$

called the standard representation of $\mathrm{SO}_{2n+1}(\mathbb{C})$ and the spinor representation of $\mathrm{GSpin}_{2n+1}(\mathbb{C})$, respectively. In particular, the modular form F has two (incomplete) L-functions

$$L(s, F, \mathrm{std}) = L_{\mathrm{fin}}(s, \pi_F, \mathrm{std}) \quad \text{and} \quad L(s, F, \mathrm{spin}) = L_{\mathrm{fin}}(s, \pi_F, \mathrm{spin}).$$

Note that when $n = 1$, the Hecke L-function $L(s, f)$ of $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is the shift of the spinor L-function $L(s - k + 1/2, f, \mathrm{spin})$.

Arthur [1, Theorem 1.5.2] proved the Langlands conjecture for Sp_n replacing irreducible representations of $\mathcal{L}_{\mathbb{Q}}$ with irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. Using this result, Chenevier–Lannes [3, Theorem* 8.5.2] gave a general decomposition of the space $\mathcal{S}_{\rho_{\mathbf{k}}}(\mathrm{Sp}_n(\mathbb{A}_{\mathbb{Q}}))$ of holomorphic cusp forms on $\mathrm{Sp}_n(\mathbb{A}_{\mathbb{Q}})$, which are automorphic forms obtained by holomorphic Siegel cusp forms. In particular, they showed the **multiplicity one theorem** [3, Corollary* 8.5.4] for $S_{\rho_{\mathbf{k}}}(\mathrm{Sp}_n(\mathbb{Z}))$ such that $\mathbf{k} = (k_1, \dots, k_n)$ with $k_n > n$.

3 Lifting theorems

Retranslating this decomposition into modular forms, we can obtain three types of lifting theorems with respect to the standard L-functions. Let $g \in S_{\rho_{\mathbf{k}}}(\mathrm{Sp}_n(\mathbb{Z}))$ with $\mathbf{k} = (k_1, \dots, k_n)$ such that $k_n > n$. When $n = 0$, we interpret the standard L-function $L(s, g, \mathrm{std})$ for $g \in S_{\rho_{\mathbf{k}}}(\mathrm{Sp}_n(\mathbb{Z}))$ to be the Riemann zeta function $\zeta(s)$.

Theorem 3.1 (Ikeda–Miyawaki type)**.** *For positive integers k and d, we assume one of the following:*

- $k + d - 1 < k_n - n$*,* $k > d$ *and* $k \equiv d + n \bmod 2$*; or*
- $k - d > k_1 - 1$*,* $k > d$ *and* $k \equiv d \bmod 2$*.*

Define $\mathbf{k}' = (k'_1, \dots, k'_{n+2d}) \in \mathbb{Z}^{n+2d}$ *so that* $k'_1 \geq \cdots \geq k'_{n+2d}$ *and*

$$\{k'_i - i \mid 1 \leq i \leq n + 2d\} = \{k_i - i \mid 1 \ \leq i \leq n\} \cup \{k + i \mid -d \leq i \leq d - 1\}.$$

Then for $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$*, there exists* $F_{f,g} \in S_{\rho_{\mathbf{k}'}}(\mathrm{Sp}_{n+2d})(\mathbb{Z})$ *such that*

$$L(s, F_{f,g}, \mathrm{std}) = L(s, g, \mathrm{std}) \prod_{i=1}^{2d} L\left(s + k + d - i, f\right).$$

Theorem 3.2 (Ibukiyama type)**.** *For positive integers k, j and d, we assume all of the following:*

- $k \equiv j \equiv 0 \bmod 2$*;*
- $k > 2d + 1$ *and* $j > 2d - 1$*;*
- $k_i - i \leq (j/2) - d$ *or* $k_i - i \geq (j/2) + k + d - 1$ *for* $1 \leq i \leq n$*.*

Define $\mathbf{k}' = (k'_1, \dots, k'_{n+4d}) \in \mathbb{Z}^{n+2d}$ *so that* $k'_1 \geq \cdots \geq k'_{n+4d}$ *and*

$$\begin{aligned} &\{k'_i - i \mid 1 \leq i \leq n + 4d\} \\ &= \{k_i - i \mid 1 \ \leq i \leq n\} \cup \left\{\frac{j}{2} + k + i - 1, \frac{j}{2} + i + 1 \;\middle|\; -d \leq i \leq d - 1\right\}. \end{aligned}$$

Then for $f \in S_{k,j}(\mathrm{Sp}_2(\mathbb{Z}))$*, there exists* $F_{f,g} \in S_{\rho_{\mathbf{k}'}}(\mathrm{Sp}_{n+4d})(\mathbb{Z})$ *such that*

$$L(s, F_{f,g}, \mathrm{std}) = L(s, g, \mathrm{std}) \prod_{i=1}^{2d} L\left(s + d + \frac{1}{2} - i, f, \mathrm{spin}\right).$$

Theorem 3.3 (Yoshida type)**.** *For positive integers k, k' and d, we assume all of the following:*

- n *is odd;*
- g *is locally tempered, which means that the local L-factor* $L_p(s, g, \mathrm{std})$ *at p is holomorphic for* $\mathrm{Re}(s) \geq 1/2$*;*
- $k \geq k' \geq d$*;*
- $k - k' < k_i - i < k + k' - 1$ *for any* $1 \leq i \leq n$*.*

Define $\mathbf{k}' = (k'_1, \dots, k'_{n+4d}) \in \mathbb{Z}^{n+2d}$ *so that* $k'_1 \geq \cdots \geq k'_{n+4d}$ *and*

$$\begin{aligned} &\{k'_i - i \mid 1 \leq i \leq n+4d-2\} \\ &= \{k_i - i \mid 1 \ \leq i \leq n\} \cup \{k + k' + i - 1, k - k' + i \mid -d+1 \leq i \leq d-1\}\,. \end{aligned}$$

Then for $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ *and* $f' \in S_{2k'}(\mathrm{SL}_2(\mathbb{Z}))$*, there exists* $F_{f,f',g} \in S_{\rho_{\mathbf{k}'}}(\mathrm{Sp}_{n+4d-2})(\mathbb{Z})$ *such that*

$$L(s, F_{f,f',g}, \mathrm{std}) = L(s, g, \mathrm{std}) \prod_{i=1}^{2d-1} L\left(s + d - i, f \times f'\right).$$

Here, $L(s, f \times f')$ *is the Rankin–Selberg L-function, which is degree* 4 *and satisfies a functional equation with respect to* $s \leftrightarrow 1 - s$*.*

Remark 3.4. *1. By the multiplicity-one theorem [3, Corollary⋆ 8.5.4], the liftings are unique up to scalar multiplications. It is not known how to construct the liftings in general.*

2. Several special cases are (conjecturally) well-known. For example:

- *When* $n = 0$*, the lifting in Theorem 3.1 is called the Duke–Imamoglu–Ibukiyama–Ikeda lifting, or shortly, the Ikeda lifting of* f *[8].*
- *Miyawaki's conjecture [9, Conjectures 4.3, 4.5] is Theorem 3.1 when* $(n, \mathbf{k}, k, d) = (1, (k-2), k-1, 1)$ *with even* $k \geq 12$ *and when* $(n, \mathbf{k}, k, d) = (1, (k), k-2, 1)$ *with even* $k \geq 14$*. See also [9, Corollary 6.2]. His conjecture was generalized by Heim [5].*
- *When* $(n, \mathbf{k}, k, j, d) = (0, \emptyset, n+2, 2m-2n-2, n/2)$ *with even* m, n *such that* $m > 2n \geq 4$*, one can obtain Ibukiyama's conjecture [6, Conjecture 3.1] from Theorem 3.2.*
- *When* $(n, \mathbf{k}, k, k', d) = (1, (b+3), a+c+5, a-c+3, 1)$ *with* $a \geq b \geq c \geq 0$ *in Theorem 3.3, the lifting is the one predicted by Bergström, Faber and van der Geer [2, Conjecture 7.7 (i)]. See also [7, §4]. Note that the second condition in Theorem 3.3 holds automatically when* $n = 1$*.*

3. If g *is given more explicitly (e.g., if* $n \leq 2$*), one can obtain more liftings. For example, we can prove Ibukiyama's conjecture [6, Conjecture 3.2].*

4. By the same method, one can also prove some non-existence of liftings. For example, when $n = 0$ *in Theorem 3.3, one can check that there is no such lifting. This is compatible with the well-known fact that there is no (classical) Yoshida lifting of level one.*

Remark 3.5. *In several conjectures, the liftings* F *are determined not by the standard L-function* $L(s, F, \mathrm{std})$ *but by the spinor L-function* $L(s, F, \mathrm{spin})$*. For example, see [9, Conjectures 4.3, 4.5], [6, Conjectures 2.1, 2.2] and [2, Conjecture 7.7]. To obtain the behavior of spinor L-functions by the same method, one would need the Langlands conjecture for the spinor representation*

$$\mathrm{spin}\colon \mathrm{GSpin}_{2n+1}(\mathbb{C}) \to \mathrm{GL}_{2^n}(\mathbb{C})$$

of the dual group of GSp_n*. This is a special case of the* ***beyond endoscopy****, which is a widely open problem.*

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