Some remarks on the relative polar variety and the Brasselet number *

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Introduction

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be an analytic function defined in a neighborhood of the origin and Σf the critical locus of f. The Milnor fiber $F_{f,0}$ is given by $f^{-1}(\delta)\cap B_\epsilon$, where δ is a regular value of f, $0<|\delta|\ll\epsilon\ll 1$. In [15], Milnor proved that, if f has an isolated singularity, $F_{f,0}$ has the homotopy type of a wedge of $\mu(f)$ spheres of dimension $n-1$, where $\mu(f)$ is the Milnor number of f. Also, $\mu(f)$ is the number of Morse points in a Morsefication of f in a neighborhood of the origin.

In [6], Hamm generalized Milnor's results for complete intersections with isolated singularity $F=(f_1,\ldots,f_k):(\mathbb{C}^n,0)\to(\mathbb{C}^k,0), 1<k<n$, proving that the Milnor fiber $F^{-1}(\delta)\cap B_\epsilon, 0<|\delta|\ll\epsilon\ll 1$, has the homotopy type of a wedge of $\mu(F)$ spheres of dimension $n-k$. In this context, Lê [7] and Greuel [5] proved that $\mu(F)+\mu(F')=dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{\mathbb{C}^n,0}}{I}\right)$, where $F':(\mathbb{C}^n,0)\to(\mathbb{C}^{k-1},0)$ is the map with components $f_1,\ldots,f_{k-1}$ and I is the ideal generated by $f_1,\ldots,f_{k-1}$ and the $(k\times k)-$ minors $\frac{\partial(f_1,\ldots,f_k)}{\partial(x_{i_1},\ldots,x_{i_k})}$. Notice that the number $dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{\mathbb{C}^n,0}}{I}\right)$ is the number of critical points of a Morsefication of f_k appearing on the Milnor fibre of F'.

If f is defined over a complex analytic space X and f has an isolated singularity at the origin, a generalization for the Milnor number is the Euler obstruction of the function f, introduced in [2], by Brasselet, Massey, Parameswaran and Seade. In [17], Seade, Tibăr and Verjovsky proved that, up to sign, this number is the number of Morse critical points of a stratified Morsefication of f appearing in the regular part of X.

In a more general context, if f is defined over a complex analytic space X equipped with a good stratification $\mathcal{V}$ of X relative to f (see Definition 2.1) and the function f does not have isolated singularity at the origin, a way to describe the generalized Milnor fiber $X\cap f^{-1}(\delta)\cap B_\epsilon$ is to use the Brasselet number of f at the origin, $B_{f,X}(0)$, introduced by Dutertre and Grulha, in [3], and that generalizes the Milnor number to this more general setting and the local Euler obstruction, introduced by MacPherson, in [10], in his proof for the Deligne-Grothendieck conjecture. In [3], the authors presented several formulas to compute Brasselet numbers couting number of stratified Morse critical points. For example, they present a Lê-Greuel type formula for the Brasselet number: if $g:X\to\mathbb{C}$ is prepolar with respect to $\mathcal{V}$ at the origin (see Definition 2.4) and $0<|\delta|\ll\epsilon\ll 1$, then

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$$B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{d-1} n_q,$$

where n_q is the number of Morse critical points of a partial Morsefication of $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$ appearing in the regular part of X, and $X^g = X \cap \{g = 0\}$.

They also proved results about the topology of functions with isolated singularity defined over an analytic complex Whitney stratified variety X. If X is equidimensional, let $f, g : X \to \mathbb{C}$ be analytic functions with isolated singularity at the origin such that g is prepolar at the origin with respect to the good stratification induced by f (see (1)) and f is prepolar at the origin with respect to the good stratification induced by g, then

$$B_{f,X}(0) - B_{g,X}(0) = (-1)^{d-1}(n_q - m_q).$$

where $X^f = X \cap \{f = 0\}$, n_q is the number of Morse critical points of a Morsefication of $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$ appearing in the regular part of X and m_q is the number of Morse critical points of a Morsefication of $f|_{X \cap g^{-1}(\delta) \cap B_\epsilon}$ appearing in the regular part of X, for $0 < |\delta| \ll \epsilon \ll 1$.

Computing these numbers of stratified Morse critical points is directly connected to relative polar varieties. Consider l a linear form in $\mathbb{C}^n$, $\mathcal{W}$ a Whitney stratification of an open subset U of X, $\{W_i \setminus \{l = 0\}, W_i \cap \{l = 0\} \setminus \{0\}, \{0\}\}$ the good stratification of U induced by l and a function-germ $g : (U, 0) \to (\mathbb{C}, 0)$. If l sufficiently generic, the relative polar variety (curve) $\Gamma_{g,l}$ defined by Lê and Teissier, in [8], coincides with the relative polar curve defined by Massey, in [12], and with the relative polar varieties, defined by Massey, in [14] (see [11] and [13]). Each of these polar varieties are useful not only to compute polar multiplicities ([9]) and intersection numbers ([14]), but also to describe critical loci of pair of functions defined over X ([12], [3]), which is the approach we are interested the most.

In this work, we use relative polar varieties to compute a number of stratified Morse critical points of a specific type of Morsefication of a function-germ, aiming to obtain informations about the Brasselet number of this germ.

1 Local Euler obstruction and Euler obstruction of a function

We begin presenting the local Euler obstruction, a singular invariant defined by MacPherson and used as one of the main tools in his proof for the Deligne-Grothendieck conjecture about the existence and uniqueness of Chern classes for singular varities.

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an equidimensional reduced complex analytic germ of dimension d in a open set $U \subset \mathbb{C}^n$. Consider a complex analytic Whitney stratification $\mathcal{V} = \{V_\lambda\}$ of U adapted to X such that $\{0\}$ is a stratum. We choose a small representative of $(X, 0)$, denoted by X, such that 0 belongs to the closure of all strata. We write $X = \cup_{i=0}^{q} V_i$, where $V_0 = \{0\}$ and $V_q = X_{reg}$, where X_{reg} is the regular part of X. We suppose that $V_0, V_1, \ldots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \overline{V_1}, \ldots, \overline{V_q}$ are reduced. We write $d_i = dim(V_i),\ i \in \{1, \ldots, q\}$. Note that $d_q = d$.

Let $G(d, N)$ be the Grassmannian manifold, $x \in X_{reg}$ and consider the Gauss map $\phi : X_{reg} \to U \times G(d, N)$ given by $x \mapsto (x, T_x(X_{reg}))$.

Definition 1.1. The closure of the image of the Gauss map ϕ in $U \times G(d, N)$, denoted by $\tilde{X}$, is called **Nash modification** of X. It is a complex analytic space endowed with an analytic projection map $\nu : \tilde{X} \to X$.

Consider the extension of the tautological bundle $\mathcal{T}$ over $U \times G(d,N)$. Since $\tilde{X} \subset U \times G(d,N)$, we consider $\tilde{T}$ the restriction of $\mathcal{T}$ to $\tilde{X}$, called the **Nash bundle**, and $\pi : \tilde{T} \to \tilde{X}$ the projection of this bundle.

In this context, denoting by φ the natural projection of $U \times G(d,N)$ at U, we have the following diagram:

$$\begin{array}{ccc} \tilde{T} & \longrightarrow & \mathcal{T} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & U \times G(d,N) \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \longrightarrow & U \subseteq \mathbb{C}^N \end{array}$$

Considering $||z|| = \sqrt{z_1\overline{z_1} + \cdots + z_N\overline{z_N}}$, the 1-differential form $w = d||z||^2$ over $\mathbb{C}^N$ defines a section in $T^*\mathbb{C}^N$ and its pullback $\varphi^* w$ is a 1- form over $U \times G(d,N)$. Denote by $\tilde{w}$ the restriction of $\varphi^* w$ over $\tilde{X}$, which is a section of the dual bundle $\tilde{T}^*$.

Choose ϵ small enough for $\tilde{w}$ be a non zero section over $\nu^{-1}(z), 0 < ||z|| \leqslant \epsilon$, let B_ϵ be the closed ball with center at the origin with radius ϵ and denote:

1. $Obs(\tilde{T}^*, \tilde{w}) \in \mathbb{H}^{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z})$ as the obstruction for extending $\tilde{T}$ from $\nu^{-1}(S_\epsilon)$ to $\nu^{-1}(B_\epsilon)$;

2. $O_{\nu^{-1}(B_\epsilon),\nu^{-1}(S_\epsilon)}$ as the fundamental class in $\mathbb{H}_{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z})$.

Definition 1.2. The **local Euler obstruction** of X at 0, $Eu_X(0)$, is given by the evaluation

$$Eu_X(0) = \langle Obs(\tilde{T}^*, \tilde{w}), O_{\nu^{-1}(B_\epsilon),\nu^{-1}(S_\epsilon)} \rangle.$$

In [1], Brasselet, Lê and Seade proved a formula to make the calculation of the Euler obstruction easier.

Theorem 1.3. *(Theorem 3.1 of [1]) Let $(X,0)$ and $\mathcal{V}$ be given as before, then for each generic linear form l, there exists ϵ_0 such that for any ϵ with $0 < \epsilon < \epsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X,0)$ is equal to*

$$Eu_X(0) = \sum_{i=1}^{q} \chi(V_i \cap B_\epsilon \cap l^{-1}(\delta)).Eu_X(V_i),$$

where χ is the Euler characteristic, $Eu_X(V_i)$ is the Euler obstruction of X at a point of V_i, $i = 1, \ldots, q$ and $0 < |\delta| \ll \epsilon \ll 1$.

Let us give the definition of another invariant introduced by Brasselet, Massey, Parameswaran and Seade in [2]. Let $f : X \to \mathbb{C}$ be a holomorphic function with isolated singularity at the origin given by the restriction of a holomorphic function $F : U \to \mathbb{C}$ and denote by $\overline{\nabla} F(x)$ the conjugate of the gradient vector field of F in $x \in U$,

$$\overline{\nabla} F(x) := \left(\overline{\frac{\partial F}{\partial x_1}}, \ldots, \overline{\frac{\partial F}{\partial x_n}} \right).$$

Since f has an isolated singularity at the origin, for all $x \in X \setminus \{0\}$, the projection $\hat{\zeta}_i(x)$ of $\overline{\nabla} F(x)$ over $T_x(V_i(x))$ is non-zero, where $V_i(x)$ is a stratum containing x. Using this projection, the authors constructed, in [2], a stratified vector field over X, denoted by $\overline{\nabla} f(x)$. Let $\tilde{\zeta}$ be the lifting of $\overline{\nabla} f(x)$ as a section of the Nash bundle $\tilde{T}$ over $\tilde{X}$, without singularity over $\nu^{-1}(X \cap S_\epsilon)$.

Let $\mathcal{O}(\tilde{\zeta}) \in \mathbb{H}^{2n}(\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_\epsilon))$ be the obstruction cocycle for extending $\tilde{\zeta}$ as a non zero section of $\tilde{T}$ inside $\nu^{-1}(X \cap B_\epsilon)$.

Definition 1.4. The **local Euler obstruction of the function** f, $Eu_{f,X}(0)$ is the evaluation of $\mathcal{O}(\tilde{\zeta})$ on the fundamental class $[\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_\epsilon)]$.

The next theorem compares the Euler obstruction of a space X with the Euler obstruction of function defined over X.

Theorem 1.5. *(Theorem 3.1 of [2]) Let $(X,0)$ and $\mathcal{V}$ be given as before and let $f : (X,0) \to (\mathbb{C},0)$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \epsilon \ll 1$, we have*

$$Eu_{f,X}(0) = Eu_X(0) - \sum_{i=1}^{q} \chi(V_i \cap B_\epsilon \cap f^{-1}(\delta)).Eu_X(V_i).$$

In [17], Seade, Tibăr and Verjovsky proved that the Euler obstruction of a function f is also related to the number of Morse critical points of a Morsefication of f. Before we state their result, let us see the definition of a general point.

Definition 1.6. Let $(X,0) \subset (U,0)$ be a germ of complex analytic space in $\mathbb{C}^n$ equipped with a Whitney stratification and let $f : (X,0) \to (\mathbb{C},0)$ be an analytic function, given by the restriction of an analytic function $F : (U,0) \to (\mathbb{C},0)$. Then 0 is said to be a **generic point** of f if the hyperplane $Ker(d_0 F)$ is transverse in $\mathbb{C}^n$ to all limit of tangent spaces $\lim_{n\to\infty} T_{x_n}(V_\alpha)$, for all V_α and sequence of points $x_n \in V_\alpha$ converging to 0.

Now, let us see the definition of a Morsefication of a function.

Definition 1.7. A function $f : (X,0) \to (\mathbb{C},0)$ is said to be **Morse stratified** if for all strata V_α, with $dim V_\alpha \geq 1, 0$ is a generic point of the restriction $f|_{V_\alpha}$ and for $V_0 = \{0\}$, 0 is a Morse point of $f|_{V_0}$.

A stratified Morsefication of a germ of analytic function $f : (X,0) \to (\mathbb{C},0)$ is a deformation $\tilde{f}$ of f such that $\tilde{f}$ is Morse stratified.

Proposition 1.8. *(Proposition 2.3 of [17]) Let $f : (X,0) \to (\mathbb{C},0)$ be a germ of analytic function with isolated singularity at the origin. Then,*

$$Eu_{f,X}(0) = (-1)^d n_{reg},$$

where n_{reg} is the number of Morse points in X_{reg} in a stratified Morsefication of f.

In the case where we have a function with several number of isolated critical points, one can be interested in a deformation of this function which is a Morsefication around each one of these singularities. This is what Dutertre and Grulha called a partial Morsefication.

Definition 1.9. A **partial Morsefication** of $g : f^{-1}(\delta) \cap X \cap B_\epsilon \to \mathbb{C}$ is a function $\tilde{g} : f^{-1}(\delta) \cap X \cap B_\epsilon \to \mathbb{C}$ (not necessarily holomorphic) which is a local Morsefication of all isolated critical points of g in $f^{-1}(\delta) \cap X \cap \{g \neq 0\} \cap B_\epsilon$ and which coincides with g outside a small neighborhood of these critical points.

2 Brasselet number

We present now the two most important tools in this work: the relative polar variety, defined by Massey in [12], and the Brasselet number, introduced by Dutertre and Grulha in [3]. We also present formulas proved by Dutertre and Grulha to compute Brasselet numbers by counting numbers of stratified Morse critical points.

Let X be a reduced complex analytic space (not necessarily equidimensional) of dimension d in an open set $U \subseteq \mathbb{C}^n$ and let $f : (X, 0) \to (\mathbb{C}, 0)$ be an analytic map. We write $V(f) = f^{-1}(0)$.

Definition 2.1. A **good stratification of** X **relative to** f is a stratification $\mathcal{V}$ of X which is adapted to $V(f)$ such that $\{V_\lambda \in \mathcal{V}, V_\lambda \nsubseteq V(f)\}$ is a Whitney stratification of $X \setminus V(f)$ and such that for any pair (V_λ, V_γ) such that $V_\lambda \nsubseteq V(f)$ and $V_\gamma \subseteq V(f)$, the (a_f)-Thom condition is satisfied, that is, if $p \in V_\gamma$ and $p_i \in V_\lambda$ are such that $p_i \to p$ and $T_{p_i}V(f|_{V_\lambda} - f|_{V_\lambda}(p_i))$ converges to some $\mathcal{T}$, then $T_pV_\gamma \subseteq \mathcal{T}$.

If $f : X \to \mathbb{C}$ has a stratified isolated critical point and $\mathcal{V}$ is a Whitney stratification of X, then

$$\{V_\lambda \setminus X^f, V_\lambda \cap X^f \setminus \{0\}, \{0\}, V_\lambda \in \mathcal{V}\} \tag{1}$$

is a good stratification of X relative to f, called the good stratification induced by f.

Definition 2.2. The **critical locus of** f **relative to** $\mathcal{V}$, $\Sigma_{\mathcal{V}} f$, is given by the union

$$\Sigma_{\mathcal{V}} f = \bigcup_{V_\lambda \in \mathcal{V}} \Sigma(f|_{V_\lambda}).$$

Definition 2.3. If $\mathcal{V} = \{V_\lambda\}$ is a stratification of X, the **symmetric relative polar variety of** f **and** g **with respect to** $\mathcal{V}$, $\tilde{\Gamma}_{f,g}(\mathcal{V})$, is the union $\cup_\lambda \tilde{\Gamma}_{f,g}(V_\lambda)$, where $\Gamma_{f,g}(V_\lambda)$ denotes the closure in X of the critical locus of $(f, g)|_{V_\lambda \setminus (X^f \cup X^g)}$, $X^f = X \cap \{f = 0\}$ and $X^g = X \cap \{g = 0\}$.

Definition 2.4. Let $\mathcal{V}$ be a good stratification of X relative to a function $f : (X, 0) \to (\mathbb{C}, 0)$. A function $g : (X, 0) \to (\mathbb{C}, 0)$ is **prepolar with respect to** $\mathcal{V}$ **at the origin** if the origin is a stratified isolated critical point, that is, 0 is an isolated point of $\Sigma_{\mathcal{V}} g$.

Definition 2.5. A function $g : (X, 0) \to (\mathbb{C}, 0)$ is **tractable at the origin with respect to a good stratification** $\mathcal{V}$ **of** X **relative to** $f : (X, 0) \to (\mathbb{C}, 0)$ if $dim_0\, \tilde{\Gamma}^1_{f,g}(\mathcal{V}) \leq 1$ and, for all strata $V_\alpha \subseteq X^f$, $g|_{V_\alpha}$ has no critical points in a neighbourhood of the origin except perhaps at the origin itself.

Another concept useful for this work is the notion of constructible functions. Consider a Whitney stratification $\mathcal{W} = \{W_1, \dots, W_q\}$ of X such that each stratum W_i is connected.

Definition 2.6. A constructible function with respect to the stratification $\mathcal{W}$ of X is a function $\beta : X \to \mathbb{Z}$ which is constant on each stratum W_i, that is, there exist integers $t_1, \dots, t_q$, such that $\beta = \sum_{i=1}^q t_i.1_{W_i}$, where 1_{W_i} is the characteristic function of W_i.

Definition 2.7. The Euler characteristic $\chi(X, \beta)$ of a constructible function $\beta : X \to \mathbb{Z}$ with respect to the stratification $\mathcal{W}$ of X, given by $\beta = \sum_{i=1}^q t_i.1_{W_i}$, is defined by $\chi(X, \beta) = \sum_{i=1}^q t_i.\chi(W_i)$.

Before we state Dutertre and Grulha results, we need to introduce some definitions about normal Morse data. We cite as main references [4] and [18]. The first concept we present is the complex link, an object analogous to the Milnor fibre, important in the study of complex stratified Morse theory.

Let V be a stratum of the stratification $\mathcal{V}$ of X and let x be a point of V. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic complex function-germ such that the differential form $Dg(x)$ is not a degenerate covector of $\mathcal{V}$ at x. Let N be a normal slice to V at x, that is, N is a closed complex submanifold of $\mathbb{C}^n$ which is transversal to V at x and $N \cap V = \{x\}$.

Definition 2.8. Let B_ϵ be the closed ball of radius ϵ centered at x. The **complex link** l_V of V is defined by $l_V = X \cap N \cap B_\epsilon \cap \{g = \delta\}$, where $0 < |\delta| \ll \epsilon \ll 1$.

The **normal Morse datum** $NMD(V)$ of V is the pair of spaces

$$NMD(V) = (X \cap N \cap B_\epsilon, X \cap N \cap B_\epsilon \cap \{g = \delta\}).$$

In Part II, section 2.3 of [4], the authors explained why this two notions are independent of all choices made.

Definition 2.9. Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the stratification $\mathcal{V}$. Its normal Morse index $\eta(V, \beta)$ along V is defined by

$$\eta(V, \beta) = \chi(NMD(V), \beta) = \chi(X \cap N \cap B_\epsilon, \beta) - \chi(l_V, \beta).$$

In the case where the constructible function is the local Euler obstruction, the following identities are valid ([18], page 34):

$$\eta(V', Eu_{\overline{V}}) = 1, \text{ if } V' = V \text{ and } \eta(V', Eu_{\overline{V}}) = 0, \text{ if } V' \neq V.$$

We present now the definition of the Brasselet number and the main theorems of [3], used as inspiration for this work.

Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a complex analytic function germ and let $\mathcal{V}$ be a good stratification of X relative to f. We denote by $V_1, \ldots, V_q$ the strata of $\mathcal{V}$ that are not contained in $\{f = 0\}$ and we assume that $V_1, \ldots, V_{q-1}$ are connected and that $V_q = X_{reg} \setminus \{f = 0\}$. Note that V_q could be not connected.

Definition 2.10. Suppose that X is equidimensional. Let $\mathcal{V}$ be a good stratification of X relative to f. The **Brasselet number** of f at the origin, $B_{f,X}(0)$, is defined by

$$B_{f,X}(0) = \textstyle\sum_{i=1}^{q} \chi(V_i \cap f^{-1}(\delta) \cap B_\epsilon) Eu_X(V_i),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Remark: If V_q^i is a connected component of V_q, $Eu_X(V_q^i) = 1$.

Notice that if f has a stratified isolated singularity at the origin, then $B_{f,X}(0) = Eu_X(0) - Eu_{f,X}(0)$ (see Theorem 1.5).

In [3], Dutertre and Grulha proved interesting formulas describing the topological relation between the Brasselet number and a number of certain critical points of a special type of deformation of functions. Let us now present some of these results.

Let $g : (X, 0) \to (\mathbb{C}, 0)$ be a complex analytic function which is tractable at the origin with respect to $\mathcal{V}$ relative to f. Then $\tilde{\Gamma}_{f,g}$ is a complex analytic curve and for $0 < |\delta| \ll 1$ the critical points of $g|_{f^{-1}(\delta) \cap X}$ in B_ϵ lying outside $\{g = 0\}$ are isolated. Let $\tilde{g}$ be a partial Morsefication of $g : f^{-1}(\delta) \cap X \cap B_\epsilon \to \mathbb{C}$ and, for each $i \in \{1, \ldots, q\}$, let n_i be the number of stratified Morse critical points of $\tilde{g}$ appearing on $V_i \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$.

Theorem 2.11. *(Theorem 4.2 of [3]) Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the stratification $\mathcal{V}$. Suppose that $g : (X, 0) \to (\mathbb{C}, 0)$ is a complex analytic function tractable at the origin with respect to $\mathcal{V}$ relative to f. For $0 < |\delta| \ll \epsilon \ll 1$, we have*

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) - \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \sum_{i=1}^{q} (-1)^{d_i - 1} n_i \eta(V_i, \beta).$$

In the case that $\beta = Eu_X$, the last theorem implies the following.

Corollary 2.12. *(Corollary 4.3 of [3]) Suppose that X is equidimensional and that g is tractable at the origin with respect to $\mathcal{V}$ relative to f. For $0 < |\delta| \ll \epsilon \ll 1$, we have*

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, Eu_X) - \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\epsilon, Eu_X) = (-1)^{d-1} n_q.$$

If one supposes, in addition, that g is prepolar, a consequence of this result is a Lê-Greuel type formula for the Brasselet number.

Theorem 2.13. *(Theorem 4.4 of [3]) Suppose that X is equidimensional and that g is prepolar with respect to $\mathcal{V}$ at the origin. For $0 < |\delta| \ll \epsilon \ll 1$, we have*

$$B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{d-1} n_q,$$

where n_q is the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\epsilon$ appearing in a Morsefication of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$.

In [3], the authors also related the topology of the generalized Minor fibres of f and g and some number of Morse points.

Theorem 2.14. *Suppose that g (resp. f) is prepolar with respect to the good stratification induced by f (resp. g) at the origin. Let $\beta : X \to \mathbb{Z}$ be a constructible function with respect to the Whitney stratification $\mathcal{V}$. For $0 < |\delta| \ll \epsilon \ll 1$,*

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) - \chi(X \cap g^{-1}(\delta) \cap B_\epsilon, \beta) = \sum_{i=1}^{q} (-1)^{d_i - 1} (n_i - m_i) \eta(V_i, \beta),$$

where n_i (resp. m_i) is the number of stratified Morse critical points on the stratum $V_i \cap f^{-1}(\delta) \cap B_\epsilon$ (resp. $V_i \cap g^{-1}(\delta) \cap B_\epsilon$) appearing in a Morsefication of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ (resp. $f : X \cap g^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$).

In the case where $\beta = Eu_X$, the last theorem implies the following result.

Corollary 2.15. *Suppose that X is equidimensional and that g (resp. f) is prepolar with respect to the good stratification induced by f (resp. g) at the origin. Then*

$$B_{f,X}(0) - B_{g,X}(0) = (-1)^{d-1}(n_q - m_q),$$

where n_q (resp. m_q) is the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\epsilon$ (resp. $V_q \cap g^{-1}(\delta) \cap B_\epsilon$) appearing in a Morsefication of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ (resp. $f : X \cap g^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$).

3 Brasselet numbers and empty relative polar varieties

In this final section, we present relations between Brasselet numbers of two function-germs in the case where the relative polar variety associated to these germs is empty.

Let $(X,0)$ be a complex analytic space with ambient space $U \subset \mathbb{C}^n$. Let $f, g : (X,0) \to (\mathbb{C},0)$ be germs of holomorphic functions and let $\mathcal{V}$ be a good stratification of U relative to f. Suppose that $\Sigma_{\mathcal{V}} g \cap \{f=0\} = \{0\}$. We aim to obtain information about the Brasselet number of f and the Brasselet number of g in the case where the relative polar variety $\Gamma_{f,g}(\mathcal{V})$ is empty.

We begin with a description of two relevant subsets of the relative polar variety $\Gamma_{f,g}(\mathcal{V})$.

Proposition 3.1. *The stratified critical set $\Sigma_{\mathcal{V}} g$ of g and the symmetric relative polar variety $\tilde{\Gamma}_{f,g}(\mathcal{V})$ are subsets of $\Gamma_{f,g}(\mathcal{V})$.*

Proof. If $x \in \Sigma_{\mathcal{V}} g$, $d_x\tilde{g}|_{V_\alpha} = 0$, for a stratum $V_\alpha \in \mathcal{V}$ containing x and an analytic extension $\tilde{g}$ of g in a neighborhood of x. If $V_\alpha \subset \{f=0\}$, then $x=0$, since $\Sigma_{\mathcal{V}} g \cap \{f=0\} = \{0\}$. If $V_\alpha \subset X \setminus \{f=0\}$,

$$rk(d_x\tilde{f}|_{V_\alpha}, d_x\tilde{g}|_{V_\alpha}) \leq 1,$$

where $\tilde{f}$ is an analytic extension of f in a neighborhood of x, that is $x \in \Sigma(f,g)|_{V_\alpha} = \Sigma(f,g)|_{V_\alpha \setminus \{f=0\}}$. Therefore, $x \in \Gamma_{f,g}(\mathcal{V}_\alpha)$.

Furthermore, $\tilde{\Gamma}_{f,g}(V_\alpha)$ is given by components of $\Gamma_{f,g}(V_\alpha)$ no contained in $\{g=0\}$, that is, $\tilde{\Gamma}_{f,g}(V_\alpha) = \overline{\Gamma_{f,g}(V_\alpha) \setminus \{g=0\}} \subseteq \Gamma_{f,g}(V_{\alpha}.)$ Therefore,

$$\tilde{\Gamma}_{f,g}(\mathcal{V}) \cup \Sigma_{\mathcal{V}} g \subseteq \Gamma_{f,g}(\mathcal{V}).$$

□

Using this proposition, we obtain the following useful information about the behavior of g with respect to the stratification $\mathcal{V}$.

Corollary 3.2. *If $\Gamma_{f,g}(\mathcal{V})$ is empty, then g is prepolar at the origin with respect to the good stratification $\mathcal{V}$ of X relative by f.*

Proof. By Proposition 3.1, if $\Gamma_{f,g}(\mathcal{V})$ is empty, $\Sigma_{\mathcal{V}} g$ is empty, that is, g has no stratified critical point with respect to $\mathcal{V}$. □

Let n_i be the number of stratified Morse critical points of a Morsefication of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ in $V_i \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$, for each $i \in \{1, \ldots, q\}$. The next proposition uses the relative polar variety $\Gamma_{f,g}(\mathcal{V})$ for counting the numbers n_i.

Proposition 3.3. *If $\Gamma_{f,g}(\mathcal{V})$ is empty, then $n_i = 0$, for all $i \in \{1, \ldots, q\}$.*

Proof. Let V_i be a stratum of $\mathcal{V}$ and x be a critical point of $g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon}$. Then, if $\tilde{f}$ and $\tilde{g}$ are analytic extensions of f and g in a neighborhood of x, respectively, $x \in V_i \cap f^{-1}(\delta) \cap B_\epsilon$ and $rk(d_x\tilde{f}|_{V_i}, d_x\tilde{g}|_{V_i}) \leq 1$, that is,

$$x \in \big(V_i \cap f^{-1}(\delta) \cap B_\epsilon\big) \cap \big(\Sigma_{\mathcal{V}} f \cup \Sigma_{\mathcal{V}} g \cup \tilde{\Gamma}_{f,g}(\mathcal{V})\big).$$

By Proposition 1.3 of [12], $\Sigma_{\mathcal{V}} f \subset \{f=0\}$. Therefore,

$$\Sigma g|_{V_i\cap f^{-1}(\delta)\cap B_\epsilon} = V_i \cap f^{-1}(\delta) \cap B_\epsilon \cap \left(\tilde{\Gamma}_{f,g}(V_i) \cup \Sigma g|_{V_i}\right) \subseteq V_i \cap f^{-1}(\delta) \cap B_\epsilon \cap \Gamma_{f,g}(V_i).$$

Since $\Gamma_{f,g}(\mathcal{V})$ is empty, by Proposition 3.1, $\Sigma g|_{V_i\cap f^{-1}(\delta)\cap B_\epsilon}$ is empty . Therefore, $n_i = 0$, for all $i \in \{1, \dots, q\}$. □

In [3], Dutertre and Grulha proved a Lê-Greuel type formula for the Brasselet number, with which it is possible count the number of stratified Morse critical points using Brasselet numbers. We apply their result to obtain a relation between Brasselet number in the setting we already know the number of Morse points. First let us show a more general result.

Corollary 3.4. *If $\beta : X \to \mathbb{Z}$ is a constructible function with respect to the good stratification $\mathcal{V}$ of X relative to f and $\Gamma_{f,g}(\mathcal{V})$ is empty, then*

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \chi(X \cap \{g = 0\} \cap f^{-1}(\delta) \cap B_\epsilon, \beta).$$

Proof. By Corollary 3.2, since $\Gamma_{f,g}(\mathcal{V})$ is empty, g is prepolar at the origin with respect to $\mathcal{V}$ and, by Proposition 1.12 of [12], tractable at the origin with respect to $\mathcal{V}$. Then, by Theorem 4.2 of [3], we obtain

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \chi(X \cap \{g = 0\} \cap f^{-1}(\delta) \cap B_\epsilon, \beta) + \sum_{i=1}^{q} (-1)^{d-1} n_i \eta(V_i, \beta),$$

where n_i is the number of stratified Morse critical points of a Morsefication of $g|_{V_i\cap f^{-1}(\delta)\cap B_\epsilon}$ appearing in $V_i \cap f^{-1}(\delta) \cap B_\epsilon$. Using again that $\Gamma_{f,g}(\mathcal{V})$ is empty, $n_i = 0$, for all $i \in \{1, \dots, q\}$ and we conclude the equality. □

If the constructible function β is giving by the local Euler obstruction, one obtains a relation between Brasselet numbers.

Corollary 3.5. *If X is equidimensional and $\Gamma_{f,g}(\mathcal{V})$ is empty, then $B_{f,X}(0) = B_{f,X^g}(0)$.*

Proof. By Theorem 4.4 of [3], $B_{f,X}(0) = B_{f,X^g}(0) + (-1)^{d-1} n_q$, where n_q is the number of stratified Morse criticalpoints of a Morsefication of $g|_{X\cap f^{-1}(\delta)\cap B_\epsilon}$ appearing on $V_q \cap f^{-1}(\delta) \cap B_\epsilon$. Since $\Gamma_{f,g}(\mathcal{V})$ is empty, n_q is zero and the equality holds. □

When f is a generic linear form on $\mathbb{C}^n$, $B_{f,X}(0) = Eu_X(0)$ and $B_{f,X^g}(0) = Eu_{X^g}(0)$. Therefore, Corollary 3.5 implies the following consequence.

Corollary 3.6. *If X is equidimensional and $\Gamma_{f,g}(\mathcal{V})$ is empty, then $Eu_X(0) = Eu_{X^g}(0)$.*

When both f and g has isolated singularity at the origin, Dutertre and Grulha proved several formulas about the Brasselet numbers of f and g. Using this formulas, one can obtain further information about these numbers if $\Gamma_{f,g}(\mathcal{V})$ is empty. If that is the case, then g is prepolar at the origin with respect to the good stratification $\mathcal{V}$ of X induced by f, given as a refinement of a Whitney stratification $\mathcal{W} = \{W_i\}_i$ of X. By Corollary 6.1 of [3], f is prepolar at the origin with respect to the good stratification $\overline{\mathcal{V}}$ induced by g, also given by a refinement of $\mathcal{W}$. Applying Proposition 1.12 of [12], we obtain that $\Gamma_{f,g}(\mathcal{V}) = \tilde{\Gamma}_{f,g}(\mathcal{V})$ and $\Gamma_{g,f}(\overline{\mathcal{V}}) = \tilde{\Gamma}_{g,f}(\overline{\mathcal{V}})$. But

$$\begin{aligned}
\tilde{\Gamma}_{f,g}(\mathcal{V}) &= \bigcup_{V_i \in \mathcal{V}} \overline{\Sigma(f,g)|_{V_i \setminus (\{f=0\} \cup \{g=0\})}} \\
&= \bigcup_{W_i \in \mathcal{W}} \overline{\Sigma(f,g)|_{(W_i \setminus \{f=0\}) \setminus \{g=0\}}} \\
&= \bigcup_{W_i \in \mathcal{W}} \overline{\Sigma(f,g)|_{(W_i \setminus \{g=0\}) \setminus \{f=0\}}} \\
&= \bigcup_{\overline{V_i} \in \overline{\mathcal{V}}} \overline{\Sigma(f,g)|_{\overline{V_i} \setminus (\{g=0\} \cup \{f=0\})}} \\
&= \tilde{\Gamma}_{g,f}(\overline{\mathcal{V}}).
\end{aligned}$$

Therefore, these four polar varieties are equal. Using this description, one concludes the following.

Proposition 3.7. *If $\beta : X \to \mathbb{Z}$ is a constructible function with respect to the Whitney stratification $\mathcal{W}$ and $\Gamma_{f,g}(\mathcal{V})$ is empty, where $\mathcal{V}$ is the good stratification of X induced by f, then*

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \chi(X \cap g^{-1}(\delta) \cap B_\epsilon, \beta).$$

Proof. Since $\Gamma_{f,g}(\mathcal{V})$ is empty, g is prepolar at the origin with respect to $\mathcal{V}$. Then, f is prepolar at the origin with respect to the good stratification $\overline{\mathcal{V}}$ induced by g. By Theorem 6.4 of [3],

$$\begin{aligned}
\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) &= \chi(X \cap g^{-1}(\delta) \cap B_\epsilon, \beta) \\
&+ \sum_{i=1}^{q} (-1)^{d_i - 1}(n_i - m_i)\eta(W_i, \beta),
\end{aligned}$$

where d_i denotes the dimension of $W_i \in \mathcal{W}$. By Proposition 3.4, if m_i is the number of stratified Morse critical points of a Morsefication of $f|_{X \cap g^{-1}(\delta) \cap B_\epsilon}$ appearing on $V_i \cap g^{-1}(\delta) \cap \{f \neq 0\} \cap B_\epsilon$, for each $i \in \{1, \ldots, q\}$, since $\Gamma_{f,g}(\mathcal{V})$ is empty, $\Gamma_{f,g}(\overline{\mathcal{V}})$ is empty and $m_i = 0$. Since the number n_i of stratified Morse critical points of a Morsefication of $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$ in $V_i \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$ is zero, for each $i \in \{1, \ldots, q\}$, the equality is proved. □

Corollary 3.8. *If X is equidimensional and $\Gamma_{f,g}(\mathcal{V})$ is empty, then $B_{g,X}(0) = B_{f,X}(0)$.*

Proof. Since $\Gamma_{f,g}(\mathcal{V})$ is empty, g (resp. f) is prepolar wih respect to the good stratification of X induced by f (resp. g). Applying Corollary 6.5 of [3], we obtain $B_{f,X}(0) = B_{g,X}(0) + (-1)^{d-1}(n_q - m_q)$, where n_q (resp. m_q) is the number of stratified Morse critical points of a Morsefication of $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$ (resp. $f|_{X \cap g^{-1}(\delta) \cap B_\epsilon}$) appearing on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\epsilon$ (resp. $V_q \cap g^{-1}(\delta) \cap B_\epsilon$). Using again that $\Gamma_{f,g}(\mathcal{V})$ is empty, we have that $n_q = m_q = 0$, what leads to the equality. □

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