A note on the number of cusps of perturbations of complex polynomials

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1. Introduction

A smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called an *excellent map* if for any $p \in \mathbb{R}^2$, there exist local coordinates (x, y) centered at p and local coordinates centered at $f(p)$ such that f is locally described in one of the following forms:

(1) $(x, y) \mapsto (x, y)$,
(2) $(x, y) \mapsto (x, y^2)$,
(3) $(x, y) \mapsto (x, y^3 + xy)$.

A point in case (1) is a regular point. Points in cases (2) and (3) are called a *fold* and a *cusp*, respectively. Denote by $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ the set of all smooth maps $\mathbb{R}^2 \to \mathbb{R}^2$ equipped with the C^∞-topology. In [7], Whitney showed that the set of excellent maps is dense in $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$. It's known that there is a relation between the topology of surfaces and the topology of the critical locus of a map, see [5, 1]. Fukuda and Ishikawa also studied the number of cusps of stable perturbations of generic map germs [1]. They showed the number of cusps modulo 2 is a topological invariant of generic map germs. Moreover, the number of cusps modulo 2 depends only on the topology of surfaces.

Let $f(z)$ be a complex polynomial such that $f(0) = 0$. Then there exist a positive integer k and a complex polynomial g such that $f(z) = z^k g(z)$ and $g(0) \neq 0$. We call k *the multiplicity of f at the origin*. We consider certain perturbations of complex polynomials and calculate explicitly the number of cusps of perturbations by using multiplicities of singularities of complex polynomials.

We identify $\mathbb{C}$ with $\mathbb{R}^2$. Then $f(z)$ defines a real polynomial map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (\Re f(x, y), \Im f(x, y)),$$

where $z = x + \sqrt{-1}y$. Assume that the origin 0 of $\mathbb{C}$ is a singularity of f. We define a *linear perturbation f_t of f* as follows:

$$f_t(z) := f(z) + t(a + ib)\bar{z},$$

where $a, b, t \in \mathbb{R}, i = \sqrt{-1}$ and $0 < |t| \ll 1$. Note that a linear perturbation f_t of f is not a complex polynomial, but is a 1-variable mixed polynomial in the sense of Oka [4]. We now regard a mixed polynomial map $f_t : \mathbb{C} \to \mathbb{C}$ as a real polynomial map $(\Re f_t, \Im f_t) : \mathbb{R}^2 \to \mathbb{R}^2$. If $f(z) = z^n$, Fukuda and Ishikawa showed that the number of cusps of a linear perturbation of f is congruent to $n + 1$ modulo 2, see [1, Example 2.3]. If a and b lie outside the union of zero sets of analytic functions determined by a, b and f, f_t is an excellent map for $0 < |t| \ll 1$, see Lemma 2. The main theorem is the following [2].

Theorem 1. *Let $f(z)$ be a complex polynomial and k be the multiplicity of f the origin. Suppose that $k \geq 2$. If a linear perturbation f_t of f is an excellent map for $0 < |t| \ll 1$, then the number of cusps of $f_t|_U$ is equal to $k + 1$, where U is a sufficiently small neighborhood of the origin.*

2. Singularities of polynomial maps

Let $g = (g_1, g_2) : U \to \mathbb{R}^2$ be a polynomial map, where U is an open set. Set $J = \frac{\partial(g_1, g_2)}{\partial(x, y)}$, $G_i = \frac{\partial(g_i, J)}{\partial(x, y)}$ for $i = 1, 2$. We define the algebraic set G' as follows:

$$G' := \Big\{ (x, y) \in U \mid J(x, y) = G_1(x, y) = G_2(x, y) = \frac{\partial(G_1, J)}{\partial(x, y)} = \frac{\partial(G_2, J)}{\partial(x, y)} = 0 \Big\}.$$

In [3, Proposition 2] and [6, Proposition 2.2], Krzyżanowska and Szafraniec showed the following proposition:

Proposition 1. *The algebraic set G' is empty if and only if the set of singularities of g consists of either fold singularities or cusps. Moreover, the number of cusps of g is equal to the number of $\{(x, y) \in U \mid J(x, y) = G_1(x, y) = G_2(x, y) = 0\}$.*

3. Multiplicity with sign

Set $z = x + iy$. Then a pair of real polynomials (g_1, g_2) defines a mixed polynomial $g(z, \bar{z})$ as follows:

$$\begin{aligned}
g(z, \bar{z}) &= g_1(x, y) + ig_2(x, y) \\
&= g_1\Big(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\Big) + ig_2\Big(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\Big).
\end{aligned}$$

Suppose that w is a mixed singularity of a mixed polynomial g, i.e., the gradient vectors of g_1 and g_2 at w are linearly dependent over $\mathbb{R}$. Then we have

$$\Big| \frac{\partial g}{\partial z}(w) \Big| = \Big| \frac{\partial g}{\partial \bar{z}}(w) \Big|,$$

see [4]. Let $\alpha \in \mathbb{C}$ be an isolated root of $g(z, \bar{z}) = 0$. Put

$$S_\varepsilon^1(\alpha) := \{ z \in \mathbb{C} \mid |z - \alpha| = \varepsilon \},$$

where ε is a sufficiently small positive real number. We define the *multiplicity with the sign of the root α* by the mapping degree of the normalized function

$$\frac{g}{|g|} : S_\varepsilon^1(\alpha) \to S^1.$$

We denote the multiplicity with the sign of the root α by $m_s(g, \alpha)$.

We say that α is a *positive simple root* if α satisfies

$$\Big| \frac{\partial g}{\partial z}(\alpha) \Big| > \Big| \frac{\partial g}{\partial \bar{z}}(\alpha) \Big|.$$

Similarly, α is a *negative simple root* if α satisfies

$$\Big| \frac{\partial g}{\partial z}(\alpha) \Big| < \Big| \frac{\partial g}{\partial \bar{z}}(\alpha) \Big|.$$

In [4, Proposition 15], α is a positive (resp. negative) simple root if and only if $m_s(g, \alpha) = 1$ (resp. $m_s(g, \alpha) = -1$).

Consider a family of mixed polynomials $g_t(z, \bar{z}) = 0$ for $g_0 = g$ and $t \in \mathbb{R}$. Oka showed the following proposition, see [4, Proposition 16].

Proposition 2. *Let $\{P_1(t), \ldots, P_\nu(t)\}$ be the roots of $g_t(z, \bar{z}) = 0$ which are bifurcating from $z = \alpha$. Then we have*

$$\textstyle\sum_{j=1}^{\nu} m_s(g_t, P_j(t)) = m_s(g, \alpha).$$

4. The existence of linear perturbations which are excellent maps

Let $f(z)$ be a complex polynomial. Assume that $f(0) = 0$ and the origin of $\mathbb{C}$ is a singularity of f. Set $f_1 = \Re f$ and $f_2 = \Im f$. We take $a, b \in \mathbb{R}$. Then a linear perturbation f_t of f is defined by $f_t(z) = f(z) + t(a + ib)\bar{z}$, where $0 < |t| \ll 1$. Note that f_t is equal to

$$
\begin{aligned}
f_t(z) &= f(z) + t(a + ib)\bar{z} \\
&= f_1(z) + t(ax + by) + i\{f_2(z) + t(bx - ay)\}.
\end{aligned}
$$

Then f_t defines a real polynomial map from $\mathbb{R}^2$ to $\mathbb{R}^2$ as follows:

$$
f_t : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (f_1(x, y) + t(ax + by), f_2(x, y) + t(bx - ay)).
$$

We calculate J, G_1 and G_2 of f_t. By the Cauchy–Riemann equations $\frac{\partial f_2}{\partial x} = -\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial y} = \frac{\partial f_1}{\partial x}$, J is modified as

$$
\begin{aligned}
J &= \det \begin{pmatrix} \frac{\partial f_1}{\partial x} + ta & \frac{\partial f_1}{\partial y} + tb \\ \frac{\partial f_2}{\partial x} + tb & \frac{\partial f_2}{\partial y} - ta \end{pmatrix} \\
&= \left| \frac{\partial f}{\partial z} \right|^2 - t^2(a^2 + b^2).
\end{aligned}
$$

Since f is a complex valued harmonic function, $\frac{\partial f_1}{\partial x \partial x} = -\frac{\partial f_1}{\partial y \partial y}$. Then we have

$$
\begin{aligned}
G_1 &= \det \begin{pmatrix} \frac{\partial f_1}{\partial x} + ta & \frac{\partial f_1}{\partial y} + tb \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y} \end{pmatrix} \\
&= 2\left(\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_1}{\partial y}\right)^2 \right) \frac{\partial^2 f_1}{\partial x \partial y} + 4 \frac{\partial f_1}{\partial x} \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial y \partial y} \\
&\quad + 2t\left\{ a\left(\frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial y \partial y} \right) - b\left(-\frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial y \partial y} + \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial x \partial y} \right) \right\}, \\
G_2 &= \det \begin{pmatrix} -\frac{\partial f_1}{\partial y} + tb & \frac{\partial f_1}{\partial x} - ta \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y} \end{pmatrix} \\
&= 2\left(\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_1}{\partial y}\right)^2 \right) \frac{\partial^2 f_1}{\partial y \partial y} - 4 \frac{\partial f_1}{\partial x} \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial x \partial y} \\
&\quad + 2t\left\{ a\left(-\frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial y \partial y} + \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial x \partial y} \right) + b\left(\frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial y \partial y} \right) \right\}.
\end{aligned}
$$

If G_1 and G_2 are equal to 0 at (x, y), then (x, y) satisfies the following equation:

$$
\begin{aligned}
a&\left[\left(-3\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 \right) \frac{\partial f_1}{\partial y} \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y}\right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y}\right)^2 \right\} \right. \\
&\left. + 2\left(-\left(\frac{\partial f_1}{\partial x}\right)^2 + 3\left(\frac{\partial f_1}{\partial y}\right)^2 \right) \frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial^2 f_1}{\partial y \partial y} \right] \\
+ b&\left[\left(-\left(\frac{\partial f_1}{\partial x}\right)^2 + 3\left(\frac{\partial f_1}{\partial y}\right)^2 \right) \frac{\partial f_1}{\partial x} \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y}\right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y}\right)^2 \right\} \right. \\
&\left. - 2\left(-3\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 \right) \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial^2 f_1}{\partial y \partial y} \right] \\
=& \ 0.
\end{aligned} \tag{1}
$$

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Set real polynomials ϕ_1, ϕ_2 and Φ as follows:

$$\phi_1 := \left(-3\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2\right)\frac{\partial f_1}{\partial y}\left\{\left(\frac{\partial^2 f_1}{\partial y \partial y}\right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y}\right)^2\right\}$$
$$+ 2\left(-\left(\frac{\partial f_1}{\partial x}\right)^2 + 3\left(\frac{\partial f_1}{\partial y}\right)^2\right)\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x \partial y}\frac{\partial^2 f_1}{\partial y \partial y},$$

$$\phi_2 := \left(-\left(\frac{\partial f_1}{\partial x}\right)^2 + 3\left(\frac{\partial f_1}{\partial y}\right)^2\right)\frac{\partial f_1}{\partial x}\left\{\left(\frac{\partial^2 f_1}{\partial y \partial y}\right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y}\right)^2\right\}$$
$$- 2\left(-3\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2\right)\frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x \partial y}\frac{\partial^2 f_1}{\partial y \partial y},$$

$$\Phi := a\phi_1 + b\phi_2.$$

Suppose that G_1 and G_2 are equal to 0 at (x, y). By the equation (1) and the definitions of ϕ_1, ϕ_2 and Φ, $\Phi(x, y)$ is also equal to 0. To show the existence of linear perturbations which are excellent maps, we consider the intersection of $\phi_1^{-1}(0)$ and $\phi_2^{-1}(0)$.

Lemma 1. *Let U be a sufficiently small neighborhood of the origin 0 of $\mathbb{C}$. Assume that U satisfies $\{w \in U \mid \frac{\partial f}{\partial z}(w) = 0\} = \{0\}$ and $\{w \in U \mid \frac{\partial^2 f}{\partial z \partial z}(w) = 0\} \subset \{0\}$. Then the intersection of $\phi_1^{-1}(0), \phi_2^{-1}(0)$ and U is equal to $\{0\}$.*

To study singularities of f_t, we define the mixed polynomial G_t as follows:

$$G_t := G_1 + iG_2$$
$$= \left(\frac{\partial f}{\partial z} + t(a + ib)\right)\frac{\partial J}{\partial y} - i\left(\frac{\partial f}{\partial z} - t(a + ib)\right)\frac{\partial J}{\partial x}.$$

Since $\frac{\partial J}{\partial z}$ is equal to $\frac{1}{2}\left(\frac{\partial J}{\partial x} - i\frac{\partial J}{\partial y}\right)$, $\frac{\partial J}{\partial x}$ and $\frac{\partial J}{\partial y}$ are equal to

$$\frac{\partial J}{\partial x} = 2\Re\frac{\partial J}{\partial z} = 2\Re\frac{\partial^2 f}{\partial z \partial z}\frac{\overline{\partial f}}{\partial z} = \frac{\partial^2 f}{\partial z \partial z}\frac{\overline{\partial f}}{\partial z} + \frac{\overline{\partial^2 f}}{\partial z \partial z}\frac{\partial f}{\partial z},$$
$$\frac{\partial J}{\partial y} = -2\Im\frac{\partial J}{\partial z} = -2\Im\frac{\partial^2 f}{\partial z \partial z}\frac{\overline{\partial f}}{\partial z} = i\left(\frac{\partial^2 f}{\partial z \partial z}\frac{\overline{\partial f}}{\partial z} - \frac{\overline{\partial^2 f}}{\partial z \partial z}\frac{\partial f}{\partial z}\right),$$

where $z = x + iy$. Thus G_t is equal to

$$-2i\left(\frac{\partial f}{\partial z}\right)^2\frac{\overline{\partial^2 f}}{\partial z \partial z} + 2ti(a + ib)\frac{\partial^2 f}{\partial z \partial z}\frac{\overline{\partial f}}{\partial z}.$$

Suppose that z satisfies $G_t(z) = 0$ and $\frac{\partial f}{\partial z}(z)\frac{\partial^2 f}{\partial z \partial z}(z) \neq 0$. By the above equation, z satisfies $J(z) = 0$. Since the multiplicity k of f at the origin is greater than 1, $G_t(0) = 0$ and $\frac{\partial f}{\partial z}(0)\frac{\partial^2 f}{\partial z \partial z}(0) = 0$. Thus we have

$$\left\{z \in U \mid G_t(z) = 0, \frac{\partial f}{\partial z}(z) \neq 0, \frac{\partial^2 f}{\partial z \partial z}(z) \neq 0\right\}$$
$$= \{z \in U \setminus \{0\} \mid G_t(z) = 0\} \subset J^{-1}(0).$$

Similarly, we define the following mixed polynomial:

$$H_t := \det\begin{pmatrix} \frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y} \end{pmatrix} + i\det\begin{pmatrix} \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y} \end{pmatrix}$$
$$= \left(\frac{\partial G_1}{\partial x} + i\frac{\partial G_2}{\partial x}\right)\frac{\partial J}{\partial y} - \left(\frac{\partial G_1}{\partial y} + i\frac{\partial G_2}{\partial y}\right)\frac{\partial J}{\partial x}.$$

The differentials of G_t satisfy the following equations:
$$\frac{\partial G_t}{\partial z} = \frac{1}{2}\Big(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\Big), \quad \frac{\partial G_t}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial G_1}{\partial x} - \frac{\partial G_2}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial G_2}{\partial x} + \frac{\partial G_1}{\partial y}\Big).$$
Then we have
$$H_t = \Big(\frac{\partial G_t}{\partial z} + \frac{\partial G_t}{\partial \bar{z}}\Big)\frac{\partial J}{\partial y} - i\Big(\frac{\partial G_t}{\partial z} - \frac{\partial G_t}{\partial \bar{z}}\Big)\frac{\partial J}{\partial x}$$
$$= \frac{\partial G_t}{\partial z}\Big(\frac{\partial J}{\partial y} - i\frac{\partial J}{\partial x}\Big) + \frac{\partial G_t}{\partial \bar{z}}\Big(\frac{\partial J}{\partial y} + i\frac{\partial J}{\partial x}\Big).$$
Since $\frac{\partial J}{\partial y} - i\frac{\partial J}{\partial x} = -2i\frac{\partial J}{\partial \bar{z}}$ and $\frac{\partial J}{\partial y} + i\frac{\partial J}{\partial x} = 2i\frac{\partial J}{\partial z}$, H_t is equal to
$$H_t = -4\Big(\frac{\partial f}{\partial z}\Big)^2 \frac{\partial^2 f}{\partial z\partial z}\overline{\Big\{2\Big(\frac{\partial^2 f}{\partial z\partial z}\Big)^2 - \frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z\partial z\partial z}\Big\}}$$
$$+ 4t(a+ib)\overline{\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial z\partial z}}\Big\{-\Big(\frac{\partial^2 f}{\partial z\partial z}\Big)^2 + \frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z\partial z\partial z}\Big\}.$$

Note that $J(0) = |\frac{\partial f}{\partial z}(0)|^2 - t^2(a^2 + b^2) \neq 0$ for $t \neq 0$ and $(a,b) \neq (0,0)$. By the definitions of G_t and H_t, we have
$$\{z \in U \setminus \{0\} \mid G_t(z) = H_t(z) = 0\}$$
$$= \Big\{z \in U \mid J(z) = G_1(z) = G_2(z) = \frac{\partial(G_1, J)}{\partial(x,y)}(z) = \frac{\partial(G_2, J)}{\partial(x,y)}(z) = 0\Big\}.$$

By using Lemma 1, we show the existence of a linear perturbation f_t of f which is an excellent map for generic (a,b).

Lemma 2. *For a generic choice of (a,b), $f_t|_U$ is an excellent map.*

Let w be a singularity of f and U_w be a sufficiently small neighborhood of w. By changing coordinates of U_w and $f(U_w)$, we may assume that $w = 0$ and $f(w) = 0$. So we can apply Lemma 2 to any singularity of f. Thus we can check that f_t is an excellent map for $0 < |t| \ll 1$ if a and b are generic.

5. Calculation of the number of cusps

To calculate the number of cusps of f_t, we study zero points of G_t and differentials of G_t.

Lemma 3. *The set $\{z \in U \mid G_t(z) = 0, z \neq 0\}$ is the set of positive simple roots of G_t for $(a,b) \neq (0,0)$ and $0 < |t| \ll 1$.*

Assume that f_t is an excellent map for $0 < |t| \ll 1$. We calculate the number of cusps of $F_t|_U$. By Proposition 1, the number of cusps of $f_t|_U$ is equal to
$$\sharp\{z \in U \mid G_t(z) = 0, z \neq 0\}.$$
Set $\{z \in U \mid G_t(z) = 0, z \neq 0\} = \{w_1, \ldots, w_\nu\}$. We denote the multiplicity of sign by $m_s(G_t, w_j)$ for $j = 1, \ldots, \nu$. By Proposition 2 and Lemma 3, we have
$$\Big(\sum_{j=1}^{\nu} m_s(G_t, w_j)\Big) + m_s(G_t, 0) = \nu + m_s(G_t, 0) = m_s(G_0, 0).$$
The multiplicity $m_s(G_0, 0)$ is equal to
$$\deg\Big(-2i\Big(\frac{\partial f}{\partial z}\Big)^2 \overline{\frac{\partial^2 f}{\partial z\partial z}}\Big)\Big/\Big|-2i\Big(\frac{\partial f}{\partial z}\Big)^2 \overline{\frac{\partial^2 f}{\partial z\partial z}}\Big| : S_\varepsilon^1(0) \to S^1\Big)$$
$$= 2(k-1) - (k-2) = k,$$

where $S^1_\varepsilon(0) = \{z \in U \mid |z| = \varepsilon\}$ and $0 < \varepsilon \ll 1$. By the definition of G_t, for any $t \neq 0$, $m_s(G_t, 0)$ is equal to

$$\deg\left(2ti(a+ib)\frac{\partial^2 f}{\partial z \partial z}\frac{\overline{\partial f}}{\partial z} \Big/ \left|2ti(a+ib)\frac{\partial^2 f}{\partial z \partial z}\frac{\overline{\partial f}}{\partial z}\right| : S^1_{\varepsilon_t}(0) \to S^1\right)$$
$$= k - 2 - (k-1) = -1,$$

where $0 < \varepsilon_t \ll \varepsilon$. Thus the number ν of cusps of $f_t|_U$ is equal to $k+1$.

We estimate the number of cusps of f_t in $\mathbb{R}^2$.

Corollary 1. *Let f_t be a liner perturbation of a complex polynomial f in Theorem 1 and $n = \deg f$. Assume that $n \geq 2$. Then the number of cusps of f_t belongs to $[n+1, 3n-3]$. In particular, the number of cusps of f_t is at least three.*

6. Examples

In this section, we construct a perturbation of a complex polynomial which has $(n+1)$-cusps and also a perturbation which has $(3n-3)$-cusps.

Example 1. *Let $f(z) = z^n$ and $f_t(z) = z^n + t(a+ib)\bar{z}$ be a perturbation of f which is an excellent map. Then $G_t(z)$ is equal to*

$$G_t(z) = -2in^3(n-1)z^{2n-2}\bar{z}^{n-2} + 2tn^2(n-1)(a+ib)z^{n-2}\bar{z}^{n-1}$$
$$= -2in^2(n-1)|z|^{2n-4}\{nz^n - t(a+ib)\bar{z}\}.$$

Set $z = re^{i\theta}$ and $a + ib = \tau e^{i\iota}$, where $\tau > 0$. Then we have

$$-2in^2(n-1)r^{2n-4}\{nr^n e^{ni\theta} - t\tau r e^{i(\iota-\theta)}\}.$$

Assume that $z \neq 0$ and $G_t(z) = 0$. Then z satisfies

$$r = \left(\frac{t\tau}{n}\right)^{\frac{1}{n-1}}, \quad \theta = \frac{\iota + 2j\pi}{n+1},$$

for $j = 0, \ldots, n$. Thus the number of cusps of f_t is equal to $n+1$.

Example 2. *Let $f(z) = z^n + z$. Then the number of singularities of f is equal to $n-1$ and the multiplicity at each singularity of f is equal to 2. Let $f_t(z) = z^n + z + t(a+ib)\bar{z}$ be a perturbation of f which is an excellent map. By the same argument as in the proof of Corollary 1, the number of cusps of f_t is equal to $3n-3$.*

7. Non-linear perturbations

7.1. Perturbations of f_t. Let f_t be a linear perturbation of f which is an excellent map. We fix a, b and t. Let $g(z, \bar{z})$ be a mixed polynomial which satisfies $\frac{\partial g}{\partial z}(0) = \frac{\partial g}{\partial \bar{z}}(0) = 0$. In this subsection, we study a perturbation of f_t:

$$f_{t,s}(z) := f(z) + t(a+ib)\bar{z} + sg(z, \bar{z}),$$

where $0 < |s| \ll |t| \ll 1$. Since $|s|$ is sufficiently small, we can show the following theorem.

Theorem 2. *The set of singularities of $f_{t,s}$ consists of either fold singularities or cusps and the number of cusps of $f_{t,s}$ is constant for $0 \leq |s| \ll |t| \ll 1$.*

7.2. Lower bounds of the numbers of cusps of non-linear perturbations.

Let $h(z, \bar{z})$ be a mixed polynomial which satisfies $h(0) = 0$ and $|\frac{\partial h}{\partial z}(0)| \neq |\frac{\partial h}{\partial \bar{z}}(0)|$. We define a perturbation $f_{t,h}$ of a complex polynomial f as follows:

$$f_{t,h}(z) := f(z) + th(z, \bar{z}),$$

where $0 < |t| \ll 1$. Set $h_1 = \Re h, h_2 = \Im h$ and

$$J_{t,h} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x} + t\frac{\partial h_1}{\partial x} & \frac{\partial f_1}{\partial y} + t\frac{\partial h_1}{\partial y} \\ -\frac{\partial f_1}{\partial y} + t\frac{\partial h_2}{\partial x} & \frac{\partial f_1}{\partial x} + t\frac{\partial h_2}{\partial y} \end{pmatrix}.$$

Then any singularity of $f_{t,h}$ belongs to $J_{t,h}^{-1}(0)$. Assume that $f_{t,h}$ satisfies the following conditions:

(i) $f_{t,h}$ is an excellent map for $0 < |t| \ll 1$,

(ii) any cusp of $f_{t,h}$ is a simple root of $G_{t,h}$, where

$$G_{t,h} := \det \begin{pmatrix} \frac{\partial f_1}{\partial x} + t\frac{\partial h_1}{\partial x} & \frac{\partial f_1}{\partial y} + t\frac{\partial h_1}{\partial y} \\ \frac{\partial J_{t,h}}{\partial x} & \frac{\partial J_{t,h}}{\partial y} \end{pmatrix} + i \det \begin{pmatrix} -\frac{\partial f_1}{\partial y} + t\frac{\partial h_2}{\partial x} & \frac{\partial f_1}{\partial x} + t\frac{\partial h_2}{\partial y} \\ \frac{\partial J_{t,h}}{\partial x} & \frac{\partial J_{t,h}}{\partial y} \end{pmatrix}$$

$$= -2i\left(\frac{\partial f}{\partial z} + t\frac{\partial h}{\partial z}\right)\overline{\frac{\partial J_{t,h}}{\partial z}} + 2ti\frac{\partial h}{\partial \bar{z}}\frac{\partial J_{t,h}}{\partial z}.$$

Since $f_{t,h}$ is an excellent map, the intersection of $J_{t,h}^{-1}(0)$ and $(\frac{\partial J_{t,h}}{\partial z})^{-1}(0)$ is empty by Proposition 1. Let U be a sufficiently small neighborhood of the origin. Then the number of cusps of $f_{t,h}|_U$ is equal to the number of $\{z \in U \mid G_{t,h}(z) = 0, \frac{\partial J_{t,h}}{\partial z}(z) \neq 0\}$. We define

$$\delta = \begin{cases} 1 & |\frac{\partial h}{\partial z}(0)| > |\frac{\partial h}{\partial \bar{z}}(0)| \\ -1 & |\frac{\partial h}{\partial z}(0)| < |\frac{\partial h}{\partial \bar{z}}(0)| \end{cases}.$$

Theorem 3. *Let $f_{t,h}$ a perturbation of a complex polynomial f which satisfies the condition (i) and the condition (ii). Then the number of cusps of $f_{t,h}|_U$ is greater than or equal to $k - \delta$, where k is the multiplicity of f at the origin.*

REFERENCES

[1] T. Fukuda, G. Ishikawa, On the number of cusps of stable perturbations of a plane-to-plane singularity, Tokyo J. Math. **10** (1987), 375–384.

[2] K. Inaba, On the number of cusps of deformations of complex polynomials, arxiv: 1811.01189.

[3] I. Krzyżanowska, Z. Szafraniec, On polynomial mappings from the plane to the plane, J. Math. Soc. Japan. **66** (2014), 805–818.

[4] M. Oka, Intersection theory on mixed curves, Kodai Math. J. **35** (2012), 248–267.

[5] J. R. Quine, A global theorem for singularities of maps between oriented 2-manifolds, Trans. Amer. Math. Soc. **236** (1978), 307–314.

[6] Z. Szafraniec, On bifurcations of cusps, arXiv:1710.00591, to appear in J. Math. Soc. Japan.

[7] H. Whitney, On singularities of mapping of Euclidean spaces. I. Mappings of the plane into the plane, Ann. of Math. (2), **62** (1955), 374–410.

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