THE CHERN-SCHWARTZ-MACPHERSON CLASS OF GENERIC DETERMINANTAL VARIETIES

TERENCE GAFFNEY, NIVALDO G. GRULHA JR. AND MARIA A. S. RUAS

Abstract. This is an expository work, based on a talk given by the second author at the RIMS conference 2018, on results of [8] about the computation of the local Euler obstruction of generic determinantal varieties and applications of this result to compute the Chern–Schwartz–MacPherson class of such varieties.

Introduction

In [17] MacPherson proved the existence and uniqueness of Chern classes for possibly singular complex algebraic varieties. The local Euler obstruction, defined by MacPherson in that paper, was one of the main ingredients in his proof.

In order to understand these ideas better, some authors worked on some more specific situations. For example, in the special case of toric surfaces, an interesting formula for the Euler obstruction was proved by Gonzalez–Sprinberg [10], this formula was generalized by Matsui and Takeuchi for normal toric varieties [18].

A natural class of singular varieties to investigate the local Euler obstruction and the generalizations of the characteristic classes is the class of generic determinantal varieties (Def. 1.10). Roughly speaking, generic determinantal varieties are sets of matrices with a given upper bound on their ranks. Their significance comes, for instance, from the fact that many examples in algebraic geometry are of this type, such as the Segre embedding of a product of two projective spaces. Independently, in recent work [25], Zhang computed the Chern-Mather-MacPherson Class of projectivized determinantal varieties, in terms of the trace of certain matrices associated with the push forward of the MacPherson-Schwartz class of the Tjurina transform of the singularity.

In [8] we prove a surprising formula that allow us to compute the local Euler obstruction of generic determinantal varieties using only Newton binomials. Using this formula we also compute the Chern–Schwartz–MacPherson classes of such varieties. This results is presented in sequel.

2010 Mathematics Subject Classification. Primary 14C17, 32S15, 55S35; Secondary 14J17, 58K05, 32S60.

Key words and phrases. Local Euler obstruction, Chern-Schwartz-MacPherson class, generic determinantal varieties.

Acknowledgments

The first author was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, Brazil, grant PVE 401565/2014-9. The second author was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP, Brazil, grants 2015/16746-7 and 2017/09620-2 and Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, Brazil, grants 474289/2013-3, 303641/2013-4 and 303046/2016-3 . The third author was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP, Brazil, grant 2014/00304-2 and Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, Brazil, grant 305651/2011-0.

1. The Chern–Schwartz–MacPherson class of generic determinantal varieties

In [17] MacPherson proved the existence and uniqueness of Chern classes for possibly singular complex algebraic varieties, which was conjectured earlier by Deligne and Grothendieck. These are homology classes which for nonsingular varieties are the Poincaré duals of the usual Chern classes. Some time later, Brasselet and Schwartz proved in [4], using Alexander's duality that the Schwartz class, stated before the Deligne–Grothendieck conjecture, coincides with MacPherson's class, and therefore this class is called the Chern–Schwartz–MacPherson class.

In his proof, MacPherson used the language of constructible sets and functions. A constructible set in an algebraic variety is one obtained from the subvarieties by finitely many of the usual set-theoretic operations. A constructible function on a variety is one for which the variety has a finite partition into constructible sets such that the function is constant on each set. MacPherson proved the following result.

Proposition 1.1 ([17, Prop. 1]). *There is a unique covariant functor* $\mathbf{F}$ *from compact complex algebraic varieties to the abelian group whose value on a variety is the group of constructible functions from that variety to the integers and whose value* f_* *on a map* f *satisfies*

$$f_*(1_W)(p) = \chi(f^{-1}(p) \cap W),$$

where 1_W *is the function that is identically one on the subvariety* W *and zero elsewhere, and where* χ *denotes the topological Euler characteristic.*

Theorem 1 of [17] is the main result of that paper. As we mentioned before, the result was conjectured by Deligne and Grothendieck and we write it below.

Theorem 1.2. *There exist a natural transformation from the functor* $\mathbf{F}$ *to homology which, on a nonsingular variety* X*, assigns to the constant function* 1_X *the Poincaré dual of the total Chern class of* X*.*

In other words, the theorem asserts that we can assign to any constructible function α on a compact complex algebraic variety X an element $c_*(\alpha)$ of $H_*(X)$ satisfying the following three conditions:

(1) $f_*c_*(\alpha) = c_*f_*(\alpha)$
(2) $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$
(3) $c_*(1_X) = \mathrm{Dual}\, c(X)$ if X smooth.

As mentioned in [17], this is exactly Deligne's definition of the total Chern class of any compact variety X; the total Chern class is c_* applied to the constant function 1_X on X. The compactness restriction may be dropped with minor modifications of the proof if all maps are taken to be proper and Borel–Moore homology (homology with locally finite supports) is used.

Let us now introduce some objects in order to define the Chern–Schwartz–MacPherson class. For more details on these concepts we suggest [3, 4, 15, 17].

Suppose X is a representative of a d-dimensional analytic germ $(X, 0) \subset (\mathbb{C}^n, 0)$, such that $X \subset U$, where U is an open subset of $\mathbb{C}^n$. Let $G(d, n)$ denote the Grassmannian of complex d-planes in $\mathbb{C}^n$. On the regular part X_{reg} of X the Gauss map $\phi : X_{\mathrm{reg}} \to U \times G(d, n)$ is well defined by $\phi(x) = (x, T_x(X_{\mathrm{reg}}))$.

Definition 1.3. *The Nash transformation (or Nash blow up) of X denoted by $N(X)$ is the closure of the image $\mathrm{Im}(\phi)$ in $U \times G(d, n)$. It is a (usually singular) complex analytic space endowed with an analytic projection map $\nu : N(X) \to X$ which is biholomorphic away from $\nu^{-1}(\mathrm{Sing}(X))$.*

The fiber of the tautological bundle $\mathcal{T}$ over $G(d, n)$, at point $P \in G(d, n)$, is the set of vectors v in the d-plane P. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d, n)$. Let $N(T)$ be the restriction of $\mathcal{T}$ to $N(X)$, with projection map π. The bundle $N(T)$ on $N(X)$ is called *the Nash bundle* of X.

An element of $N(T)$ is written (x, P, v) where $x \in U$, P is a d-plane in $\mathbb{C}^n$ based at x and v is a vector in P. We have the following diagram:

$$\begin{array}{ccc} N(T) & \hookrightarrow & \mathcal{T} \\ \pi \downarrow & & \downarrow \\ N(X) & \hookrightarrow & U \times G(d, n) \\ \nu \downarrow & & \downarrow \\ X & \hookrightarrow & U. \end{array}$$

Mather has defined an extension of Chern classes to singular varieties by the formula

$$c_{CM}(X) = \nu_* \,\mathrm{Dual}\, c(N(T)),$$

where $c(N(T))$ denotes the total Chern class in cohomology of the Nash bundle, the Dual denotes the Poincaré duality map defined by capping with the fundamental (orientation) homology class.

An algebraic cycle on a variety X is a finite formal linear sum $\sum n_i[X_i]$ where the n_i are integers and the X_i are irreducible subvarieties of X. We may define c_{CM} on any algebraic cycle of X by

$$c_{CM}(\sum n_i[X_i]) = \sum n_i c_{CM}(X_i),$$

where by abuse of notation we denote $\mathrm{incl}_{i*}c_{CM}(X_i)$ by $c_{CM}(X_i)$.

An important object introduced by MacPherson in his work is the local Euler obstruction. This invariant was deeply investigated by many authors, and for an overview about it see [2]. Brasselet and Schwartz presented in [4] an alternative definition for the local Euler obstruction using stratified vector fields.

Definition 1.4. *Let us denote by $TU|_X$ the restriction to X of the tangent bundle of U. A stratified vector field v on X means a continuous section of $TU|_X$ such that if $x \in V_\alpha \cap X$ then $v(x) \in T_x(V_\alpha)$.*

By Whitney condition (a) one has the following:

Lemma 1.5 (See [4]). *Every stratified vector field v nowhere zero on a subset $A \subset X$ has a canonical lifting as a nowhere zero section $\tilde{v}$ of the Nash bundle $N(T)$ over $\nu^{-1}(A) \subset N(X)$.*

Now consider a stratified radial vector field $v(x)$ in a neighborhood of $\{0\}$ in X, *i.e.*, there is ε_0 such that for every $0 < \varepsilon \leq \varepsilon_0$, $v(x)$ is pointing outwards the ball B_ε over the boundary $S_\varepsilon := \partial B_\varepsilon$.

The following interpretation of the local Euler obstruction has been given by Brasselet and Schwartz in [4].

Definition 1.6. *Let v be a radial vector field on $X \cap S_\varepsilon$ and $\tilde{v}$ the lifting of v on $\nu^{-1}(X \cap S_\varepsilon)$ to a section of the Nash bundle.*

The local Euler obstruction (or simply the Euler obstruction), denoted by $\mathrm{Eu}_0(X)$, is defined to be the obstruction to extending $\tilde{v}$ as a nowhere zero section of $N(T)$ over $\nu^{-1}(X \cap B_\varepsilon)$.

More precisely, let

$$\mathcal{O}(\tilde{v}) \in H^{2d}(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon); \mathbb{Z})$$

be the obstruction cocycle to extending $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$. The Euler obstruction $\mathrm{Eu}_0(X)$ is defined as the evaluation of the cocycle $\mathcal{O}(\tilde{v})$ on the fundamental class of the topological pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$.

Note that if $0 \in X$ is a smooth point we have $\mathrm{Eu}_0(X) = 1$, but the converse is false, this was first observed by Piene in [19] (Example, pp. 28–29).

In this paper we use an interesting formula for the local Euler obstruction due to Brasselet, Lê and Seade, that shows that the Euler obstruction, as a constructible function, satisfies the Euler condition relative to generic linear forms.

Theorem 1.7 ([3, Theo. 3.1]). *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an equidimensional reduced complex analytic germ of dimension d. Let us consider $X \subset U \subset \mathbb{C}^n$ a sufficiently small representative of the germ, where U is an open subset of*

$\mathbb{C}^n$. We consider a complex analytic Whitney stratification $\mathcal{V} = \{V_i\}$ of U adapted to X and we assume that $\{0\}$ is a 0-dimensional stratum. We also assume that 0 belongs to the closure of all the strata. Then for each generic linear form l, there is ε_0 such that for any ε with $0 < \varepsilon < \varepsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X, 0)$ is equal to:

$$\mathrm{Eu}_0(X) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap l^{-1}(\delta)\big) \cdot \mathrm{Eu}_{V_i}(X),$$

where $\mathrm{Eu}_{V_i}(X)$ is the value of the Euler obstruction of X at any point of V_i, $i = 1, \ldots, q$, and $0 < |\delta| \ll \varepsilon \ll 1$, where χ denotes the topological Euler characteristic.

With the aid of Gonzalez–Sprinberg's purely algebraic interpretation of the local Euler obstruction ([11]), Lê and Teissier in [15] showed that the local Euler obstruction is an alternating sum of the multiplicities of the local polar varieties. This is an important formula for computing the local Euler obstruction, and we use it in this paper.

Theorem 1.8 ([15, Cor. 5.1.4])**.** *Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be the germ of an equidimensional reduced analytic space of dimension d. Then*

$$\mathrm{Eu}_0(X) = \sum_{i=0}^{d-1} (-1)^{d-i-1} m_{d-i-1}(X, 0),$$

where $m_i(X, 0)$ is the polar multiplicity of the local polar varieties $P_i(X, 0)$.

The local polar variety $P_i(X, 0)$ is the local version of the global polar variety $P_i(X)$.

Using the local Euler obstruction, MacPherson defined a map T from the algebraic cycles on X to the constructible functions on X by

$$T\big(\sum n_i X_i\big)(p) = \sum n_i \,\mathrm{Eu}_p(X_i).$$

And he proved that (Lemma 2 and Theorem 2 of [17]):

Theorem 1.9. *T is a well-defined isomorphism from the group of algebraic cycles to the group of constructible functions and that $c_{CM}T^{-1}(1_X)$ satisfies the requirements for c_* in the Deligne–Grothendieck conjecture.*

In this section we prove a formula to compute the local Euler obstruction of a generic determinantal variety, and applying MacPherson's definition, we find the Chern–Schwartz–MacPherson class of this variety.

First, let us recall the definition of the generic determinantal variety.

Definition 1.10. *Let $n, k, s \in \mathbb{Z}$, $n \geq 1$, $k \geq 0$ and $\mathrm{Mat}_{(n,n+k)}(\mathbb{C})$ be the set of all $n \times (n+k)$ matrices with complex entries, $\Sigma^s \subset \mathrm{Mat}_{(n,n+k)}(\mathbb{C})$ the subset formed by matrices that have rank less than s, with $1 \leq s \leq n$. The set Σ^s is called the generic determinantal variety.*

Remark 1.11. *The following properties of the generic determinantal varieties are fundamental in this work.*

(1) *Σ^s is an irreducible singular algebraic variety.*

(2) *The codimension of* Σ^s *in the ambient space is* $(n-s+1)(n+k-s+1)$.
(3) *The singular set of* Σ^s *is exactly* Σ^{s-1}.
(4) *The stratification of* Σ^s *given by* $\{\Sigma^t \setminus \Sigma^{t-1}\}$, *with* $1 \leq t \leq s$ *is locally analytically trivial and hence it is a Whitney stratification of* Σ^s.

As references for these topics we recommend, chapter 2, section 2, of [1] *and the book* [5].

Remark 1.12. *Every element of* $\mathrm{Mat}_{(n,n+k)}(\mathbb{C})$ *can be seen as a linear map from* $\mathbb{C}^n$ *to* $\mathbb{C}^{n+k}$, *or from* $\mathbb{C}^{n+k}$ *to* $\mathbb{C}^n$, *then we will also refer to the space of matrices as* $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ *or* $\mathrm{Hom}(\mathbb{C}^{n+k}, \mathbb{C}^n)$.

The next result is a very important in this paper. To state it let us fix some notations. Let $\overline{\chi}$ denote the reduced Euler characteristic, that is $\overline{\chi} = \chi - 1$, where χ denotes the topological Euler characteristic.

Let us also recall the notion of normal slice this notion is related to the complex link and normal Morse datum. The complex link is an important object in the study of the topology of complex analytic sets. It is analogous to the Milnor fibre and was studied first in [13]. It plays a crucial role in complex stratified Morse theory (see [9]) and appears in general bouquet theorems for the Milnor fibre of a function with isolated singularity (see [14, 21, 24]). It is related to the multiplicity of polar varieties and also to the local Euler obstruction (see [15, 16]).

Definition 1.13. *Let* V *be a stratum of a Whitney stratification of* X, *a small representative of the analytic germ* $(X,0) \subset (\mathbb{C}^n,0)$, *and* x *be a point in* V. *We call* N *a normal slice to* V *at* x, *if* N *is a closed complex submanifold of* $\mathbb{C}^n$ *which is transversal to* V *at* x *and* $N \cap V = \{x\}$.

Proposition 1.14 ([7, Prop. 3])**.** *Let* $\ell : \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k}) \to \mathbb{C}$ *be a generic linear form. Then, for* $s \leq n$, *one has*

$$\overline{\chi}(\Sigma^s \cap \ell^{-1}(1)) = (-1)^s \binom{n-1}{s-1}.$$

In order to find the Chern–Schwartz–MacPherson class of a generic determinantal variety, first we calculate its local Euler obstruction to apply MacPherson's result.

Theorem 1.15. *Let* $\Sigma^s \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ *be a generic determinantal variety defined as above, we have*

$$e(s,n) = \binom{n}{s-1},$$

for $1 \leq s \leq n$.

Remark 1.16. *The last result has a pretty accompanying graphic. For this part, fix* $k \in \mathbb{Z}^+$, $k \geq 1$, *and for* $i \in \mathbb{Z}^+$, $1 \leq i \leq n+1$, *let us denote* $\Sigma^i \subset \mathrm{Hom}(n, n+k)$ *by* Σ^i_n.

On the one hand we have a triangle of spaces and maps. In the apex of the triangle (row zero) we have $0 \in \mathrm{Hom}(\mathbb{C}^0, \mathbb{C}^k)$. *Row* 1 *is* $0 \in \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^{k+1})$ *and* $\mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^{k+1})$. *We have maps from the element in row* 0 *to each* *element in row 1 given by the inclusions of $\mathbb{C}^k$ to $\mathbb{C}^{k+1}$, and projection of $\mathbb{C}^1$ to $\mathbb{C}^0$. Row 2 is $\Sigma^1 = \{0\} \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{k+2})$, $\Sigma^2 \subset \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{k+2})$, and $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{k+2})$.*

Again there are maps given by projection and inclusion from elements of row 1 into adjacent pairs of elements of row 2. Then row n consists of the spaces $\Sigma^i_n \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$, $1 \leq i \leq n+1$, with maps from the previous row to adjacent pairs of elements of this row. The other triangle is Pascal's triangle. Then our theorem says that the local Euler obstruction takes the triangle of spaces to Pascal's triangle.

$$
\begin{array}{ccccccccc}
 & & & & \Sigma^1_0 & & & & \\
 & & & \Swarrow & & \Searrow & & & \\
 & & & \Sigma^1_1 & & \Sigma^2_1 & & & \\
 & & \Swarrow & \Searrow & & \Swarrow & \Searrow & & \\
 & & \Sigma^1_2 & & \Sigma^2_2 & & \Sigma^3_2 & & \\
 & \Swarrow & \Searrow & \Swarrow & \Searrow & \Swarrow & \Searrow & & \\
 & & & & \vdots & & & & \\
 & \Swarrow & & \Searrow & & \Swarrow & & \Searrow & \\
\Sigma^1_n & & \cdots & & \Sigma^i_n & & \cdots & & \Sigma^{n+1}_n \\
\Swarrow\ \Searrow & & & & \Swarrow\ \Searrow & & & & \Swarrow\ \Searrow
\end{array}
\qquad\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & & & & \vdots & & & & \\
1 & & \cdots & & \binom{n}{i-1} & & \cdots & & 1
\end{array}
$$

FIGURE 1. Triangles

Remark 1.17. *The Euler obstruction is a constructible function,*

$$e_p(s,n) = \sum_{i=1}^{s} \alpha_i \mathbb{1}_{(\Sigma^i \setminus \Sigma^{i-1})}(p),$$

where $\mathbb{1}_{(\Sigma^i \setminus \Sigma^{i-1})}(p)$ is 1 on $\Sigma^i \setminus \Sigma^{i-1}$, 0 elsewhere and α_i is the value of the Euler obstruction of Σ^n at any point of $\Sigma^i \setminus \Sigma^{i-1}$. As $\alpha_i = e(s-i+1, n-i+1)$. Using our formula to calculate α_i we have the next corollary.

Corollary 1.18. *In the above setup we have*

$$e_p(s,n) = \sum_{i=1}^{s} \binom{n-i+1}{s-i} \mathbb{1}_{(\Sigma^i \setminus \Sigma^{i-1})}(p).$$

Recall that an algebraic cycle on a variety X is a finite formal linear sum $\sum n_i[X_i]$ where the n_i are integers and the X_i are irreducible subvarieties of X. Taking $X = \Sigma^s$ and remembering that all $\Sigma^i \subset \Sigma^s$, where $i = 1, \ldots, s$ are irreducible subvarieties and using Theorem 1.15 we get a formula for the local Chern–Schwartz–MacPherson cycle of Σ^n, denoted by $[\mathrm{csm}(\Sigma^s)]$.

The next result is an interesting property of alternating sums of binomial products, whose elements are chosen in the Pascal's triangle in a "V" distribution. This lemma is essential to define the Chern–Schwartz–MacPherson cycle.

Theorem 1.19. *The local Chern–Schwartz–MacPherson cycle of the algebraic variety $\Sigma^s \subset \mathrm{Hom}(n, n+k)$ is*

$$[\mathrm{csm}(\Sigma^s)] = \sum_{i=0}^{s-1} (-1)^{s-1+i} \binom{n-i-1}{s-i-1} [\Sigma^{i+1}]$$

Using this last proposition and by MacPherson definition, we can calculate the Chern–Schwartz–MacPherson class of Σ^s as follows.

Theorem 1.20. *In the same setting as above, we have that the total Chern–Schwartz–MacPherson class of Σ^s is*

$$c_{\mathrm{CSM}}(\Sigma^s) = \sum_{j=0}^{s-1} (-1)^{s-1+j} \binom{n-j-1}{s-j-1} c_{\mathrm{CM}}(\Sigma^{j+1})$$

where $c_{\mathrm{CM}}(X)$ denotes the total Chern–Mather class of a variety X.

References

[1] E. Arbarello, M. Cornualba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves.* Vol. I, volume 267 of Fundamental Principles of Mathematical Sciences, Springer Verlag, New York, 1985.

[2] J. -P. Brasselet, *Local Euler obstruction, old and new*, *XI Brazilian Topology Meeting (Rio Claro, 1998)*, 140–147, World Sci. Publishing, River Edge, NJ, 2000.

[3] J. -P. Brasselet, Lê D. T. and J. Seade, *Euler obstruction and indices of vector fields.* Topology 39 (2000), no. 6, 1193–1208.

[4] J. -P. Brasselet M.-H. Schwartz *Sur les classes de Chern d'un ensemble analytique complexe*, *Astérisque* 82-83 (1981) 93-147.

[5] W. Bruns, U. Vetter, *Determinantal Rings*, *Springer-Verlang*, *New York*, (1998).

[6] Chuan Chong Chen, Khee Meng Koh *Principles and Techniques in Combinatorics* World Scientific 1992.

[7] W. Ebeling, and S. M. Gusein-Zade, *On indices of* 1*-forms on determinantal singularities*, Proc. Steklov Inst. Math. 267 (2009), no. 1, 113–124.

[8] T. Gaffney, N. G. Grulha Jr. and M. Ruas,*The local Euler obstruction and topology of the stabilization of associated determinantal varieties*, Mathematische Zeitschrift (2019) Vol. 291, Issue 3–4, pp 905–930.

[9] M. Goresky and R. MacPherson, *Stratified Morse theory*, *Springer-Verlag*, Berlin, 1988.

[10] G. Gonzalez-Springberg, *Calcul de l'invariant local d'Euler pour les singularités quotient de surfaces*, *C. R. Acad. Sci. Paris* Ser. A–B 288, A989–A992 (1979).

[11] G. Gonzalez-Springberg, *L' obstruction locale d'Euler et le théorème de MacPherson*, Séminaire de géométrie analytique de l'E.N.S. 1978 –79.

[12] V. H. Jorge Pérez and M. J. Saia, *Euler obstruction, polar multiplicities and equisingularity of map germs in $\mathcal{O}(n,p), n < p$.* Internat. J. Math. 17 (2006), no. 8, 887–903.

[13] Lê D. T., *Vanishing cycles on complex analytic sets*, *Proc. Sympos., Res. Inst. Math. Sci., Kyoto, Univ. Kyoto, 1975* Sûrikaisekikenkyûsho Kókyûroku, no. 266 (1976), 299–318.

[14] Lê D. T., *Complex analytic functions with isolated singularities*, *J. Algebraic Geom.* **1** (1992), no. 1, 83–99.

[15] Lê D. T. and B. Teissier, *Variétés polaires Locales et classes de Chern des variétés singulières*, *Ann. of Math.* 114, (1981), 457–491.

[16] Lê D. T. and B. Teissier, *Limites d'espaces tangents en géométrie analytique.* Comm. Math. Helv., 63(4):540–578, 1988.

[17] R. MacPherson, *Chern classes for singular algebraic varieties*, *Ann. of Math.* 100 (1974), 423–432.

[18] Y. Matsui and K. Takeuchi, *A geometric degree formula for A-discriminants and Euler obstructions of toric varieties*, *Adv. Math.* 226, 2040–2064 (2011).

[19] R. Piene, Cycles polaires et classes de Chern pour les variétés projectives singulières, *Introduction à la théorie des singularités,II, Travaux en Cours*, (37), Hermann, Paris, (1988), 4–34.

[20] J. Schurmann and M. Tibar, *Index formula for MacPherson cycles of affine algebraic varieties*, Tohoku Math. J. (2) Volume 62, Number 1 (2010), 29–44.

[21] D. Siersma, *A bouquet theorem for the Milnor fibre*, *J. Algebraic Geom.* **4** (1995), no. 1, 51–66.

[22] N. C. C. Siesquén, *Euler obstruction of essentially isolated determinantal singularities*, *Topology and its Applications* **234** (2018), 166–177.

[23] B. Teissier, *Variétés polaires, II, Multiplicités polaires, sections planes, et conditions de Whitney*, Algebraic geometry (La Rábida, 1981), 314–491, Lecture Notes in Math. 961, Springer, Berlin, (1982).

[24] M. Tibăr, *Bouquet decomposition of the Milnor fibre*, *Topology* **35** (1996), no. 1, 227–241.

[25] X. Zhang, *Chern-Schwartz-MacPherson Class of Determinantal Varieties*, `arXiv:1605.05380 [math.AG]`.

T. Gaffney, Department of Mathematics, Northeastern University, Boston, MA 02215, t.gaffney@neu.edu

N. Grulha, Instituto de Ciências Matemáticas e de Computação - USP, Av. Trabalhador são-carlense, 400 - Centro, CEP: 13566-590 - São Carlos - SP, Brazil, njunior@icmc.usp.br

M. Ruas, Instituto de Ciências Matemáticas e de Computação - USP, Av. Trabalhador são-carlense, 400 - Centro, CEP: 13566-590 - São Carlos - SP, Brazil, maasruas@icmc.usp.br