LEVEL TWO GENERALIZATION OF ARAKAWA-KANEKO ZETA FUNCTION AND POLY-COSECANT NUMBERS

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Abstract. We present a level two generalization of Arakawa-Kaneko zeta function introduced by T. Arakawa and M. Kaneko. We prove certain formulas for Arakawa-Kaneko zeta function of level two. Also, we study the level two generalization of poly-Bernoulli numbers, which is referred to as the poly-cosecant numbers. We obtain a recurrence and two explicit formulas for poly-cosecant numbers. Moreover, we extend those formulas for multiple versions in a similar manner. This is in part a joint work with M. Kaneko and H. Tsumura.

1. Introduction

Poly-Bernoulli numbers (Kaneko 1997; Arakawa-Kaneko 1999) have two versions, namely $B_n^{(k)}$ and $C_n^{(k)}$, which were defined by Kaneko in [5] and in Arakawa-Kaneko [2] by using generating series. For any integer $k \in \mathbb{Z}$, the sequences of rational numbers $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$ are defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!},$$

where $\mathrm{Li}_k(z)$ is the poly-logarithm function (or rational function when $k \leq 0$) defined by

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z| < 1).$$

Since $\mathrm{Li}_1(z) = -\log(1-z)$, the generating functions on the left-hand sides respectively become

$$\frac{te^t}{e^t-1} \quad \text{and} \quad \frac{t}{e^t-1}$$

when $k = 1$, and hence $B_n^{(1)}$ and $C_n^{(1)}$ becomes the usual Bernoulli numbers. There are various properties of poly-Bernoulli numbers (e.g.: explicit formulas, duality relations, etc.).

In this paper, we study the level two version of poly-Bernoulli numbers, which we also call the poly-cosecant numbers (Sasaki 2012 [9]; Kaneko-M.-Tsumura 2019 [6]) $D_n^{(k)}$ defined by

$$\frac{\mathrm{A}_k(\tanh t/2)}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!}$$

for $k \in \mathbb{Z}$, where $\mathrm{A}_k(z)$ is the poly-logarithm function of level 2 defined by

$$\mathrm{A}_k(z) = 2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^k} \quad (z \in \mathbb{C};\ |z| < 1),$$

which was first studied by Sasaki (see [9, Definition 5]). In particular, for $k = 1$, we have $\mathrm{A}_1(z) = 2\tanh^{-1}(z)$. In this case, $D_n^{(1)}$ becomes the ordinary cosecant number D_n defined by

$$\frac{t}{\sinh t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}.$$

Note that $D_{2n+1}^{(k)} = 0$ for $(n \in \mathbb{Z}_{\geq 0})$.

We may define the multi-poly-cosecant numbers $D_n^{(k_1,\ldots,k_r)}$ by

$$\frac{\mathrm{A}(k_1,\ldots,k_r;\tanh(t/2))}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!},$$

where the function

$$\mathrm{A}(k_1,\ldots,k_r;z) = 2^r \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \bmod 2}} \frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}$$

for $k_1,\ldots,k_r \in \mathbb{Z}$ is 2^r times $\mathrm{Ath}(k_1,\ldots,k_r;z)$ which was introduced in [8, §5]. (Our $\mathrm{A}_k(z)$ is $\mathrm{A}(k;z)$). We can regard $D_n^{(k_1,\ldots,k_r)}$ as a level 2-version of the multi-poly-Bernoulli numbers $B_n^{(k_1,\ldots,k_r)}$ and $C_n^{(k_1,\ldots,k_r)}$.

Now we recall the following lemma.

Lemma 1.1. [8, Lemma 5.1]

(1) For $k_1,\ldots,k_r \in \mathbb{Z}_{\geq 1}$

$$\frac{d}{dt}\mathrm{A}(k_1,\ldots,k_r;z) = \begin{cases} \frac{1}{z}\mathrm{A}(k_1,\ldots,k_{r-1},k_r-1;z) & (k_r \geq 2) \\ \frac{2}{1-z^2}\mathrm{A}(k_1,\ldots,k_{r-1};z) & (k_r = 1) \end{cases}$$

(2) $\mathrm{A}(\underbrace{1,\ldots,1}_{r};z) = \frac{1}{r!}\left(\mathrm{A}_1(z)\right)^r$

In their research, Arakawa and Kaneko [2] studied the single variable function

$$\zeta(k_1,\ldots,k_{r-1};s) = \sum_{0<m_1<\cdots<m_{r-1}<m_n} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_n^s}$$

for the purpose of establishing the connection between MZVs and poly-Bernoulli numbers. This is absolutely convergent for $Re(s) > 1$. They have shown that the poly-Bernoulli numbers can be expressed as special values at negative arguments of certain combinations of these functions. Corresponding to these functions, Arakawa and Kaneko [2] defined the following zeta function which is known as Arakawa-Kaneko zeta function as

$$\xi(k_1,\ldots,k_r;s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{t^{s-1}}{e^t-1} Li_{k_1,\ldots,k_r}(1-e^{-t})dt$$

where $r, k_1,\ldots,k_r \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{C}$ with $Re(s) > 0$.

For $r = 1$ we denote $\xi(k;s)$ by $\xi_k(s)$. Note that $\xi_1(s) = s\zeta(s+1)$.

In [8] Kaneko and Tsumura defined the single variable multiple zeta function of level-2 as follows.

$$T_0(k_1,\ldots,k_{r-1},s) = \sum_{\substack{0<m_1<\cdots<m_r \\ m_i \equiv i \bmod 2}} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^s}$$

for $k_1,\ldots,k_{r-1} \in \mathbb{Z}_{\geq 1}$ and $Re(s) > 1$.

Furthermore, as its normalized version,

$$T(k_1,\ldots,k_{r-1},s) = 2^r T_0(k_1,\ldots,k_{r-1},s).$$

The values $T(k_1,\ldots,k_{r-1},k_r)(k_j \in \mathbb{N}, k_r \geq 2 : \mathit{addmisible})$ are called the multiple T-values.

When $k_r > 1$, we see that

$$\mathrm{A}(k_1, \dots, k_r; 1) = T(k_1, \dots, k_r).$$

Now according to these functions, Kaneko and Tsumura (see [8, Sction 5]) defined a level 2-version of $\xi(k_1, \dots, k_r; s)$

$$\psi(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{A}(k_1, \dots, k_r; \tanh t/2)}{\sinh(t)} dt$$

for $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ and $Re(s) > 0$.

2. Formulas on the level 2 version of Arakawa-Kaneko zeta functions

In this section, we prove certain formulas for Arakawa-Kaneko zeta functions of level two. We obtain a level two version of [2, Proposition 2] as follows.

Proposition 2.1. *(1) For $Re(s) > 1$*

$$T(k_1, \dots, k_{n-1}, s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\sinh(t)} \mathrm{A}(k_1, \dots, k_{n-1}; e^{-t}) dt.$$

(2) For $Re(s) > 1, n \geq 2, j \geq 0$

$$\int_0^\infty t^{s+j-1} \mathrm{A}(k_1, \dots, k_{n-1}; e^{-t}) dt = \Gamma(s+j) T(k_1, \dots, k_{n-2}, s + j + k_{n-1}).$$

Proof. To prove (1), we use the definition

$$\begin{aligned} T(k_1, \dots, k_{n-1}, s) &= 2^n \sum_{\substack{0<m_1<\cdots<m_n \\ m_i \equiv i \bmod 2}} \frac{1}{m_1^{k_1} \cdots m_{n-1}^{k_{n-1}} m_n^s} \\ &= 2^n \sum_{\substack{0<m_1<\cdots<m_{n-1} \\ m_i \equiv i \bmod 2}} \frac{1}{m_1^{k_1} \cdots m_{n-1}^{k_{n-1}}} \sum_{\substack{m_n = m_{n-1}+1 \\ m_n \not\equiv m_{n-1} \bmod 2}} \frac{1}{m_n^s}, \end{aligned}$$

and use the standard expression

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-nt} t^{s-1} dt \tag{2.1}$$

to convert the inner sum into the integral. Then we can get the desired results.

To obtain (2), we only need to use the definition

$$\mathrm{A}(k_1, \dots, k_{n-1}; e^{-t}) = 2^{n-1} \sum_{\substack{0<m_1<\cdots<m_{n-1} \\ m_i \equiv i \bmod 2}} \frac{e^{-m_{n-1}t}}{m_1^{k_1} \cdots m_{n-1}^{k_{n-1}}}$$

and use equation (2.1) to obtain

$$\int_0^\infty e^{-m_{n-1}t} t^{s+j-1} dt = \frac{\Gamma(s+j)}{m_{n-1}^{s+j}}.$$

□

Now we obtain the following lemma associated with the multi-poly-logarithm functions of level two, corresponding to [7, Lemma 3.5].

Lemma 2.2. *Let $\mathbf{k}$ be any index. Then we have*

$$\mathrm{A}\left(\mathbf{k}; \frac{1-z}{1+z}\right) = \sum_{\mathbf{k}', j \geq 0} C_{\mathbf{k}}(\mathbf{k}'; j) \mathrm{A}\left(\underbrace{1, \dots, 1}_{j}; \frac{1-z}{1+z}\right) \mathrm{A}(\mathbf{k}'; z)$$

where the sum on the right runs over indices $\mathbf{k}'$ *and integers* $j \geq 0$ *that satisfy* $|\mathbf{k}'| + j \leq |\mathbf{k}|$*, and* $C_{\mathbf{k}}(\mathbf{k}'; j)$ *is a* $\mathbb{Q}-$*linear combination of* $T-$*values of weight* $|\mathbf{k}| - |\mathbf{k}'| - j$.

Proof. We prove this by induction on the weight $\mathbf{k}$. When $\mathbf{k} = (1)$, we have the trivial identity

$$\mathrm{A}_1\left(\frac{1-z}{1+z}\right) = \mathrm{A}_1\left(\frac{1-z}{1+z}\right).$$

Suppose the weight $|\mathbf{k}| > 1$ and assume the statement holds for any index of weight less than $|\mathbf{k}|$.

For $\mathbf{k} = k_1, \ldots, k_r$, set $\mathbf{k}_- = (k_1, \ldots, k_{n-1}, k_n - 1)$.

First, assume that $\mathbf{k}$ is admissible. Then by the differential relation and the induction hypothesis, we get,

$$\begin{aligned}\frac{d}{dz}\mathrm{A}\left(\mathbf{k}; \frac{1-z}{1+z}\right) &= -\frac{2}{1-z^2}\mathrm{A}\left(\mathbf{k}_-; \frac{1-z}{1+z}\right)\\ &= -\frac{2}{1-z^2}\sum_{\mathbf{l}, j\geq 0} C_{\mathbf{k}_-}(\mathbf{l}; j)\mathrm{A}\left(\underbrace{1, \ldots, 1}_{j}; \frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{l}; z) \qquad (2.2)\end{aligned}$$

Let the depth of $\mathbf{l}$ be s. Again by the differential relation we see that

$$\frac{2}{1-z^2}\mathrm{A}\left(\underbrace{1, \ldots, 1}_{j}; \frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{l}; z) = \frac{d}{dz}\left(\sum_{i=0}^{j}\mathrm{A}\left(\underbrace{1, \ldots, 1}_{j}; \frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{l}, i+1; z)\right).$$

Now substitute this in (2.2). Then we get

$$\mathrm{A}\left(\mathbf{k}; \frac{1-z}{1+z}\right) = -\sum_{\mathbf{l}, j\geq 0} C_{\mathbf{k}_-}(\mathbf{l}; j)\mathrm{A}\left(\underbrace{1, \ldots, 1}_{j}; \frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{l}, i+1; z) + C$$

where C is a constant. Since,

$$\lim_{z\to 0}\mathrm{A}\left(\underbrace{1, \ldots, 1}_{j}; \frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{l}, i+1; z) = 0,$$

we have $C = T(\mathbf{k})$. Now we can obtain the desired result.

When $\mathbf{k}$ is not necessarily admissible, we write $\mathbf{k} = (\mathbf{k}_0, \underbrace{1, \ldots, 1}_{q})$ with admissible $\mathbf{k}_0$ and $q \geq 0$. Now we prove the formula by induction on q. Since, $\mathbf{k}_0$ is admissible the case $q = 0$ is already done.

Suppose $q \geq 1$ and assume the claim is true for smaller q. Then by the assumption we get

$$\mathrm{A}\left(\mathbf{k}'; \frac{1-z}{1+z}\right) = \sum_{\mathbf{m}, j\geq 0} C_{\mathbf{k}'}(\mathbf{m}; j)\mathrm{A}\left(\underbrace{1, \ldots, 1}_{j}; \frac{1-z}{1+z}\right)\mathrm{A}(\mathbf{m}; z)$$

where $\mathbf{k}' = (\mathbf{k}_0, \underbrace{1, \ldots, 1}_{q-1})$. Now multiply both sides by $\mathrm{A}_1\left(\frac{1-z}{1+z}\right)$. Then by the shuffle product, the left-hand side becomes of the form

$$q\mathrm{A}\left(\mathbf{k}; \frac{1-z}{1+z}\right) + \sum_{\mathbf{k}_0':\text{admissible}}\mathrm{A}\left(\mathbf{k}_0', \underbrace{1, \ldots, 1}_{q-1}; \frac{1-z}{1+z}\right).$$

By using the induction hypothesis, each term in the sum can be written in the claimed form. Since,

$$\mathrm{A}_1\left(\frac{1-z}{1+z}\right)\mathrm{A}\left(\underbrace{1,\ldots,1}_{j};\frac{1-z}{1+z}\right)=(j+1)\mathrm{A}\left(\underbrace{1,\ldots,1}_{j+1};\frac{1-z}{1+z}\right)$$

the right-hand side also becomes the claimed form. Hence we get the desired form. □

The following theorem shows that the function $\psi(\mathbf{k};s)$ can be written in terms of $T-$functions.

Theorem 2.3. *Let* $\mathbf{k}$ *be any index set.*

$$\psi(\mathbf{k};s)=\sum_{\mathbf{k}',j\geq 0}C_{\mathbf{k}}(\mathbf{k}';j)\binom{s+j-1}{j}T(\mathbf{k}';s+j)$$

Here, the sum is over indices $\mathbf{k}'$ *and integers* $j\geq 0$ *that satisfy* $|\mathbf{k}'|+j\leq|\mathbf{k}|$*, and* $C_{\mathbf{k}}(\mathbf{k}';j)$ *is a* $\mathbb{Q}-$*linear combination of* $T-$*values of weight* $|\mathbf{k}|-|\mathbf{k}'|-j$.

Proof. Let r, l be the depths of $\mathbf{k}$ and $\mathbf{k}'$ respectively. Put $z=e^{-t}$ in the above lemma.

$$\mathrm{A}\left(\mathbf{k};\frac{1-e^{-t}}{1+e^{-t}}\right)=\sum_{\mathbf{k}',j\geq 0}C_{\mathbb{K}}(\mathbf{k}';j)\mathrm{A}\left(\underbrace{1,\ldots,1}_{j};\frac{1-e^{-t}}{1+e^{-t}}\right)\mathrm{A}(\mathbf{k}';e^{-t}).$$

By using Lemma 1.1 we can write the above equation as

$$\mathrm{A}\left(\mathbf{k};\tanh t/2\right)=\sum_{\mathbf{k}',j\geq 0}C_{\mathbf{k}}(\mathbf{k}';j)\frac{t^j}{j!}\mathrm{A}(\mathbf{k}';e^{-t}).\tag{2.3}$$

We know the definition,

$$\psi(\mathbf{k};s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{A}(\mathbf{k};\tanh t/2)}{\sinh(t)}dt.$$

Finally, we substitute equation (2.3) and in the above equation and apply Proposition 2.1 to obtain the desired formula for $\psi(\mathbf{k};s)$.

□

3. Recurrence and explicit formulas for poly-cosecant numbers

In this section, we will obtain recurrence and explicit formulas for poly-cosecant numbers. Furthermore, we discuss about their multi-indexed versions.

The following proposition gives a recurrence formula for $D_n^(k)$ which can be derived in two ways by using definition and the iterated integral expression of the generating function. Here we only consider the proof by definition.

Note that since $\mathrm{A}_0(\tanh(t/2))=\sinh(t)$, $D_0^{(0)}=1$ and $D_n^{(0)}=0$ for all $n\geq 1$.

Proposition 3.1. *For any integers* k *and* $n\geq 0$,

$$D_n^{(k-1)}=\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2m+1}D_{n-2m}^{(k)}.$$

Proof. By the definition of poly-cosecant numbers we have that,

$$\mathrm{A}_k(\tanh(t/2))=\sinh t\sum_{n=0}^{\infty}D_n^{(k)}\frac{t^n}{n!}.$$

Differentiate with respect to t,

$$\frac{\mathrm{A}_{k-1}(\tanh t/2)}{\sinh t} = \cosh t \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} + \sinh t \sum_{n=1}^{\infty} D_n^{(k)} \frac{t^{n-1}}{(n-1)!}$$

By using the definitions we can write the above equation as,

$$\begin{aligned}\sum_{n=0}^{\infty} D_n^{(k-1)} \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} + \sum_{m=0}^{\infty} \frac{t^{2m+1}}{(2m+1)!} \sum_{n=1}^{\infty} D_n^{(k)} \frac{t^{n-1}}{(n-1)!} \\ &= \sum_{m=0}^{\infty} \sum_{n=2m}^{\infty} D_{n-2m}^{(k)} \frac{t^n}{(2m)!(n-2m)!} \\ &\quad + \sum_{m=0}^{\infty} \sum_{n=2m}^{\infty} D_{n-2m}^{(k)} \frac{t^n}{(2m+1)!(n-2m-1)!} \quad ; (n = n+2m) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} D_{n-2m}^{(k)} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m+1} D_{n-2m}^{(k)} \frac{t^n}{n!}.\end{aligned}$$

By equating the coefficients of $\frac{t^n}{n!}$ we can get the desired result.

□

When $k > 0$, we may want to write this as

$$(n+1)D_n^{(k)} = D_n^{(k-1)} - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(k)} \quad (n > 0).$$

Note that $D_0^{(k)} = 1$ for all $k \in \mathbb{Z}$.

Let $x_{(n)} = x(x-1)\cdots(x-n+1)$ and $x^{(n)} = x(x+1)\cdots(x+n-1)$. Then the Stirling numbers of the first kind is defined by

$$x^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

and the Stirling numbers of the second kind is defined by

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} x_{(k)}.$$

In the following theorem we obtain two explicit formulas for $D_n^{(k)}$.

Theorem 3.2. *For any $k \in \mathbb{Z}$ and $n \geq 0$, we have*

(1)

$$D_n^{(k)} = 4 \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=1}^{2m+1} \sum_{q=0}^{n} (-1)^n (2^{p+q+1} - 1) \binom{n}{q} \begin{Bmatrix} n-q \\ 2m \end{Bmatrix} \begin{bmatrix} 2m+1 \\ p \end{bmatrix} \frac{B_{p+q+1}}{p+q+1}.$$

and

(2)

$$D_n^{(k)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=2m+1}^{n+1} \frac{(-1)^{p+1} p!}{2^{p-1}} \binom{p-1}{2m} \begin{Bmatrix} n+1 \\ p \end{Bmatrix}.$$

To prove the first formula of Theorem 3.2, we prepare the following lemma.

Lemma 3.3. *For $n \geq 1$ we have,*

$$x^n \left(\frac{d}{dx}\right)^n = \sum_{m=1}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} \left(x\frac{d}{dx}\right)^m.$$

Proof. We can prove this by induction on n. For $n = 1$ both sides equal to $x\frac{d}{dx}$.
Suppose the formula is true for n. Then,

$$\begin{aligned}
x^{n+1}\left(\frac{d}{dx}\right)^{n+1} &= x^{n+1}\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)^n \\
&= x^{n+1}\frac{d}{dx}\left[\sum_{m=1}^{n} \frac{(-1)^{n-m}}{x^n} \begin{bmatrix} n \\ m \end{bmatrix} \left(x\frac{d}{dx}\right)^m\right] \\
&= \sum_{m=1}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} \left[-n\left(x\frac{d}{dx}\right)^m + \left(x\frac{d}{dx}\right)^{m+1}\right] \\
&= \sum_{m=1}^{n+1} (-1)^{n-m+1} \left(n\begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n \\ m-1 \end{bmatrix}\right) \left(x\frac{d}{dx}\right)^m \\
&= \sum_{m=1}^{n+1} (-1)^{n-m+1} \begin{bmatrix} n+1 \\ m \end{bmatrix} \left(x\frac{d}{dx}\right)^m.
\end{aligned}$$

Here we have used $\begin{bmatrix} n \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} n \\ n+1 \end{bmatrix} = 0$.
This shows the formula is true for $n+1$. Therefore the formula holds. □

Now we give the proof for the first formula of Theorem 3.2.

Proof of Theorem 3.2(First Formula). We write

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} \\
&= 2\sum_{m=0}^{\infty} \frac{(\tanh(t/2))^{2m+1}}{(2m+1)^k} \frac{1}{\sinh t} \\
&= 4\sum_{m=0}^{\infty} \frac{1}{(2m+1)^k} \frac{e^t(e^t-1)^{2m}}{(e^t+1)^{2m+2}}.
\end{aligned} \tag{3.1}$$

Since

$$\frac{1}{(x+1)^{n+1}} = \frac{(-1)^n}{n!}\left(\frac{d}{dx}\right)^n \frac{1}{x+1}, \tag{3.2}$$

we see by setting $x = e^t$ and using Lemma 3.3 that

$$\frac{e^{nt}}{(e^t+1)^{n+1}} = \frac{1}{n!}\sum_{p=1}^{n} (-1)^p \begin{bmatrix} n \\ p \end{bmatrix} \left(\frac{d}{dt}\right)^p \frac{1}{e^t+1}. \tag{3.3}$$

From

$$\frac{t}{e^t-1} = \sum_{q=0}^{\infty} B_q \frac{t^q}{q!}$$

and

$$\frac{1}{e^t+1} = \frac{1}{e^t-1} - \frac{2}{e^{2t}-1},$$

we have

$$\frac{1}{e^t+1} = \sum_{q=0}^{\infty}(1-2^q)B_q\frac{t^{q-1}}{q!}.$$

By taking the p-th derivative of both sides, we get

$$\left(\frac{d}{dt}\right)^p\left(\frac{1}{e^t+1}\right) = \sum_{q=p+1}^{\infty}(1-2^q)\frac{B_q}{q}\frac{t^{q-p-1}}{(q-p-1)!} = \sum_{q=p+1}^{\infty}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}$$

and we substitute this in (3.3) to obtain

$$\begin{aligned}\frac{e^{nt}}{(e^t+1)^{n+1}} &= \frac{1}{n!}\sum_{p=1}^{n}(-1)^p\begin{bmatrix}n\\p\end{bmatrix}\sum_{q=0}^{\infty}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}\\ &= \frac{1}{n!}\sum_{q=0}^{\infty}\sum_{p=1}^{n}(-1)^p\begin{bmatrix}n\\p\end{bmatrix}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.\end{aligned}$$

From this, we have

$$\begin{aligned}\frac{e^t}{(e^t+1)^{2m+2}} &= \frac{e^{-(2m+1)t}}{(e^{-t}+1)^{2m+2}}\\ &= \frac{1}{(2m+1)!}\sum_{q=0}^{\infty}\sum_{p=1}^{2m+1}(-1)^{p+q}\begin{bmatrix}2m+1\\p\end{bmatrix}(1-2^{p+q+1})\frac{B_{p+q+1}}{p+q+1}\frac{t^q}{q!}.\end{aligned}$$

Together with the well-known generating series ([1, Proposition 2.6 (7)], note that $\left\{\begin{matrix}s\\2m\end{matrix}\right\} = 0$ if $s < 2m$)

$$(e^t-1)^{2m} = (2m)!\sum_{s=0}^{\infty}\left\{\begin{matrix}s\\2m\end{matrix}\right\}\frac{t^s}{s!},$$

we obtain

$$\begin{aligned}&\frac{e^t(e^t-1)^{2m}}{(e^t+1)^{2m+2}}\\ &= \frac{1}{2m+1}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\begin{bmatrix}2m+1\\p\end{bmatrix}\left\{\begin{matrix}s\\2m\end{matrix}\right\}\frac{B_{p+q+1}}{p+q+1}\frac{t^{q+s}}{q!s!}\\ &= \frac{1}{2m+1}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\left\{\begin{matrix}n-q\\2m\end{matrix}\right\}\frac{B_{p+q+1}}{p+q+1}\frac{t^n}{n!}.\end{aligned}$$

Substituting this into (3.1), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}D_n^{(k)}\frac{t^n}{n!}\\ &= 4\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{k+1}}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\sum_{p=1}^{2m+1}(-1)^{p+q}(1-2^{p+q+1})\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\left\{\begin{matrix}n-q\\2m\end{matrix}\right\}\frac{B_{p+q+1}}{p+q+1}\frac{t^n}{n!}\\ &= 4\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{(2m+1)^{k+1}}\sum_{p=1}^{2m+1}\sum_{q=0}^{n-2m}(2^{p+q+1}-1)\binom{n}{q}\begin{bmatrix}2m+1\\p\end{bmatrix}\left\{\begin{matrix}n-q\\2m\end{matrix}\right\}\frac{B_{p+q+1}}{p+q+1}\frac{t^n}{n!}.\end{aligned}$$

(We have used the facts that $B_{p+q+1} = 0$ if $p+q \geq 1$ is even and $\left\{ {n-q \atop 2m} \right\} = 0$ if $n-q < 2m$.) By equating the coefficients of $t^n/n!$ on both sides, we obtain the desired result. □

We can easily prove the second formula of Theorem 3.2 by using the definition of the n-th tangent numbers of order k, $T_{n,m}$ for the non negative integers n and k, by the generating relation (see [3, P. 259]).

$$\frac{\tan^k t}{k!} = \sum_{n=k}^{\infty} T_{n,m} \frac{t^n}{n!}, \tag{3.4}$$

and the formula in [4, Proposition 9]

$$T_{n,m} = (-1)^{\frac{n-k}{2}} (-1)^n \sum_{m=k}^{n} (-1)^m 2^{n-m} \left\{ {n \atop m} \right\} \binom{m-1}{k-1} \frac{m!}{k!}. \tag{3.5}$$

Proof of Theorem 3.2(Second Formula). From the definition we have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \frac{\mathrm{A}_k(\tanh(t/2))}{\sinh t} = \frac{d}{dt} \mathrm{A}_{k+1}(\tanh(t/2)) \\ &= 2 \frac{d}{dt} \sum_{m=0}^{\infty} \frac{(\tanh(t/2))^{2m+1}}{(2m+1)^{k+1}}. \end{aligned} \tag{3.6}$$

By using $\tanh t = -i \tan(it)$ and equations (3.4) and (3.5), we can write

$$\begin{aligned} (\tanh(t/2))^m &= (-i)^m m! \sum_{n=m}^{\infty} T_{n,m} \frac{i^n}{2^n} \frac{t^n}{n!} \\ &= (-i)^m (-1)^{\frac{n-m}{2}} \sum_{n=m}^{\infty} \sum_{p=m}^{n} (-2)^{n-p} p! \binom{p-1}{m-1} \left\{ {n \atop p} \right\} \frac{i^n}{2^n} \frac{t^n}{n!} \\ &= (-1)^m \sum_{n=m}^{\infty} \sum_{p=m}^{n} (-1)^p \frac{p!}{2^p} \binom{p-1}{m-1} \left\{ {n \atop p} \right\} \frac{t^n}{n!}. \end{aligned}$$

We therefore have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(k)} \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{k+1}} \sum_{n=2m}^{\infty} \sum_{p=2m}^{n} (-1)^p \frac{(p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2m+1)^{k+1}} \sum_{p=2m}^{n} \frac{(-1)^p (p+1)!}{2^p} \binom{p}{2m} \left\{ {n+1 \atop p+1} \right\} \frac{t^n}{n!}. \end{aligned}$$

By equating the coefficients of $t^n/n!$, we complete the proof of the theorem. □

Remark 3.4. In very similar manners, by using the definition of multi-poly cosecant numbers

$$\frac{\mathrm{A}(k_1, \ldots, k_r; \tanh t/2)}{\sinh t} = \sum_{n=0}^{\infty} D_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!},$$

we obtain the recurrence and explicit formulas for multi-poly-cosecant numbers as follows.

Notations: For any index set $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}^r$, put

$$\mathbf{k}_- = (k_1, \ldots, k_{r-1}, k_r - 1).$$

Proposition 3.5. *For any admissible index* $\mathbf{k}$ *and* $n \geq 0$,

$$D_n^{(\mathbf{k}-)} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2m+1} D_{n-2m}^{(\mathbf{k})}$$

Theorem 3.6. *(1) For any index set* $\mathbf{k}$ *and* $n \geq 0$,

$$D_n^{(\mathbf{k})} = 2^{r+1} \sum_{\substack{0<m_1<\cdots<m_{r-1}<m_r<n+2 \\ m_i \equiv i \ mod 2}} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{r-1} m_r^{k_r+1}} \sum_{p=1}^{m_r} \sum_{q=0}^{n-m_r+1} (-1)^n (2^{p+q+1}-1) \binom{n}{q} \times \left\{ {n-q \atop m_r-1} \right\} \left[{m_r \atop p} \right] \frac{B_{p+q+1}}{p+q+1}.$$

(2) For any index set $\mathbf{k}$ *and* $n \geq 0$,

$$D_n^{(\mathbf{k})} = \sum_{\substack{0<m_1<\cdots<m_r<n+2 \\ m_i \equiv i \ mod 2}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \sum_{p=m_r}^{n+1} \frac{(-1)^{p+m_r} p!}{2^{p-r}} \binom{p-1}{m_r-1} \left\{ {n+1 \atop p} \right\}.$$

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