Forcing and combinatorics of names

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Abstract

This is a lecture notes style paper about nice names, constructed via antichains, and their combinatorics. Also, we propose a simpler vision of completion of posets. These topics ease the transition from the forcing theorem to more advanced topics like forcing iterations.

1 Introduction

This text presents the author's vision about nice names (particularly of functions between ground model objects) and their use to calculate cardinalities in forcing generic extensions. The notion of (maximal) antichain is essential to construct these objects and to understand their combinatorics. Also, an alternative simpler definition of completion of a poset is proposed, which turn out to be useful when relating the forcing relation of a poset with its completion.

The topics proposed here ease the transition, in the study of forcing theory, from the forcing theorem to more advanced topics. Thereafter, this text only requires the knowledge of the Forcing Theorem 1.1 (and its proof) as in e.g. [Kun80, Kun11]. For the section about completions, we also assume some knowledge about (complete) Boolean algebras, e.g. [BM77]. At the end some facts about two steps iteration are included, which the author learned from professor Brendle's 2010 seminar "How to force it".

Notation

Throughout all the text, we work inside a countable transitive model V of ZF (or ZFC, depending on what is indicated in each result or section), unless otherwise indicated. A *poset* (also called *forcing notion*, or just a *forcing*) is a pair $\langle \mathbb{P}, \leq \rangle$ where $\leq$ is a reflexive and transitive relation on $\mathbb{P}$. In contrast with most set theory texts, we do not assume that $\mathbb{P}$ has a maximum condition $\mathbb{1}$.

Let $\mathbb{P}$ be a poset. For $p, q \in \mathbb{P}$, $q \leq p$ is often read *q is stronger than p*. The relation $p \perp q$ denotes that p and q are *incompatible*, that is, $\neg\exists r \in \mathbb{P}(r \leq p \wedge r \leq q)$, and $p \parallel q$ denotes that they are *compatible*.

Recall that τ is a $\mathbb{P}$-*name* if τ is a relation and $\forall(\sigma, p) \in \tau(\sigma$ is a $\mathbb{P}$-name and $p \in \mathbb{P})$. Denote by $V^{\mathbb{P}}$ the class of $\mathbb{P}$-names in V.

A filter G on $\mathbb{P}$ (usually living outside V) is $\mathbb{P}$-generic over V if it intersects every dense set $D \in V$ in $\mathbb{P}$. For $\tau \in V^{\mathbb{P}}$ define

$$\tau[G] := \{\sigma[G] : \exists p \in G((\sigma, p) \in \mathbb{P})\}$$

and $V[G] := \{\tau[G] : \tau \in V^{\mathbb{P}}\}$. We usually refer to V as the *ground model*, and to $V[G]$ as a $\mathbb{P}$*-generic extension of* V. When we say that φ is a *formula in the forcing language of* $\mathbb{P}$ we mean that φ has the form $\varphi(\tau_0, \dots, \tau_{n-1})$ for some $\mathbb{P}$-names $\tau_0, \dots, \tau_{n-1}$ (in the ground model) and some formula $\varphi(x_0, \dots, x_{n-1})$. We usually abbreviate the list $\tau_0, \dots, \tau_{n-1}$ by $\bar{\tau}$.

Concerning the forcing relation, $\Vdash_{\mathbb{P}} \varphi$ means that $p \Vdash_{\mathbb{P}} \varphi$ for all $p \in \mathbb{P}$. Recall:

Theorem 1.1 (The Forcing Theorem)**.** *Let $\varphi(x_0, \dots, x_{n-1})$ be a formula. Assume that V is a countable transitive model of* ZF*, $\mathbb{P} \in V$ and $\tau_0, \dots, \tau_{n-1} \in V^{\mathbb{P}}$. Then:*

(a) **Definability Lemma.** *Whenever $p \in \mathbb{P}$, $p \Vdash \varphi(\tau_0, \dots, \tau_{n-1})$ iff, for any $\mathbb{P}$-generic G over V, if $p \in G$ then $\varphi^{V[G]}(\tau_0[G], \dots, \tau_{n-1}[G])$.*

(b) **Truth Lemma.** *For any $\mathbb{P}$-generic G over V,*

$$\varphi^{V[G]}(\tau_0[G], \dots, \tau_{n-1}[G]) \text{ iff } \exists p \in G(p \Vdash_V \varphi(\tau_0, \dots, \tau_{n-1})).$$

Recall that, whenever G is $\mathbb{P}$-generic over V, $V[G]$ is a transitive model of ZF extending V such that V and $V[G]$ have the same ordinals. Even more, $V \models$ AC (the axiom of choice) implies $V[G] \models$ AC.

Results without proof in this paper can be consulted in the references, or are left to the reader. Details in some proofs are omitted as well.

2 Nice names

In this section we work in ZF, unless otherwise stated, and fix an arbitrary forcing notion $\mathbb{P}$. Since we do not assume that $\mathbb{P}$ have a maximum condition $\mathbb{1}$, for $x \in V$ we define the *canonical name of* x by

$$\check{x} := \{(\check{z}, p) : z \in x \text{ and } p \in \mathbb{P}\}.$$

We also define the *canonical name of the* $\mathbb{P}$*-generic set* as $\dot{G}_{\mathbb{P}} := \{(\check{p}, p) : p \in \mathbb{P}\}$. It is not hard to check that $\dot{x}[G] = x$ and $\dot{G}_{\mathbb{P}}[G] = G$ for any $\mathbb{P}$-generic G over V. Those are the first examples of *nice* names.

Like $\check{x}$ and $\dot{G}_{\mathbb{P}}$, it is very practical to work with well defined names for certain type of objects. As an example, we look at a simple definition for names of unordered pairs, ordered pairs, and union of sets.

Definition 2.1. Let τ, σ be $\mathbb{P}$-names. Define the $\mathbb{P}$-names

$$\begin{aligned} \mathrm{up}(\tau, \sigma) &:= \{(\tau, p) : p \in \mathbb{P}\} \cup \{(\sigma, p) : p \in \mathbb{P}\}, \\ \mathrm{op}(\tau, \sigma) &:= \mathrm{up}(\mathrm{up}(\tau, \tau), \mathrm{up}(\tau, \sigma)), \\ \mathrm{un}(\tau) &:= \{(\pi, r) : r \in \mathbb{P} \wedge \exists (\sigma, p) \in \tau \exists q \in \mathbb{P}((\pi, q) \in \sigma \text{ and } r \leq p, q)\}. \end{aligned}$$

Note that these notions are absolute for transitive models of ZF.

Lemma 2.2. *If $\tau, \sigma \in V^{\mathbb{P}}$ then* *(a)* $\Vdash \mathrm{up}(\tau,\sigma) = \{\tau,\sigma\}$. *(b)* $\Vdash \mathrm{op}(\tau,\sigma) = (\tau,\sigma)$. *(c)* $\Vdash \mathrm{un}(\tau) = \bigcup\tau$.

The following type of sets are essential for forcing combinatorics.

Definition 2.3. Let $C \subseteq \mathbb{P}$ and $p_0 \in \mathbb{P}$.

(1) C is *open (in* $\mathbb{P}$) if $\forall p \in C \forall q \leq p (q \in C)$.

(2) C is an *antichain (in* $\mathbb{P}$*)* if $\forall p, q \in C(p \neq q \Rightarrow p \perp q)$.

(3) C is *predense (in* $\mathbb{P}$*)* if $\forall p \in \mathbb{P} \exists q \in C(p \parallel q)$.

(4) C is *predense below* p_0 if $\forall p \leq p_0 \exists q \in C(p \parallel q)$.

(5) C is a *maximal antichain* if C is a predense antichain.

(6) C is a *maximal antichain below* p_0 if it is an antichain predense below p_0.

Note that these notions are absolute for transitive models of ZF (even equivalent to Δ_0-formulas).

Below we state very easy facts about these concepts.

Lemma 2.4. *Let* $p_0 \in \mathbb{P}$ *and let* φ *be a formula in the forcing language (of* $\mathbb{P}$*).*

(a) The set $D_\varphi := \{p \in \mathbb{P} : p \Vdash \varphi\}$ *is open.*

(b) Any dense set (below p_0*) is predense (below* p_0*).*

(c) Any maximal antichain (below p_0*) is predense (below* p_0*).*

(d) A subset of $\mathbb{P}$ *is open dense (below* p_0*) iff it is open predense (below* p_0*).*

(e) $p_0 \Vdash \varphi$ *iff* D_φ *is predense below* p_0*. Likewise,* $\Vdash \varphi$ *iff* D_φ *is predense.*

(f) $A \subseteq \mathbb{P}$ *is a maximal antichain iff* A *is an antichain and, for any antichain* A' *in* $\mathbb{P}$*, if* $A \subseteq A'$ *then* $A = A'$

The existence of maximal antichains can be proved by Zorn's Lemma.

Lemma 2.5 (ZFC)**.** *Let* $D \subseteq \mathbb{P}$ *be dense. If* $C \subseteq D$ *is an antichain then there exists a maximal antichain* $A \subseteq D$ *in* $\mathbb{P}$ *such that* $C \subseteq A$*.*

Antichains are very useful to define names thanks to the following property.

Lemma 2.6. *Assume that* $G \subseteq \mathbb{P}$ *is a filter and* $A \subseteq \mathbb{P}$ *is an antichain. Then* $|G \cap A| \leq 1$*.*

Proof. If $p, q \in G \cap A$ then $p \parallel q$, but since A is an antichain, $p = q$. □

The following lemma describes useful characterizations of generic sets.

Lemma 2.7. *Let* M *be a transitive model of* ZF *with* $\mathbb{P} \in M$*, and let* $G \subseteq \mathbb{P}$ *be a filter (usually outside* M*). Then the following statements are equivalent.*

(i) G *is* $\mathbb{P}$*-generic over* M*.*

(ii) For any open dense $D \in M$ (in $\mathbb{P}$), $G \cap D \neq \emptyset$.

(iii) For any predense $D \in M$ (in $\mathbb{P}$), $G \cap D \neq \emptyset$.

Even more, (iii) $\Rightarrow$ (iv) where

(iv) For any maximal antichain $A \in M$ (in $\mathbb{P}$), $G \cap A \neq \emptyset$,

and (iv) $\Rightarrow$ (i) is valid when $M \models \mathrm{AC}$.

In the case of (iv), $G \cap A \neq \emptyset$ actually means that $|G \cap A| = 1$ by Lemma 2.6. This is useful to define nice names, according to the following tool.

Lemma 2.8. *In V, let A be an antichain and $h : A \to V^{\mathbb{P}}$. Define $\mathrm{an}(h) := \mathrm{un}(\tau)$ where $\tau := \{(h(p), p) : p \in A\}$. Then:*

(a) For any $p \in A$, $p \Vdash \mathrm{an}(h) = h(p)$.

(b) If A is a maximal antichain (below p_0), $\varphi(z, x_0, \ldots, x_{n-1})$ is a formula, $\sigma_0, \ldots, \sigma_{n-1} \in V^{\mathbb{P}}$ and $\forall p \in A(p \Vdash \varphi(h(p), \bar{\sigma}))$, then $\Vdash \varphi(\mathrm{an}(h), \bar{\sigma})$ (respectively $p_0 \Vdash \varphi(\mathrm{an}(h), \bar{\sigma})$).

Proof. By Lemma 2.2(c), to prove (a) it is enough to show that $p \Vdash \tau = \{h(p)\}$ for any $p \in A$. We show this by using the Forcing Theorem. Fix $p \in A$ and let G be an arbitrary $\mathbb{P}$-generic over V with $p \in G$. Note that $\tau[G] = \{h(r)[G] : r \in A \cap G\}$. By Lemma 2.6, $|G \cap A| \leq 1$, but since $p \in G \cap A$, $G \cap A = \{p\}$, so $\tau[G] = \{h(p)[G]\}$. Therefore, by the Definability Lemma, $p \Vdash \tau = \{h(p)\}$, so $p \Vdash \mathrm{an}(h) = \bigcup \tau = h(p)$.

To see (b), we have by (a) that $p \Vdash \varphi(\mathrm{an}(h), \bar{\sigma})$ for any $p \in A$, so $\{p \in \mathbb{P} : p \Vdash \varphi(\mathrm{an}(h), \bar{\sigma})\}$ is predense. Hence $\Vdash \varphi(\mathrm{an}(h), \bar{\sigma})$ by Lemma 2.4(e). The "below p_0" fact is similar. $\square$

Now we are ready to introduce the following very useful notions of nice names.

Definition 2.9. As in Lemma 2.8, whenever h is a function such that $\mathrm{dom}h$ is an antichain in $\mathbb{P}$ and $h(p)$ is a $\mathbb{P}$-name for all $p \in \mathrm{dom}h$, we denote

$$\mathrm{an}(h) := \mathrm{un}(\{(h(p), p) : p \in \mathrm{dom}h\}).$$

Fix sets B and C.

(1) Say that $\dot{x}$ is a *nice name of a member of C* if $\dot{x} = \mathrm{an}(h)$ for some function h into $\mathrm{dom}\check{C} = \{\check{y} : y \in C\}$ such that $\mathrm{dom}h \subseteq \mathbb{P}$ is a maximal antichain.

Denote by $\mathrm{nice}(C) = \mathrm{nice}_{\mathbb{P}}(C)$ the collection of all nice names of members of C.

(2) When H is a function from B into the class of $\mathbb{P}$-names, denote

$$\mathrm{fn}(H) := \{(\mathrm{op}(\check{x}, H(x)), p) : x \in B,\ p \in \mathbb{P}\}.$$

(3) Say that $\dot{f}$ is a *nice name of a function from B into C* if $\dot{f} = \mathrm{fn}(H)$ for some function $H : B \to \mathrm{nice}(C)$. Denote $\mathrm{ncf}(B, C) = \mathrm{ncf}_{\mathbb{P}}(B, C)$ the collection of all nice names of functions from B into C.

Lemma 2.10. *If $B, H \in V$ and H is as in Definition 2.9(2) then $\Vdash$ "$\mathrm{fn}(H)$ is a function, $\mathrm{dom}(\mathrm{fn}(H)) = \check{B}$" and, for any $x \in B$, $\Vdash \mathrm{fn}(H)(\check{x}) = H(x)$.*

Proof. Let G be $\mathbb{P}$-generic over V. Set $f := \mathrm{fn}(H)[G]$. Then $f = \{(x, H(x)[G]) : x \in B\}$, which is clearly a function with domain B such that $f(x) = H(x)[G]$ for any $x \in B$. □

As expected, these nice names corresponds to objects they are actually describing (members of C and functions from B into C).

Corollary 2.11. *Let $B, C \in V$.*

(a) If $\dot{x} \in V$ is a nice name of a member of C then $\Vdash \dot{x} \in \check{C}$.

(b) If $\dot{f} \in V$ is a nice name of a function from B into C, then $\Vdash \dot{f} : \check{B} \to \check{C}$.

Proof. Let $h : \mathrm{dom}h \to \mathrm{dom}\check{C}$ be as in Definition 2.9(1) such that $\dot{x} = \mathrm{an}(h)$. For any $p \in \mathrm{dom}h$ it is clear that $p \Vdash h(p) \in \check{C}$. Hence, by Lemma 2.8(b), $\Vdash \dot{x} \in \check{C}$. This shows (a).

To see (b), assume that $\dot{f} = \mathrm{fn}(H)$ where $H : B \to \mathrm{nice}(C)$. By Lemma 2.10, $\Vdash$"$\dot{f}$ is a function with domain $\check{B}$" such that $\forall x \in B(\Vdash \dot{f}(x) = H(x))$. If $x \in B$ then, by (a), $\Vdash H(x) \in \check{C}$, thus $\Vdash$"$\dot{f} : \check{B} \to \check{C}$". □

Lemma 2.12. *Let $\varphi(y, x_0, \dots, x_{n-1})$ be a formula. In V, let $\tau_0, \dots, \tau_{n-1} \in V^{\mathbb{P}}$ and let K be a class of $\mathbb{P}$-names. If*

$$\forall p \in \mathbb{P}\big((p \Vdash \exists z \varphi(z, \bar{\tau})) \Rightarrow \exists q \leq p \exists \rho \in K(q \Vdash \varphi(\rho, \bar{\tau}))\big)$$

then the following set is open dense:

$$D := \{r \in \mathbb{P} : \exists \rho \in K(r \Vdash \varphi(\rho, \bar{\tau})) \vee r \Vdash \neg\exists z \varphi(z, \bar{\tau})\}$$

Proof. It is clear that D is open. Let $p \in \mathbb{P}$. If $p \nVdash \neg\exists z\varphi(z, \bar{\tau})$ then $p' \Vdash \exists z \varphi(z, \bar{\tau})$ for some $p' \leq p$. So, by the hypothesis, there is some $q \leq p'$ in D. □

Theorem 2.13 (ZFC)**.** *In V, with the same hypothesis as in Lemma 2.12, if $K \neq \emptyset$ then there is some maximal antichain $A \subseteq \mathbb{P}$ and some $h : A \to K$ such that*

$$\Vdash \varphi(\mathrm{an}(h), \bar{\tau}) \Leftrightarrow \exists z \varphi(z, \bar{\tau}).$$

Proof. Choose some $\sigma_0 \in K$, and let D be as in Lemma 2.12. By Lemma 2.5, there is some maximal antichain $A \subseteq D$ in $\mathbb{P}$. Define $h : A \to K$ as follows: for $p \in A$, when $p \Vdash \neg\exists z\varphi(z, \bar{\tau})$ set $h(p) := \sigma_0$; otherwise, $\exists \rho \in K(p \Vdash \varphi(\rho, \bar{\tau}))$ because $p \in D$, so choose some $h(p) \in K$ such that $p \Vdash \varphi(h(p), \bar{\tau}))$. By Lemma 2.8, h is as required. □

By application of the previous theorem to $K = V^{\mathbb{P}}$, we obtain the well-known

Theorem 2.14 (Maximal Principle (ZFC))**.** *Let $\varphi(y, x_0, \dots, x_{n-1})$ be a formula. In V, if $\tau_0, \dots, \tau_{n-1} \in V^{\mathbb{P}}$ then there is some $\sigma \in V^{\mathbb{P}}$ such that $\Vdash \varphi(\sigma, \bar{\tau}) \Leftrightarrow \exists z \varphi(z, \bar{\tau})$.*

Remark 2.15. In ZF, the maximal principle is equivalent to AC. Let A be a set such that $\forall a, b \in A(a \neq \emptyset \wedge (a \neq b \Rightarrow a \cap b = \emptyset))$. Define $\mathbb{P} = \bigcup A$ ordered by $q \leq p$ iff $\exists a \in A(p, q \in a)$. Use the maximal principle to find a $\mathbb{P}$-name $\dot{p}$ such that $\Vdash \dot{p} \in \dot{G}_{\mathbb{P}}$. Define $c := \{q \in \mathbb{P} : \exists r \in \mathbb{P}(r \Vdash \dot{p} = \check{q})\}$, so $\forall a \in A(|a \cap c| = 1)$.

The following result is the reason we call these objects "nice names".

Theorem 2.16 (ZFC). *In V, let B and C be sets, $p \in \mathbb{P}$ and $\sigma \in V^{\mathbb{P}}$. Then:*

(a) If $p \Vdash \sigma \in \check{C}$ then there is some nice name $\dot{x}$ of a member of C such that $p \Vdash \sigma = \dot{x}$.

(b) If $p \Vdash \sigma : \check{B} \to \check{C}$ then there is some nice name $\dot{f}$ of a function from B into C such that $p \Vdash \sigma = \dot{f}$.

Both results are also valid when omitting p (e.g. if $\Vdash \sigma \in \check{C}$ then... such that $\Vdash \sigma = \dot{x}$).

Proof. Work in V. For (a), note that

$$\forall p' \in \mathbb{P}\big((p' \Vdash \exists z(z \in \check{C} \wedge z = \sigma)) \Rightarrow \exists q \leq p' \exists \rho \in \mathrm{dom}\check{C}(q \Vdash \text{"}\rho \in \check{C} \wedge \rho = \sigma\text{"})\big).$$

Hence, by application of Theorem 2.13 to $K = \mathrm{dom}\check{C}$ ($\mathrm{dom}\check{C} \neq \emptyset$ because $p \Vdash \sigma \in \check{C}$), we obtain a function $h : A \to \mathrm{dom}\check{C}$ with A maximal antichain in $\mathbb{P}$ such that

$$\Vdash \mathrm{an}(h) = \sigma \Leftrightarrow \sigma \in \check{C}.$$

Hence $p \Vdash \dot{x} = \sigma$ where $\dot{x} := \mathrm{an}(h) \in \mathrm{nice}(C)$.

For (b), assume that $p \Vdash \sigma : \check{B} \to \check{C}$. Fix $b \in B$ and set $\varphi(y, \check{b})$:"$\sigma : \check{B} \to \check{C}$ and $y = \sigma(\check{b})$". Note that

$$\forall p' \in \mathbb{P}\big((p' \Vdash \exists z \varphi(z, \check{b})) \Rightarrow \exists q \leq p' \exists \rho \in \mathrm{dom}\check{C}(q \Vdash \varphi(\rho, \check{b})\big)$$

so, by application of Theorem 2.13 to $K = \mathrm{dom}\check{C}$, we obtain a function $h_b : A_b \to \mathrm{dom}\check{C}$ with A_b maximal antichain in $\mathbb{P}$ such that

$$\Vdash \sigma : \check{B} \to \check{C} \Rightarrow \mathrm{an}(h_b) = \sigma(b).$$

Set $H(b) := \mathrm{an}(h_b)$. It is clear that $H : B \to \mathrm{nice}(C)$, and that $\dot{f} := \mathrm{fn}(H)$ is as required by Lemma 2.10. $\square$

3 Combinatorics of names

In this section we work in ZFC, unless otherwise stated. Fix an arbitrary forcing notion $\mathbb{P}$. Recall:

Definition 3.1. Let κ be a cardinal. Say that $\mathbb{P}$ *has the κ-chain condition*, abbreviated *κ-cc*, if every antichain in $\mathbb{P}$ has size $< \kappa$.

Say that $\mathbb{P}$ *has the countable chain condition*, abbreviated *ccc*, if it has the $\aleph_1$-cc, that is, if every antichain in $\mathbb{P}$ is countable.

It is well-known that κ-cc posets preserve cofinalities (and cardinalities) above κ. Although these proofs are in [Kun80, Kun11], we present shorter proofs using the tools of the previous section.

Lemma 3.2. *In V, let κ be a cardinal and assume that $\mathbb{P}$ has the κ-cc. Let $p \in \mathbb{P}$, $\dot{x} \in V^{\mathbb{P}}$, and let B and C be sets.*

(a) If $p \Vdash \dot{x} \in \check{C}$ then there is some $K \in [C]^{<\kappa}$ such that $p \Vdash \dot{x} \in \check{K}$.

(b) If $p \Vdash \dot{x} : \check{B} \to \check{C}$ then there is some $F : B \to [C]^{<\kappa}$ such that, for any $b \in B$, $p \Vdash \dot{x}(\check{b}) \in F(b)\check{}$.

These results are also valid when omitting p.

Proof. For (a), by Theorem 2.16(a) we may assume that $\dot{x}$ is a nice name of a member of C, that is, $\dot{x} = \mathrm{an}(h)$ where $h : A \to \mathrm{dom}\check{C}$ and A is a maximal antichain in $\mathbb{P}$. Let $K := \mathrm{ran}h$. Since $\mathbb{P}$ has the κ-cc, $|K| \leq |A| < \kappa$. On the other hand, for any $q \in A$, $q \Vdash \dot{x} = h(q) \in \check{K}$, hence $\Vdash \dot{x} \in \check{K}$ by Lemma 2.8(b).

For (b), by Theorem 2.16(b) we may assume that $\dot{x}$ is a nice name of a function from B into C, that is, $\dot{x} = \mathrm{fn}(H)$ for some $H : B \to \mathrm{nice}(C)$. Fix $b \in B$. By (a), there is some $F(b) \in [C]^{<\kappa}$ such that $\Vdash \dot{x}(\check{b}) = H(b) \in F(b)\check{}$, which yields the desired function F. □

Theorem 3.3. *Let κ be an infinite cardinal and assume that $\mathbb{P}$ has the κ-cc. Then $\mathbb{P}$ preserves regular cardinals $\geq \kappa$. In particular, if $\mathbb{P}$ has the ccc then it preserves all regular cardinals.*

Proof. Let $\lambda \geq \kappa$ be a regular cardinal. It is sufficient to show that

$$\Vdash \forall \alpha < \check{\lambda} \forall f : \alpha \to \check{\lambda} \exists \beta < \check{\lambda} (\mathrm{ran} f \subseteq \beta).$$

Let $p \in \mathbb{P}$ and assume that $\dot{a}, \dot{f} \in V^{\mathbb{P}}$ such that $p \Vdash$"$\dot{a} < \check{\lambda}$ and $\dot{f} : \dot{a} \to \check{\lambda}$". Then, there are $q \leq p$ and $\alpha < \lambda$ such that $q \Vdash \dot{a} = \check{\alpha}$. It is enough to show that there is some $\beta < \lambda$ such that $q \Vdash \mathrm{ran}\dot{f} \subseteq \check{\beta}$. By Lemma 3.2(b), there is some function $F : \alpha \to [\lambda]^{<\kappa}$ such that, for any $\xi < \alpha$, $q \Vdash \dot{f}(\check{\xi}) \in F(\xi)\check{}$. Since λ is regular and $|F(\xi)| < \kappa \leq \lambda$, we have that $\sup(F(\xi)) < \lambda$. Define $f^* : \alpha \to \lambda$ by $f^*(\xi) := \sup(F(\xi))$. Again, because λ is regular, there is some $\beta < \lambda$ such that $f^*(\xi) < \beta$ for any $\xi < \alpha$, so $F(\xi) \subseteq \beta$. This implies $\Vdash F(\xi)\check{} \subseteq \check{\beta}$, so $q \Vdash \dot{f}(\check{\xi}) < \check{\beta}$. Therefore, $q \Vdash \mathrm{ran}\dot{f} \subseteq \check{\beta}$. □

To calculate cardinal exponentiation in generic extensions, that is, the size of $|^{\kappa}\lambda|$, by Theorem 2.16 we can estimate it by the number of nice names of functions $\kappa \to \lambda$ we have. The following results gives a bound of this number.

Theorem 3.4. *Let B and C be sets, and let κ be an infinite cardinal. If $\mathbb{P}$ has the κ-cc then*

(a) $|\mathrm{nice}_{\mathbb{P}}(C)| \leq |\mathbb{P}|^{<\kappa} \cdot |C|^{<\kappa}$.

(b) $|\mathrm{ncf}_{\mathbb{P}}(B, C)| \leq (|\mathbb{P}|^{<\kappa} \cdot |C|^{<\kappa})^{|B|}$.

Proof. For (a), let $\mathcal{A} := \{A \subseteq \mathbb{P} : A \text{ is a maximal antichain}\}$, so

$$\mathrm{nice}(C) = \{\mathrm{an}(h) : h : A \to \mathrm{dom}\check{C} \wedge A \in \mathcal{A}\}.$$

Hence

$$|\mathrm{nice}(C)| \leq |\{h : A \to \mathrm{dom}\check{C} : A \in \mathcal{A}\}| = \left| \bigcup_{A \in \mathcal{A}} |^{A}C| \right|$$

Since $\mathbb{P}$ has the κ-cc, $\mathcal{A} \subseteq [\mathbb{P}]^{<\kappa}$, so $|\mathcal{A}| \leq |\mathbb{P}|^{<\kappa}$. On the other hand, for any $A \in \mathcal{A}$, $|^{A}C| = |C|^{|A|} \leq |C|^{<\kappa}$. Therefore

$$|\mathrm{nice}(C)| \leq \left| \bigcup_{A \in \mathcal{A}} |^{A}C| \right| \leq |\mathcal{A}| \cdot |C|^{<\kappa} \leq |\mathbb{P}|^{<\kappa} \cdot |C|^{<\kappa}.$$

To see (b), note that $\mathrm{ncf}(B,C) = \{\mathrm{fn}(H) : H \in {}^{B}\mathrm{nice}(C)\}$, so

$$|\mathrm{ncf}(B,C)| \leq |\mathrm{nice}(C)|^{|B|},$$

and the result follows by (a). □

To finish this section, we show the use of the maximal principle (Theorem 2.14) to obtain nice names for members of $\dot{Q}[G]$ for an arbitrary $\dot{Q} \in V^{\mathbb{P}}$.

Lemma 3.5. *In V, if $\dot{Q} \in V^{\mathbb{P}}$ then there is a cardinal μ such that $\Vdash |\dot{Q}| \leq |\check{\mu}|$.*

Proof. Set $\varphi(z,\dot{Q})$:"z is a cardinal and $|\dot{Q}| = z$" and $K := \{\check{\kappa} : \kappa \text{ is a cardinal}\}$. Note that

$$\forall p \in \mathbb{P}\big((p \Vdash \exists z \varphi(z,\dot{Q})) \Rightarrow \exists q \leq p \exists \rho \in K(q \Vdash \varphi(\rho,\dot{Q}))\big)$$

so, by Theorem 2.13, there is some $h : A \to K$ with A maximal antichain in $\mathbb{P}$ such that

$$\Vdash |\dot{Q}| = \mathrm{an}(h).$$

For each $p \in A$ there is a (unique) cardinal μ_p such that $h(p) = \check{\mu}_p$. Set $\mu := \sup_{p \in A} \mu_p$, which is as required. □

Theorem 3.6. *In V let κ be an infinite cardinal, assume that $\mathbb{P}$ has the κ-cc, $\dot{Q} \in V^{\mathbb{P}}$, μ is a cardinal and $\Vdash 0 < |\dot{Q}| \leq |\check{\mu}|$. Then there is a set $\langle\dot{Q}\rangle_{\mathbb{P}}$ of $\mathbb{P}$-names of size $\leq |\mathbb{P}|^{<\kappa} \cdot \mu^{<\kappa}$ such that*

(a) $\forall \dot{q} \in \langle\dot{Q}\rangle_{\mathbb{P}} (\Vdash \dot{q} \in \dot{Q})$,

(b) whenever $p \in \mathbb{P}$, $\sigma \in V^{\mathbb{P}}$ and $p \Vdash \sigma \in \dot{Q}$, there is some $\dot{q} \in \langle\dot{Q}\rangle_{\mathbb{P}}$ such that $p \Vdash \sigma = \dot{q}$.

Proof. By the maximal principle (Theorem 2.14) there is some $\mathbb{P}$-name $\dot{f}$ such that $\Vdash$"$\dot{f} : \check{\mu} \to \dot{Q}$ is onto". Using the same principle again, there is a function $H : \mathrm{nice}(\mu) \to V^{\mathbb{P}}$ such that, for any $\dot{\alpha} \in \mathrm{nice}(\mu)$, $\Vdash \dot{f}(\dot{\alpha}) = H(\dot{\alpha})$. Set $\langle\dot{Q}\rangle_{\mathbb{P}} := \mathrm{ran} H$. Hence $|\langle\dot{Q}\rangle_{\mathbb{P}}| \leq |\mathrm{nice}(\mu)| \leq |\mathbb{P}|^{<\kappa} \cdot \mu^{<\kappa}$ by Lemma 3.4.

Item (a) is clear. For (b), assume that $p \Vdash \sigma \in \dot{Q}$. By Theorem 2.13 applied to $\varphi(z,\sigma)$:"$z \in \check{\mu}$ and $\dot{f}(z) = \sigma$" and $K = \mathrm{dom}\check{\mu}$, there is some $h : A \to K$ with A a maximal antichain in $\mathbb{P}$ such that

$$\Vdash \dot{f}(\mathrm{an}(h)) = \sigma \Leftrightarrow \sigma \in \dot{Q}.$$

Clearly, $\dot{\alpha} := \mathrm{an}(h)$ is in $\mathrm{nice}(\mu)$ and $p \Vdash \sigma = \dot{f}(\dot{\alpha})$. Hence $\dot{q} := H(\dot{\alpha})$ is as desired. □

4 Completions via subsets and antichains

In this section we work in ZF, unless otherwise indicated, and we fix two arbitrary posets $\mathbb{P}$ and $\mathbb{Q}$. We first review the basic notions of complete and dense embeddings between posets.

Definition 4.1. Let $i : \mathbb{P} \to \mathbb{Q}$.

(1) The map i is a *complete embedding* if it fulfils:

(i) $\forall p, p' \in \mathbb{P}(p' \leq_{\mathbb{P}} p \Rightarrow i(p') \leq_{\mathbb{Q}} i(p))$,

(ii) $\forall p, p' \in \mathbb{P}(p \perp_{\mathbb{P}} p' \Rightarrow i(p) \perp_{\mathbb{Q}} i(p'))$,

(iii) for any $q \in \mathbb{Q}$ there is some $p \in \mathbb{P}$ such that $\forall p' \leq_{\mathbb{P}} p(i(p') \parallel_{\mathbb{Q}} q)$. Such a p is called a *reduction of* q.

(2) The map i is a *dense embedding* if it satisfies (i), (ii) above, and

(iii') $\mathrm{ran}(i)$ is dense in $\mathbb{Q}$.

We associate with i the transformation $i^* : V^{\mathbb{P}} \to V^{\mathbb{Q}}$ defined by recursion as

$$i^*(\tau) := \{(i^*(\sigma), i(p)) : (\sigma, p) \in \tau\}$$

Note that:

Lemma 4.2. *Let* $i : \mathbb{P} \to \mathbb{Q}$*.*

(a) If i *is a dense embedding, then it is a complete embedding.*

(b) i *is a complete embedding iff* i *satisfies (i), (ii) and, for any predense* $D \subseteq \mathbb{P}$, $i[D]$ *is predense in* $\mathbb{Q}$*.*

The importance of these type of embeddings is illustrated in the following result.

Theorem 4.3. *Assume that* $i : \mathbb{P} \to \mathbb{Q}$ *is a complete embedding and* H *is* $\mathbb{Q}$*-generic over* V*. Then*

(a) $G := i^{-1}[H]$ *is* $\mathbb{P}$*-generic over* V *and* $V[G] \subseteq V[H]$*. Even more, if* i *is a dense embedding in* V*, then* $V[G] = V[H]$*.*

(b) If $\tau \in V^{\mathbb{P}}$ *and* $\sigma := i^*(\tau)$ *then* $\sigma[H] = \tau[G]$*.*

(c) If $p \in \mathbb{P}$, $\tau_0, \ldots, \tau_{n-1} \in V^{\mathbb{P}}$ *and* $\varphi(x_0, \ldots, x_{n-1})$ *is an absolute formula between transitive models of* ZF *(or* ZFC *when* $V \models \mathrm{AC}$*), then*

$$p \Vdash \varphi(\tau_0, \ldots, \tau_{n-1}) \text{ iff } i(p) \Vdash \varphi(i^*(\tau_0), \ldots, i^*(\tau_{n-1})).$$

(d) If i *is a dense embedding, then (c) is valid without the absoluteness requirement for* φ*.*

(e) If i *is a dense embedding and* G' *is* $\mathbb{P}$*-generic over* V*, then*

$$H' := i\langle G\rangle = \{r \in \mathbb{Q} : \exists q \in i[G](q \leq r)\}$$

is $\mathbb{Q}$*-generic over* V *and* $V[H'] = V[G']$*. Even more,* $i\langle i^{-1}[H]\rangle = H$ *and* $i^{-1}[i\langle G'\rangle] = G'$*.*

When we look at a Boolean algebra $\mathbb{B}$ as a poset, we must exclude its minimum element $0_{\mathbb{B}}$. So, when we deal with $\mathbb{B}$ in the context of forcing, we are actually looking at $\mathbb{B} \smallsetminus \{0_{\mathbb{B}}\}$. For example, when we say that "$i : \mathbb{P} \to \mathbb{B}$ is a complete embedding" we mean that "$i : \mathbb{P} \to \mathbb{B} \smallsetminus \{0_{\mathbb{B}}\}$ is a complete embedding".

Recall that a *completion* of $\mathbb{P}$ is a complete Boolean algebra $\mathbb{B}$ such that there is a dense embedding $i : \mathbb{P} \to \mathbb{B}$. It is known that any poset has a completion, and that it is unique modulo isomorphism. The typical completion $\mathrm{ro}(\mathbb{P})$ of a poset $\mathbb{P}$ is defined as the set of *regular open* sets in the topology of $\mathbb{P}$ whose base is $\{O(p) : p \in \mathbb{P}\}$, where $O(p) := \{r \in \mathbb{P} : r \leq p\}$.[1] Recall that, given a topological space X, U is *regular open* in X iff $U = \mathrm{int}(\mathrm{cl}(U))$.

Lemma 4.4. *Let $A \subseteq \mathbb{P}$. Then:*

(a) $p \in \mathrm{int}(\mathrm{cl}(A))$ iff A is dense below p.

(b) A is regular open iff, for any $p \in \mathbb{P}$, $p \in A \Leftrightarrow A$ is dense below p.

Proof. It is enough to show (a). Assume $p \in \mathrm{int}(\mathrm{cl}(A))$, that is, $O(p) \subseteq \mathrm{cl}(A)$. If $q \leq p$ then $q \in \mathrm{cl}(A)$, so $[q] \cap A \neq \emptyset$, that is, $\exists r \leq q (r \in A)$. Hence A is dense below p.

For the converse, assume that A is dense below p. It is enough to show that $O(p) \subseteq \mathrm{cl}(A)$. Let $q \leq p$, so there is some $r \in A$ with $r \leq q$, that is, $O(q) \cap A \neq \emptyset$. Since every open neighborhood of q contains $O(q)$, we conclude that $q \in \mathrm{cl}(A)$. □

We present an alternative way to define the completion of a poset, which is more natural and also more practical when using completions.

Definition 4.5. Let $p, q \in \mathbb{P}$, $\mathcal{F} \subseteq \mathcal{P}(\mathbb{P})$ and let A, B be subsets of $\mathbb{P}$. Define:

(1) $A \leq^*_{\mathbb{P}} B$ iff
$$\forall p \in A \forall p' \leq p \exists q \in B (p' \parallel q),$$
that is, B is predense below any $p \in A$.

(2) $A =^*_{\mathbb{P}} B$ iff $A \leq^*_{\mathbb{P}} B$ and $B \leq^*_{\mathbb{P}} A$.

(3) $p \leq^*_{\mathbb{P}} q$ iff $\{p\} \leq^*_{\mathbb{P}} \{q\}$.

(4) $p =^*_{\mathbb{P}} q$ iff $p \leq^*_{\mathbb{P}} q$ and $q \leq^*_{\mathbb{P}} p$.

(5) $A \perp_{\mathbb{P}} B$ iff $\forall p \in A \forall q \in B (p \perp q)$.

(6) ${\sim} A := \{p \in \mathbb{P} : \{p\} \perp_{\mathbb{P}} A\}$.

(7) $\bigwedge \mathcal{F} := \{p \in \mathbb{P} : \forall X \in \mathcal{F} (\{p\} \leq^*_{\mathbb{P}} X)\}$.

(8) $A \wedge B := \bigwedge \{A, B\}$.

(9) Define the poset $\mathrm{Pw}(\mathbb{P}) := \mathcal{P}(\mathbb{P}) \smallsetminus \{\emptyset\}$ ordered by $\leq^*_{\mathbb{P}}$

(10) Say that $\mathbb{P}$ is *separative* if, for any $p, q \in \mathbb{P}$, $q \leq^*_{\mathbb{P}} p$ iff $q \leq_{\mathbb{P}} p$.

The subindex $\mathbb{P}$ is omitted when clear from the context.

For example, any Boolean algebra is separative, that is, if $\mathbb{B}$ is a Boolean algebra and $b, b' \in \mathbb{B}$, then $b' \leq b$ iff $\forall a \in \mathbb{B} \smallsetminus \{0_{\mathbb{B}}\} (a \leq b' \Rightarrow a \wedge b \neq 0_{\mathbb{B}})$.

Below we list some properties of the order $\leq^*$ and the operations defined above. The proof is left as exercise for the reader.

Lemma 4.6. *Let $A, B, C \in \mathcal{P}(\mathbb{P})$ and $\mathcal{F} \subseteq \mathcal{P}(\mathbb{P})$.*

[1] In fact, $C \subseteq \mathbb{P}$ is open in this topology iff it is open in the sense of Definition 2.3(1).

(a) If $A \subseteq B$ then $A \leq^* B$.

(b) $A \leq^* B$ iff $\forall p \in A(\{p\} \leq^* B)$.

(c) If $A \leq^* B$ and $B \leq^* C$ then $A \leq^* C$.

(d) $A \perp {\sim}A$.

(e) $A \leq^* \mathbb{P}$, and $A =^* \mathbb{P}$ iff A is predense in $\mathbb{P}$.

(f) $A \cup {\sim}A =^* \mathbb{P}$.

(g) $\emptyset \leq^* A$, and $A =^* \emptyset$ iff $A = \emptyset$.

(h) $A \wedge B =^* \emptyset$ iff $A \perp B$.

(i) $\forall A \in \mathcal{F}(A \leq^* \bigcup \mathcal{F})$.

(j) If $\forall A \in \mathcal{F}(A \leq^* B)$ then $\bigcup \mathcal{F} \leq^* B$.

(k) $\forall A \in \mathcal{F}(\bigwedge \mathcal{F} \leq^* A)$.

(l) If $\forall A \in \mathcal{F}(B \leq^* A)$ then $B \leq^* \bigwedge \mathcal{F}$.

(m) $A \wedge \bigcup \mathcal{F} =^* \bigcup_{X \in \mathcal{F}} A \wedge X$.

There properties indicate that, under $\leq^*$, $\mathcal{P}(\mathbb{P})$ becomes a complete Boolean algebra modulo the equivalence relation $=^*$. We show that this is a completion of $\mathbb{P}$.

Definition 4.7. Let $\mathbb{B}_{\mathbb{P}} := \mathcal{P}(\mathbb{P})/ =^*$ ordered by $[A] \leq [B]$ iff $A \leq^* B$, where $[A]$ denotes the $=^*$-equivalence class of A.

Theorem 4.8. *The partial order $\mathbb{B}_{\mathbb{P}}$ is a complete Boolean algebra and $\pi_{\mathbb{P}} : \mathrm{Pw}(\mathbb{P}) \to \mathbb{B}_{\mathbb{P}}$, $\pi_{\mathbb{P}}(A) := [A]$, is a dense embedding.*

The previous result actually shows that $\mathbb{B}_{\mathbb{P}}$ is a completion of $\mathrm{Pw}(\mathbb{P})$. Since $\mathbb{P}$ densely embeds into $\mathrm{Pw}(\mathbb{P})$, $\mathbb{B}_{\mathbb{P}}$ is a completion of $\mathbb{P}$.

Lemma 4.9. *Let $A, B \in \mathrm{Pw}(\mathbb{P})$*

(a) A and B are incompatible in $\mathrm{Pw}(\mathbb{P})$ iff $A \perp B$ (as in Definition 4.5(5)).

(b) $\mathrm{Pw}(\mathbb{P})$ is separative.

(c) The map $\iota_{\mathbb{P}} : \mathbb{P} \to \mathrm{Pw}(\mathbb{P})$, $\iota_{\mathbb{P}}(p) := \{p\}$ is a dense embedding. In particular $h_{\mathbb{P}} := \pi_{\mathbb{P}} \circ \iota_{\mathbb{P}} : \mathbb{P} \to \mathbb{B}_{\mathbb{P}} \smallsetminus \{[\emptyset]\}$ is a dense embedding.

Proof. For (a), first assume that A and B are compatible in $\mathrm{Pw}(\mathbb{P})$, that is, there is some $C \in \mathrm{Pw}(\mathbb{P})$ such that $C \leq^* A$ and $C \leq^* B$. Since $C \neq \emptyset$, choose some $q \in C$. Then $q \parallel p$ for some $p \in A$, so choose some $r \leq p, q$. Now, since $C \leq^* B$ and $r \leq q$, $r \parallel p'$ for some $p' \in B$. Hence $\exists p \in A \exists p' \in B(p \parallel p')$.

To see the converse, assume that $\exists p \in A \exists q \in B(p \parallel q)$, so there is some $r \leq p, q$. It is easy to see that $\{r\} \leq^* A$ and $\{r\} \leq^* B$.

To see (b), assume that every $Z \leq^* A$ is compatible with B. It is enough to show that $A \leq^* B$. Let $p \in A$ and $p' \leq p$. It is clear that $\{p'\} \leq^* A$, so $\{p'\}$ is compatible with B, that is, by (a) p' is compatible with some $q \in B$.

For (c): it is clear that, for $p, q \in \mathbb{P}$, $q \leq p$ implies $\{q\} \leq^* \{p\}$ and, according to (a), $p \perp_{\mathbb{P}} q$ implies $\{p\} \perp_{\mathrm{Pw}(\mathbb{P})} \{q\}$. Also, $\mathrm{ran} i^*$ is dense in $\mathrm{Pw}(\mathbb{P})$ because $p \in A$ implies $\{p\} \leq^* A$. □

Our results also allow to show that $\mathrm{ro}(\mathbb{P})$, with the order $\subseteq$, is a completion of $\mathbb{P}$.

Theorem 4.10. *Assume that $U \subseteq \mathbb{P}$ is regular open.*

(a) If $U' \subseteq \mathbb{P}$ is regular open, then $U \leq^* U' \Leftrightarrow U \subseteq U'$.

(b) For any $A \subseteq \mathbb{P}$ there is a unique regular open $U' \subseteq \mathbb{P}$ such that $A =^* U'$.

In particular, the map $\pi_{\mathbb{P}}{\restriction}\mathrm{ro}(\mathbb{P}) : \langle \mathrm{ro}(\mathbb{P}), \subseteq\rangle \to \mathbb{B}_{\mathbb{P}}$ is an isomorphism.

In the previous sections we showed how the combinatorics of antichains is very helpful in the practice of forcing. In connection to this, it is important to know that $\mathbb{B}_{\mathbb{P}}$ can be determined by antichains as follows.

Theorem 4.11 (ZFC)**.** *For any $C \in \mathbb{B}_{\mathbb{P}}$ there is an antichain A in $\mathbb{P}$ such that $C = [A]$.*

Proof. Assume that $C = [X]$ for some $X \subseteq \mathbb{P}$. Note that $\mathrm{int}(X) := \{q \in \mathbb{P} : \exists p \in X(q \leq p)\}$. It is clear that $X =^* \mathrm{int}(X)$. By Zorn's Lemma, find a maximal A with respect to the following properties:

(i) $A \subseteq \mathrm{int}(X)$,

(ii) A is an antichain in $\mathbb{P}$.

It is not hard to see that $A =^* \mathrm{int}(X)$. □

As a consequence, we can easily estimate the size of the completion of a poset.

Corollary 4.12 (ZFC)**.** *If κ is an infinite cardinal and $\mathbb{P}$ is κ-cc, then $|\mathbb{B}_{\mathbb{P}}| \leq |\mathbb{P}|^{<\kappa}$.*

Proof. Clear because $|\mathbb{B}_{\mathbb{P}}| = |\{[A] : A \text{ antichain in } \mathbb{P}\}| \leq |[\mathbb{P}]^{<\kappa}|$. □

In the practice, it is very common to deal with the notion of $\|\varphi\|$ for a formula φ in the forcing language of a complete Boolean algebra. More precisely, if $\mathbb{B}$ is a complete Boolean algebra and φ is a formula in the forcing language of $\mathbb{B}$,[2]

$$\|\varphi\| := \bigvee\{x \in \mathbb{B} : x \Vdash_{\mathbb{B}} \varphi\}.$$

Here $\|\varphi\|$ is not just a supremum but a maximum, that is, for any $x \in \mathbb{B}$, $x \Vdash_{\mathbb{B}} \varphi$ iff $x \leq \|\varphi\|$.

Discussions about $\|\varphi\|$ are much more clear and practical when working with $\mathrm{Pw}(\mathbb{P})$ and $\mathbb{B}_{\mathbb{P}}$ instead.

Definition 4.13. Let $\varphi(\bar{\tau})$ be a formula in the forcing language of $\mathbb{P}$. Recall from Lemma 2.4 the open set $D_{\varphi(\bar{\tau})} := \{p \in \mathbb{P} : p \Vdash_{\mathbb{P}} \varphi(\bar{\tau})\}$. Define $\|\varphi(\bar{\tau})\| := [D_{\varphi(\bar{\tau})}]$ (equivalece class in $\mathbb{B}_{\mathbb{P}}$).

Lemma 4.14. *Let $p \in \mathbb{P}$, $A \in \mathrm{Pw}(\mathbb{P})$, and $\varphi(\bar{\tau})$ a formula in the forcing language of $\mathbb{P}$.*

(a) $A \leq^* D_{\varphi(\bar{\tau})}$ iff $\forall p \in A(p \Vdash \varphi(\bar{\tau}))$.

(b) $p \Vdash \varphi(\bar{\tau})$ iff $\{p\} \leq^* D_{\varphi(\bar{\tau})}$.

(c) $A \Vdash_{\mathrm{Pw}(\mathbb{P})} \varphi(\iota^*_{\mathbb{P}}(\tau_0), \ldots, \iota^*_{\mathbb{P}}(\tau_{n-1}))$ iff $A \leq^* D_{\varphi(\bar{\tau})}$.

(d) $\|\varphi(\bar{\tau})\| = \bigvee\{x \in \mathbb{B}_{\mathbb{P}} : x \Vdash_{\mathbb{B}_{\mathbb{P}}} \varphi(h^*_{\mathbb{P}}(\tau_0), \ldots, h^*_{\mathbb{P}}(\tau_{n-1}))\}$

[2] The statement $0_{\mathbb{B}} \Vdash \varphi$ is considered true, always. This is because, in connection with the forcing theorem, no $\mathbb{B} \smallsetminus \{0_{\mathbb{B}}\}$-generic filter contains $0_{\mathbb{B}}$.

Proof. For (a), $A \leq^* D_{\varphi(\bar{\tau})}$ iff $D_{\varphi(\bar{\tau})}$ is predense below p for all $p \in A$, but this is equivalent to "$p \Vdash \varphi(\bar{\tau})$ for all $p \in A$" by Lemma 2.4(e). Item (b) is immediate from (a).

For (c), since $\{\iota_{\mathbb{P}}(p) : p \in A\}$ is dense below A in $\mathrm{Pw}(\mathbb{P})$,

$$A \Vdash_{\mathrm{Pw}(\mathbb{P})} \varphi(\iota^*_{\mathbb{P}}(\tau_0), \ldots, \iota^*_{\mathbb{P}}(\tau_{n-1})) \Leftrightarrow \forall p \in A(\iota_{\mathbb{P}}(p) \Vdash_{\mathrm{Pw}(\mathbb{P})} \varphi(\iota^*_{\mathbb{P}}(\tau_0), \ldots, \iota^*_{\mathbb{P}}(\tau_{n-1}))).$$

Hence, by Lemma 4.3, this is equivalent to $\forall p \in A(p \Vdash \varphi(\bar{\tau}))$, so the result follows by (a). Item (d) is a direct consequence of (c) and Lemma 4.3. □

As an example, we show with our notation that random forcing is ${}^\omega\omega$-bounding.

Example 4.15 (ZFC)**.** Let $\mathbb{P}$ be random forcing, that is, $\mathbb{P}$ is the set of Borel subsets of $[0,1]$ with positive Lebesgue measure, ordered by $\subseteq$. We show that $\mathbb{P}$ is *${}^\omega\omega$-bounding*, that is, for any $p \in \mathbb{P}$ and any $\mathbb{P}$-name $\dot{x}$, if $p \Vdash \dot{x} : \omega \to \omega$, then there is some $r \leq p$ and $y \in {}^\omega\omega$ (in the ground model) such that $r \Vdash \forall n < \omega(\dot{x}(n) \leq \check{y}(n))$.

Denote the Lebesgue measure by Lb. First note that random forcing is ccc. Assume $\{p_\alpha < \omega_1\} \subseteq \mathbb{P}$. We can find an uncountable $S \subseteq \omega_1$ and an $n < \omega$ such that $\mathrm{Leb}(p_\alpha) \geq \frac{1}{n+1}$ for all $\alpha \in S$. If $\{p_\alpha < \omega_1\}$ were an antichain then, by taking any $F \subseteq S$ of size $n+2$ we would have

$$\mathrm{Leb}\left(\bigcup_{\alpha \in F} p_\alpha\right) = \sum_{\alpha \in F} \mathrm{Leb}(p_\alpha) \geq \frac{|F|}{n+1} > 1,$$

which contradicts that $\bigcup_{\alpha \in F} p_\alpha \subseteq [0,1]$.

Random forcing is very special when dealing with its completion. For any $X \in \mathrm{Pw}(\mathbb{P})$ there is some $p_X \in \mathbb{P}$ such that $X =^* \{p_X\}$. Indeed, by Lemma 4.11, there is some antichain $A \subseteq \mathbb{P}$ such that $A =^* X$, and A is countable because $\mathbb{P}$ is ccc. Hence $p_X := \bigcup A$ is Borel of positive measure (A is non-empty by Lemma 4.6(g)), so $p_X \in \mathbb{P}$ is as desired.

For $n, k \in \omega$, define $A_{n,k} := D_{\dot{x}(\check{n}) = \check{k}} = \{q \in \mathbb{P} : q \Vdash \dot{x}(\check{n}) = \check{k}\}$ and set $p_{n,k} := p_{A_{n,k}}$, so $A_{n,k} =^* \{p_{n,k}\}$. Also, by Lemma 4.14, for any $q \in \mathbb{P}$, $q \Vdash \dot{x}(\check{n}) = \check{k}$ iff $q \leq^* p_{n,k}$.

On the other hand, $C_n := \{p_{n,k} : k < \omega\}$ is a maximal antichain in $\mathbb{P}$, which implies that $q_n := \bigcup C_n$ has measure 1 (we left these details to the reader), so $q := \bigcap_{n<\omega} q_n$ also has measure 1.

Let $p' := p \cap q$. It is clear that $\mathrm{Lb}(p') = \mathrm{Lb}(p)$. Also, for any $n < \omega$ there is some $y(n) < \omega$ such that

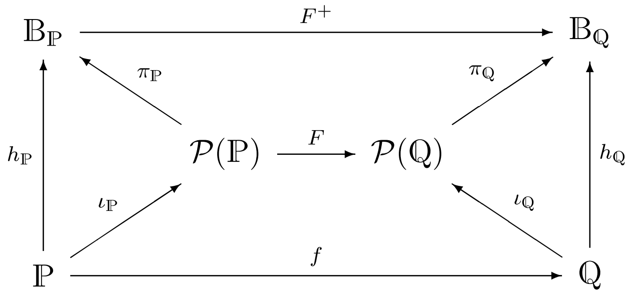
$$\mathrm{Lb}\left(p' \cap \bigcup_{k > y(n)} p_{n,k}\right) < 2^{-(n+2)} \mathrm{Lb}(p').$$

Let $r := p' \cap \bigcap_{n<\omega} \bigcup_{k \leq y(n)} p_{n,k}$. It is clear that it is Borel and $r \subseteq p'$, also

$$\mathrm{Lb}(p' \smallsetminus r) \leq \mathrm{Lb}\left(\bigcup_{n<\omega} \bigcup_{k > y(n)} (p' \cap p_{n,k})\right) \leq \frac{1}{2} \mathrm{Lb}(p')$$

so $r \in \mathbb{P}$. For $n < \omega$, since r is incompatible with $p_{n,k}$ for all $k > y(n)$, $r \leq^* \bigcup_{k \leq y(n)} p_{n,k}$, thus $r \Vdash \dot{x}(\check{n}) \leq y(n)\check{}$

To finish this section, we present some facts about liftings of complete embeddings. The proofs are left to the reader. The commutative diagram below illustrates the idea of these results.



Lemma 4.16. *Assume that $F : \mathrm{Pw}(\mathbb{P}) \to \mathrm{Pw}(\mathbb{Q})$ is a map. Stipulate $F(\emptyset) := \emptyset$. Then:*

(a) F is a complete embedding iff, for any $A, B \in \mathcal{P}(\mathbb{P})$ and $\mathcal{F} \subseteq \mathbb{P}$,

- *(i) $A \leq^* B$ iff $F(A) \leq^* F(B)$,*
- *(ii) $F\big(\bigcup \mathcal{F}\big) =^* \bigcup F[\mathcal{F}]$,*
- *(iii) $F({\sim}A) =^* {\sim}F(A)$.*

(b) If F and $G : \mathrm{Pw}(\mathbb{P}) \to \mathrm{Pw}(\mathbb{Q})$ are complete embeddings and $\forall p \in \mathbb{P}(F(\{p\}) =^* G(\{p\}))$, then $\forall A \in \mathrm{Pw}(\mathbb{P})(F(A) =^* G(A))$.

(c) F is a dense embedding iff (a)(i) holds for all $A, B \subseteq \mathbb{P}$ and $\forall Y \subseteq \mathbb{Q} \exists X \subseteq \mathbb{P}(Y =^* F(X))$.

(d) There is at most one map $F^+ : \mathbb{B}_{\mathbb{P}} \to \mathbb{B}_{\mathbb{Q}}$ satisfying $F^+ \circ \pi_{\mathbb{P}} = \pi_{\mathbb{Q}} \circ F$, and it exists iff $\forall A, B \subseteq \mathbb{P}(A =^* B \Rightarrow F(A) =^* F(B))$, in which case $F^+([A]) = [F(A)]$ for any $A \subseteq \mathbb{P}$.

(e) If F^+ can be defined then the following statements are equivalent.

- *(i) F is a complete embedding.*
- *(ii) F^+ is a complete embedding of posets.*
- *(iii) F^+ is a complete embedding of Boolean algebras, that is, for any $x, y \in \mathbb{B}_{\mathbb{P}}$ and $\mathcal{F} \subseteq \mathbb{B}_{\mathbb{P}}$:*
  - *$x \leq y$ iff $F^+(x) \leq F^+(y)$,*
  - *$F^+\big(\bigvee \mathcal{F}\big) = \bigvee F^+[\mathcal{F}]$,*
  - *$F^+({\sim}x) = {\sim}F^+(x)$.*

(f) If F^+ can be defined then the following statements are equivalent.

- *(i) F is a dense embedding.*
- *(ii) F^+ is a dense embedding.*
- *(iii) F^+ is an isomorphism.*

Theorem 4.17. *Let $f : \mathbb{P} \to \mathbb{Q}$ be a complete embedding. Define $F : \mathcal{P}(\mathbb{P}) \to \mathcal{P}(\mathbb{Q})$ by $F(A) := f[A]$. Then*

(a) $F{\restriction}\mathrm{Pw}(\mathbb{P})$ is a complete embedding into $\mathrm{Pw}(\mathbb{Q})$ and $F \circ \iota_{\mathbb{P}} = \iota_{\mathbb{Q}} \circ f$.

(b) *If f is a dense embedding the so is F.*

(c) *The function $F^+ : \mathbb{B}_\mathbb{P} \to \mathbb{B}_\mathbb{Q}$ of Theorem 4.16 can be defined.*

(d) *F^+ is the unique complete embedding $H : \mathbb{B}_\mathbb{P} \to \mathbb{B}_\mathbb{Q}$ satisfying $H \circ h_\mathbb{P} = h_\mathbb{Q} \circ f$.*

5 A comment on two step iterations

Fix a poset $\mathbb{P}$ and a $\mathbb{P}$-name of a poset $\dot{\mathbb{Q}}$. The two step iteration of $\mathbb{P}$ with $\dot{\mathbb{Q}}$ is a poset that generates the same generic extension obtained by going to a $\dot{\mathbb{Q}}[G]$-generic extension from a $\mathbb{P}$-generic extension $V[G]$. The natural idea is to define this poset as $\{(p, \dot{q}) \in \mathbb{P} \times V^\mathbb{P} : p \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$ ordered by

$$(p', \dot{q}') \leq (p, \dot{q}) \text{ iff } p' \leq p \text{ and } p' \Vdash \dot{q}' \leq \dot{q}.$$

The problem is that this collection is not a set in V. One solution, as in Kunen's book [Kun80], is to define $\mathbb{P} * \dot{\mathbb{Q}} := \{(p, \dot{q}) \in \mathbb{P} \times \mathrm{dom}(\dot{\mathbb{Q}}) : p \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$ with the same order, which is dense in the class above.

Another approach in ZFC uses combinatorics of antichains. By Lemma 3.5 and Theorem 3.6, since $\mathbb{P}$ is $\max\{\aleph_0, |\mathbb{P}|\}^+$-cc, we obtain a set of $\mathbb{P}$-names $\langle\dot{\mathbb{Q}}\rangle_\mathbb{P}$ such that

(I) $\forall \dot{q} \in \langle\dot{\mathbb{Q}}\rangle_\mathbb{P} (\Vdash \dot{q} \in \dot{\mathbb{Q}})$,

(II) whenever $p \in \mathbb{P}$, $\sigma \in V^\mathbb{P}$ and $p \Vdash \sigma \in \dot{\mathbb{Q}}$, there is some $\dot{q} \in \langle\dot{\mathbb{Q}}\rangle_\mathbb{P}$ such that $p \Vdash \sigma = \dot{q}$.

So we can define $\mathbb{P} *' \dot{\mathbb{Q}} := \mathbb{P} \times \langle\dot{\mathbb{Q}}\rangle_\mathbb{P}$ with the same order as $\mathbb{P} * \mathbb{Q}$. This is a valid approach because:

Lemma 5.1. *There is a dense embedding $f : \mathbb{P} * \dot{\mathbb{Q}} \to \mathbb{P} *' \dot{\mathbb{Q}}$.*

Proof. For $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$, by (II) we can find some $\dot{q}' \in \langle\dot{\mathbb{Q}}\rangle_\mathbb{P}$ such that $p \Vdash \dot{q} = \dot{q}'$. Set $f(p, \dot{q}) := (p, \dot{q}')$. It is easy to see that f is as required. □

The notion $\mathbb{P} * \dot{\mathbb{Q}}$ can be problematic when dealing with countable support iterations, for which $\mathbb{P} *' \dot{\mathbb{Q}}$ is more suitable. On the other hand, the disadvantage of $\mathbb{P} *' \dot{\mathbb{Q}}$ is that it could be too big: by Lemma 3.6,

Lemma 5.2. *If $\mathbb{P}$ is κ-cc (with κ infinite), μ is a cardinal and $\Vdash_\mathbb{P} |\dot{\mathbb{Q}}| \leq |\check{\mu}|$, then $|\mathbb{P} *' \dot{\mathbb{Q}}| \leq |\mathbb{P}|^{<\kappa} \cdot \mu^{<\kappa}$.*

It is possible to refine $\mathbb{P} *' \dot{\mathbb{Q}}$ even more. Define the equivalence relation on $\langle\dot{\mathbb{Q}}\rangle_\mathbb{P}$ by $\dot{q} \sim \dot{q}'$ iff $\Vdash \dot{q} = \dot{q}'$. So we can define $\lfloor\dot{\mathbb{Q}}\rfloor$ as a selector of all the equivalence classes, and $\mathbb{P} *' \dot{\mathbb{Q}}$ can be restricted to $\mathbb{P} \times \lfloor\dot{\mathbb{Q}}\rfloor$, which is dense in $\mathbb{P} *' \dot{\mathbb{Q}}$.

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