Fusions in fiber-commutative coherent configurations

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1 Introduction

Coherent configurations are a generalization of association schemes. Association schemes and coherent configurations are defined as sets of $\{0, 1\}$-matrices, called adjacency matrices, satisfying some conditions (see Definition 2.1 and [2]). Adjacency algebras of association schemes and coherent configurations are defined as algebras spanned by adjacency matrices over the complex field $\mathbb{C}$.

Commutative association schemes are association schemes satisfying that all adjacency matrices are commutative each other. Commutative association schemes have primitive idempotents and the sets of primitive idempotents are bases of adjacency algebras.

For commutative association schemes, fusion schemes are considered in [1] and [9] and these papers revealed the equivalent condition for commutative association schemes to have fusion schemes, called the Bannai-Muzychuk criterion. In the proof of this equivalence, partitions for both sets of adjacency matrices and primitive idempotents are given.

On the other hand, the sets of primitive idempotents for non-commutative association schemes and coherent configurations are not bases of adjacency algebras of them. In stead of the sets of primitive idempotents for commutative association schemes, we may consider decompositions of adjacency algebras of non-commutative association schemes and coherent configurations given by the Wedderburn theorem. The wedderburn theorem is a theorem in the representation theory of algebras and gives bases for adjacency algebras of them (see Definition 3.1 and [8]). For non-commutative association schemes, [8] revealed the sufficient condition for non-commutative association schemes to have fusion schemes.

In some sense, fiber-commutative coherent configurations (see Definition 2.2) can be regarded as a generalization of commutative association schemes. In this paper, we reveal an equivalent condition for fiber-commutative coherent configurations to have fusion configurations. To describe the equivalence, we use bases of adjacency algebras of fiber-commutative coherent configurations, called bases of matrix units (see Definition 3.1). Since the specialization for commutative association schemes of this equivalence is same as the Bannai-Muzychuk criterion (see Corollary 5.1), this equivalence is a natural generalization of the Bannai-Muzychuk criterion. Note that, bases of matrix units for adjacency algebras of coherent configurations are defined in [7] as a specialization of [4], [5]. In particular, for fiber-commutative coherent configurations, the concept of bases of matrix units for them are described in [6].

2 Coherent configurations

Let X be a finite set and $R_i \subset X \times X$ be binary relations for $i = 0, 1, \ldots, d$. For R_i, adjacency matrix A_i with respect to R_i is defined as $(A_i)_{x,y} = 1$ if $(x, y) \in R_i$ and 0 otherwise.

Let $I, J \in \mathrm{M}_X(\mathbb{C})$ be the identity matrix and the all-ones matrix, respectively.

Definition 2.1. For a finite set X, let $R_0, R_1, \ldots, R_d \subset X \times X$ be binary relations of $X \times X$ and $A_0, A_1, \ldots, A_d$ be the adjacency matrices. A *coherent configuration* $(X, \{R_i\}_{i=0}^d)$ is defined as

(i) there exists a subset $K \subset \{0, 1, \ldots, d\}$ such that $\displaystyle\sum_{i \in K} A_i = I$,

(ii) $\displaystyle\sum_{i=0}^{d} A_i = J$,

(iii) for any $i \in \{0, 1, \ldots, d\}$, there exists $i' \in \{0, 1, \ldots, d\}$ such that $A_{i'} = A_i^T$,

(iv) $A_i A_j = \sum_{i=0}^{d} p_{i,j}^k A_k$,

The algebra spanned by $A_0, A_1, \ldots, A_d$ over $\mathbb{C}$ is called the *adjacency algebra.* Coherent configurations with $|K| = 1$ are called *homogeneous coherent configurations* or *association schemes.*

For a coherent configuration $\mathcal{C} = (X, \{R_k\}_{k=0}^d)$, since the index set $\{0, 1, \ldots, d\}$ is not suitable for this paper, we rearrange it into an index set of triples as follows: By the Definition 2.1(i), $I \in \mathrm{M}_n(\mathbb{C})$ is decomposed into $\{0, 1\}$-matrices. This implies that X is decomposed into $X = \coprod_{i \in F} X_i$ for the set $F = \{1, 2, \ldots, |K|\}$. By the Definition 2.1(iv), for any $k \in \{0, 1, \ldots, d\}$, there exist $i, j \in F$ such that $R_k \subset X_i \times X_j$. For any $i, j \in F$, we denote $r_{i,j} = \#\{k \in \{0, 1, \ldots, d\} \mid R_k \subset X_i \times X_j\}$. Thus the index set $\{0, 1, \ldots, d\}$ can be replaced by $\{(i, j, a) \mid i, j \in F, a \in \{1, 2, \ldots, r_{i,j}\}\}$ and $\{R_k\}_{k=0}^d = \{R_{i,j,a} \mid i, j \in F, a \in \{1, 2, \ldots, r_{i,j}\}\}$.

Note that, the each X_i is called a *fiber.* In the rest of this paper, we always use $\{(i, j, a) \mid i, j \in F, a \in \{1, 2, \ldots, r_{i,j}\}\}$ as the index set of adjacency matrices and relations instead of $\{0, 1, \ldots, d\}$.

For brevity, we always assume that $A_{i,i,1} = I_{X_i}$ for all $i \in F$ and $A_{j,i,a} = A_{i,j,a}^T$ for all $i, j \in F$ $(i \neq j)$, where I_{X_i} is the diagonal matrix with $(I_{X_i})_{x,x} = 1$ if $x \in X_i$ and 0 otherwise.

Let $\mathfrak{A}$ be the adjacency algebra of $\mathcal{C}$. Since $\mathfrak{A}$ is a subalgebra of $\mathrm{M}_X(\mathbb{C})$, $\mathfrak{A}$ is decomposed into a direct sum of subspaces: $\mathfrak{A} = \bigoplus_{i,j \in F} \mathfrak{A}_{i,j}$, where $\mathfrak{A}_{i,j}$ are subspaces of $\mathfrak{A}$ spanned by $\{A_{i,j,a} \mid a \in \{1, 2, \ldots, r_{i,j}\}\}$ for $i, j \in F$. It is clear that $\{A_{i,j,a} \mid a \in \{1, 2, \ldots, r_{i,j}\}\}$ is a basis of $\mathfrak{A}_{i,j}$.

Definition 2.2. Let $\mathcal{C} = (X, \{R_{i,j,a}\}_{i,j,a})$ be a coherent configuration with fibers $X = \coprod_{i \in F} X_i$. For each $i \in F$, the coherent configuration $\mathcal{C}$ is *fiber-commutative* if $\mathfrak{A}_{i,i}$ is commutative for all $i \in F$. Similarly, $\mathcal{C}$ is *fiber-symmetric* if $\mathfrak{A}_{i,i}$ is symmetric for all $i \in F$. Fiber-commutative coherent configurations with $|F| = 1$ are called *commutative association schemes.*

3 Fiber-commutative coherent configurations

In the rest of this paper, we assume that all coherent configurations are fiber-commutative.

Let $\mathfrak{A}$ be the adjacency algebra of a fiber-commutative coherent configuration $\mathcal{C}$. Let $\Phi = \{\phi_s \mid s \in S\}$ be a set of representatives for all irreducible matrix representations of $\mathfrak{A}$ over $\mathbb{C}$ satisfying $\phi_s(A)^* = \phi_s(A^*)$ for any $A \in \mathfrak{A}$, where $*$ denotes the transpose-conjugate. Since $\mathfrak{A}$ is semisimple, $\mathfrak{A}$ is decomposed into

$$\mathfrak{A} = \bigoplus_{s \in S} \mathfrak{C}_s,$$

where $\mathfrak{C}_s$ is a simple two-sided ideal affording ϕ_s. Moreover, for each $s \in S$, $\mathfrak{C}_s$ is isomorphic to $\mathrm{M}_{e_s}(\mathbb{C})$, where $\mathrm{M}_{e_s}(\mathbb{C})$ is the $e_s \times e_s$ full-matrix algebra over $\mathbb{C}$.

Since $\mathcal{C}$ is fiber-commutative, for any $s \in S, i \in F_s$, $\dim(\mathfrak{A}_{i,i} \cap \mathfrak{C}_s) \leq 1$ holds (see [7, Lemma 2.9]). This assertion is also proved in the view of the representation theory by [3, Proposition 2.1]. Thus, for each $s \in S$, let

$$F_s = \{i \in F \mid \dim(\mathfrak{C}_s \cap \mathfrak{A}_{i,i}) = 1\}.$$

Then we may construct following bases (see [7, Definition 2.3]). In addition, it is clear that $e_s = |F_s|$ holds for each $s \in S$.

Definition 3.1. Let $\mathfrak{A}$ be the adjacency algebra of a fiber-commutative coherent configuration. *Bases of matrix units for* $\mathfrak{A}$ are defined as matrices $\{\varepsilon^s_{i,j} \mid s \in S, i, j \in F_s\}$ satisfying

(i) for any $s, t \in S$, $i, j \in F_s, k, l \in F_t$, $\varepsilon^s_{i,j}\varepsilon^t_{k,l} = \delta_{s,t}\delta_{j,k}\varepsilon^s_{i,l}$,

(ii) for any $s \in S$, $i, j \in F_s$, ${\varepsilon^s_{i,j}}^* = \varepsilon^s_{j,i}$,

(iii) for any $s \in S$, $i, j \in F_s, \varepsilon^s_{i,j} \in \mathfrak{A}_{i,j}$,

(iv) for any $s \in S$, there exist $\hat{s} \in S$ such that, $F_{\hat{s}} = F_s$ and, for any $i, j \in F_s$, $\overline{\varepsilon^s_{i,j}} = \varepsilon^{\hat{s}}_{i,j}$.

Note that $\{\varepsilon^s_{i,j} \mid i, j \in F_s\}$ is a basis of $\mathfrak{C}_s$ and $\{\varepsilon^s_{i,j} \mid s \in S, i, j \in F_s\}$ is a basis of $\mathfrak{A}$.

Let

$$S_{i,j} = \{s \in S \mid \dim(\mathfrak{A}_{i,j} \cap \mathfrak{C}_s) = 1\}$$

for each $i, j \in F$.

Proposition 3.2. *For* $i, j \in F$, $S_{i,j} = S_{i,i} \cap S_{j,j}$.

Proof. By $\varepsilon_{i,j}^s \in \mathfrak{A}_{i,j}$ if and only if $\varepsilon_{i,i}^s \in \mathfrak{A}_{i,i}$ and $\varepsilon_{j,j}^s \in \mathfrak{A}_{j,j}$ for $s \in S$, the result follows. $\square$

Let

$$\Lambda = \coprod_{s \in S} F_s \times F_s \times \{s\} = \coprod_{i,j \in F} \{i\} \times \{j\} \times S_{i,j} \tag{1}$$

be the set of triple indices of bases of matrix units for $\mathfrak{A}$. Since Λ is decomposed into (1), for any $i, j \in F$, $\{\varepsilon_{i,j}^s \mid s \in S_{i,j}\}$ is a basis of the subspace $\mathfrak{A}_{i,j}$.

Thus, for $i, j \in F$, $\mathfrak{A}_{i,j}$ has $\{A_{i,j,a} \mid a \in \{1, 2, \ldots, r_{i,j}\}\}$ and $\{\varepsilon_{i,j}^s \mid s \in S_{i,j}\}$ as bases and it means that these matrices are expressed as

$$A_{i,j,a} = \sum_{s \in S_{i,j}} p_{i,j,a}(s) \varepsilon_{i,j}^s,$$

$$\varepsilon_{i,j}^s = \frac{1}{\sqrt{|X_i||X_j|}} \sum_{a=1}^{r_{i,j}} q_{i,j,s}(a) A_{i,j,a}$$

for $p_{i,j,a}(s), q_{i,j,a}(s) \in \mathbb{C}$.

Definition 3.3. For a fiber-commutative coherent configuration $\mathcal{C}$, the *first and second eigenmatrices* are defined as $P = (P_{i,j})_{i,j \in F}, Q = (Q_{i,j})_{i,j \in F}$, respectively, where $(P_{i,j})_{s,a} = (p_{i,j,a}(s))$ and $(Q_{i,j})_{a,s} = q_{i,j,s}(a)$.

Note that P, Q have matrices in their entries. By this definition, $P_{i,j}, Q_{i,j} \in \mathrm{M}_{r_{i,j}}(\mathbb{C})$ satisfy $P_{i,j} Q_{i,j} = \sqrt{|X_i||X_j|} I_{r_{i,j}}$.

4 Fusions in fiber-commutative coherent configurations

Let $\mathcal{C} = (X, \{R_{i,j,a}\}_{i,j,a})$ be a fiber-commutative coherent configuration with fibers $\coprod_{i \in F} X_i$, $\mathfrak{A} = \langle A_{i,j,a} \mid i, j \in F, a \in \{1, 2, \ldots, r_{i,j}\}\rangle_{\mathbb{C}}$ be the adjacency algebra of $\mathcal{C}$ and $\{\varepsilon_{i,j}^s \mid (i, j, s) \in \Lambda\}$ be the bases of matrix units for the adjacency algebra $\mathfrak{A}$, where $\Lambda = \coprod_{i,j \in F} \{i\} \times \{j\} \times S_{i,j}$. Moreover, let $P = (P_{i,j})$ be the first eigenmatrix of $\mathcal{C}$.

In this section, we give an equivalent condition for a subalgebra of $\mathfrak{A}$ to be the adjacency algebra of a coherent configuration with the same fibers as those of $\mathcal{C}$.

Definition 4.1. A coherent configuration $\mathcal{C}' = (X, \{R'_{i,j,a}\}_{i,j,a})$ is a *fusion configuration with the same fibers as those of* $\mathcal{C}$ if $\mathcal{C}$ and $\mathcal{C}'$ have the same fibers and the adjacency algebra $\mathfrak{A}'$ of $\mathcal{C}'$ is a subalgebra of $\mathfrak{A}$.

Note that, since $\mathcal{C}$ is fiber-commutative, each fusion configuration with the same fibers as those of $\mathcal{C}$ is fiber-commutative.

Definition 4.2. A family of partitions $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ is called *admissible* if

(i) for any $i, j \in F$, $\coprod_{\delta \in \Delta_{i,j}} \delta = \{1, 2, \ldots, r_{i,j}\}$,

(ii) for any $i \in F, \{1\} \in \Delta_{i,i}$,

(iii) for any $\delta \in \Delta_{i,j}$, $\{a \in \{1, 2, \ldots, r_{j,i}\} \mid A^T_{j,i,a} = A_{i,j,b}$ for some $b \in \delta\} \in \Delta_{j,i}$.

Lemma 4.3. *If* $\mathcal{C}'$ *is a fusion configuration with the same fibers as those of* $\mathcal{C}$*, then there exists a uniquely determined admissible family of partitions* $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ *such that* $\mathcal{C}' = (X, \{R'_{i,j,\delta}\}_{i,j,\delta})$*, where* $R'_{i,j,\delta} = \coprod_{a \in \delta} R_{i,j,a}$ *for* $\delta \in \Delta_{i,j}$*. Conversely, if* $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ *is an admissible family of partitions, then the set* $\{A'_{i,j,\delta} \mid i, j \in F, \delta \in \Delta_{i,j}\}$ *satisfies conditions of* Definition 2.1 (i), (ii), (iii)*, where* $A'_{i,j,\delta} = \sum_{a \in \delta} A_{i,j,a}$*.*

Proof. Let $\{A'_{i,j,b} \mid i, j \in F, b \in \{1, 2, \ldots, r'_{i,j}\}\}$ be the set of adjacency matrices of $\mathcal{C}'$ and $\mathfrak{A}'$ be the adjacency algebra of $\mathcal{C}'$. Then $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$ and it implies that, for any $i, j \in F$ and $b \in \{1, 2, \ldots, r'_{i,j}\}$, there exists a subset $\delta_{i,j,b} \subset \{1, 2, \ldots, r_{i,j}\}$ such that

$$A'_{i,j,b} = \sum_{a \in \delta_{i,j,b}} A_{i,j,a}.$$

By Definition 2.1 (i), (ii), (iii), for all $i, j \in F$, $\Delta_{i,j} = \{\delta_{i,j,b} \mid b \in \{1, 2, \ldots, r'_{i,j}\}\}$ satisfy Definition 4.2 (i), (ii), (iii) and $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ is admissible.

The converse is clear by Definition 4.2. □

The following theorem essentially reveals the condition in Definition 2.1 (iv).

Theorem 4.4. *Let* $G = (V, E)$ *be a bipartite graph with* $V = F \cup S$ *and edge set* $E = \{(i, s) \mid (i, i, s) \in \Lambda\}$*. Then* $\mathcal{C}' = (X, \{R'_{i,j,\delta}\}_{i,j,\delta})$ *is a fusion configuration with the same fibers as those of* $\mathcal{C}$*, where* $R'_{i,j,\delta} = \coprod_{a \in \delta} R_{i,j,a}$ *for* $\delta \in \Delta_{i,j}$*, if and only if* $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ *is admissible and there exist*

(I) *diagonal matrices* $C_{i,j}$ *indexed by* $S_{i,j} \times S_{i,j}$ *with* $(C_{i,j})_{s,s} = c^s_{i,j}$ *for* $i, j \in F$,

(II) *an index set* S' *and subsets* $T_\sigma \subset S, F_\sigma \subset F$ *for* $\sigma \in S'$ *which gives a partition* $\{F_\sigma \times T_\sigma \mid \sigma \in S'\}$ *of* E *into complete bipartite edge-subgraphs;* $E = \coprod_{\sigma \in S'} F_\sigma \times T_\sigma$,

such that, for any $i, j \in F$,

(i) $|\Delta_{i,j}| = |S'_{i,j}|$,

(ii) *for any* $s \in \coprod_{\sigma \in S'_{i,j}} T_\sigma, c^s_{i,i} = 1, |c^s_{i,j}| = 1, c^s_{i,j} = \overline{c^s_{j,i}} = \overline{c^{\hat{s}}_{i,j}}$,

(iii) *for any* $\sigma \in S'_{i,j}, \delta \in \Delta_{i,j}$, *row sums of the submatrix of* $\overline{C_{i,j}} P_{i,j}$ *indexed by* $T_\sigma \times \delta$ *is a constant* $p'_{i,j,\delta}(\sigma)$ *and row sums indexed by* $O_{i,j} \times \delta$ *is* 0,

where $S'_{i,j} = \{\sigma \in S' \mid F_\sigma \ni i, j\}$ *and* $O_{i,j} = S_{i,j} \backslash \left(\coprod_{\sigma \in S'_{i,j}} T_\sigma \right)$. *Moreover, if* $\mathcal{C}'$ *is a fusion configuration with the same fibers as those of* $\mathcal{C}$, *then the first eigenmatrix* $P' = (P'_{i,j})$ *of* $\mathcal{C}'$ *with respect to bases of matrix units* $\{{\varepsilon'}^\sigma_{i,j} \mid \sigma \in S', i, j \in F_\sigma\}$ *is given by* $(P'_{i,j})_{\sigma,\delta} = p'_{i,j,\delta}(\sigma)$ *for* $\sigma \in S'_{i,j}, \delta \in \Delta_{i,j}$, *where*

$$ {\varepsilon'}^\sigma_{i,j} = \sum_{s \in T_\sigma} c^s_{i,j} \varepsilon^s_{i,j} \tag{2} $$

for $\sigma \in S', i, j \in F_\sigma$.

5 Applications

In this section, we apply Theorem 4.4 to commutative association schemes, fiber-commutative coherent configurations and the fiber-commutative coherent configuration given by $\mathbb{Z}_3^4 \rtimes S_6$.

5.1 Commutative association schemes

Commutative association schemes are defined as fiber-commutative coherent configurations with $|F| = 1$.

We assume $F = \{1\}$. For brevity, we omit indices given by F. Let $\mathfrak{X} = (X, \{R_a\}_{a=1}^r)$ be a commutative association scheme and $\{\varepsilon^s \mid s \in S\}$ be its primitive idempotents. Since $\mathfrak{X}$ has only one fiber, any fusion configuration with the same fiber as those of $\mathfrak{X}$ has also a fiber X and is called *a fusion schemes.* Then an admissible partition Δ for $\mathfrak{X}$ satisfies

(i) $\{1\} \in \Delta$,

(ii) for any $\delta \in \Delta, \{a \mid A_a^T = A_b \text{ for some } b \in \delta\} \in \Delta$.

Let S' be an index set with $|S'| = |\Delta|$ and, for $\sigma \in S'$, let $T_\sigma \subset S$. Note that, by $|F| = 1$, $F = F_\sigma$ hold for all $\sigma \in S'$. Then the bipartite graph $G = (V, E)$ defined by $\mathfrak{X}$ has vertex set $V = F \cup S = \{1\} \cup S$ and edge set $E = \{(1, s) \mid s \in S\}$. Thus the partition $\{\{1\} \times T_\sigma \mid \sigma \in S'\}$ of E can be identified with the partition $\{T_\sigma \mid \sigma \in S'\}$ of S. Moreover the first eigenmatrix P is decomposed into submatrices indexed by $T_\sigma \times \delta$ for $\sigma \in S', \delta \in \Delta$. In this case, it is clear that $c^s = 1$ for all $s \in S$ and $O = \emptyset$.

Thus Theorem 4.4 for commutative association schemes is specialized as follows.

Corollary 5.1 (Bannai-Muzychuk criterion, [1, Lemma 1] and [9]). *Let $\mathfrak{X} = (X, \{R_i\}_{i=1}^r)$ be a commutative association schemes, $\{\varepsilon^s \mid s \in S\}$ be its primitive idempotents and P be its first eigenmatix. Then $\mathfrak{X}$ has a fusion scheme $\mathfrak{X} = (X, \{R'_\delta\}_{\delta\in\Delta})$ given by a partition Δ, where $R' = \coprod_{a\in\delta} R_a$ for $\delta \in \Delta$, if and only if Δ is admissible and there exists a partition $\{T_\sigma \mid \sigma \in S'\}$ of S such that $|S'| = |\Delta|$ and for any $\sigma \in S', \delta \in \Delta$, row sums of the submatrix indexed by $T_\sigma \times \delta$ of P is constant.*

5.2 Trivial fusion configurations with the same fibers

Any fiber-commutative coherent configuration has a trivial fusion configuration with the same fibers. Let $\mathcal{C} = (X, \{R_{i,j,a}\}_{i,j,a})$ be a fiber-commutative coherent configuration and $\mathfrak{A} = \bigoplus_{i,j\in F} \mathfrak{A}_{i,j}$ be its adjacency algebra. Define an admissible partition $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ as follows:

(i) for $i \in F, \Delta_{i,i} = \{\{1\}, \{2\}, \dots, \{r_{i,i}\}\}$,

(ii) for $i, j \in F$ $(i \neq j), \Delta_{i,j} = \{\{1, 2, \dots, r_{i,j}\}\}$.

By the definition of Δ, it is trivial that Δ gives a subalgebra $\mathfrak{A}' = \bigoplus_{i,j\in F} \mathfrak{A}'_{i,j}$ such that $\mathfrak{A}'_{i,i} = \mathfrak{A}_{i,i}$ and $\mathfrak{A}'_{i,j} = \langle \sum_{a=1}^{r_{i,j}} A_{i,j,a} \rangle_{\mathbb{C}}$ for $i \neq j$ and $\mathfrak{A}'$ is the adjacency algebra of a certain fusion configuration $\mathcal{C}'$ with the same fibers as those of $\mathcal{C}$. By Theorem 4.4, the edge set E of the bipartite graph G is decomposed into

$$E = (F \times \{s_0\}) \sqcup \coprod_{\substack{(i,s)\in E \\ s\neq s_0}} \{(i,s)\}$$

where $\mathfrak{C}_{s_0} = \langle \sum_{a=1}^{r_{i,j}} A_{i,j,a} \mid i,j \in F\rangle_{\mathbb{C}}$ is the simple two-sided ideal of $\mathfrak{A}$ corresponding to $s_0 \in S$. In other words, Both $\mathfrak{A}$ and $\mathfrak{A}'$ have $\mathfrak{C}_{s_0}$ as a simple two-sided ideal. In this case, for any $i,j \in F$ $(i \neq j)$, $|S'_{i,i}| = |S_{i,i}|$ and $O_{i,j} = S_{i,j}\backslash\{s_0\}$ hold.

5.3 The fiber-commutative coherent configuration given by $\mathbb{Z}_3^4 \rtimes S_6$

There is a unique primitive permutation group $\mathcal{G}$ of degree 81 of the form $\mathcal{G} \simeq \mathbb{Z}_3^4 \rtimes S_6$, where $\mathbb{Z}_3$ is the cyclic group and S_6 is the symmetric group on 6 letters.

Since $\mathcal{G}$ has nontrivial outer automorphisms, we fix an outer automorphism x. Let $\mathcal{G}' = \{(g, g^x) \mid g \in \mathcal{G}\}$ be a permutation group of degree 162. Then the set of all orbits of $\mathcal{G}$ gives a fiber-commutative coherent configuration $\mathcal{C} = (X, \{R_{i,j,a}\}_{i,j,a})$ with $F = \{1,2\}$ and $r_{1,1} = r_{2,2} = 4, r_{1,2} = r_{2,1} = 3$. Moreover the adjacency algebra $\mathfrak{A}$ can be decomposed into

$$\mathfrak{A} = \bigoplus_{i=0}^{4} \mathfrak{C}_{s_i},$$

where $\mathfrak{C}_{s_i} \simeq \mathrm{M}_2(\mathbb{C})$ for $i = 0,1,2$ and $\mathfrak{C}_{s_i} \simeq \mathbb{C}$ for $i = 3,4$. By this decomposition, we may write $F_{s_0} = F_{s_1} = F_{s_2} = \{1,2\}, F_{s_3} = \{1\}, F_{s_4} = \{2\}$. The first eigenmatrix $P = (P_{i,j})$ can be written as

$$P_{1,1} = \begin{array}{c} \\ s_0 \\ s_1 \\ s_2 \\ s_3 \end{array} \overset{\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}}{\begin{pmatrix} 1 & 30 & 20 & 30 \\ 1 & -6 & 2 & 3 \\ 1 & 3 & -7 & 3 \\ 1 & 3 & 2 & -6 \end{pmatrix}}, \quad P_{2,2} = \begin{array}{c} \\ s_0 \\ s_1 \\ s_2 \\ s_4 \end{array} \overset{\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}}{\begin{pmatrix} 1 & 30 & 20 & 30 \\ 1 & -6 & 2 & 3 \\ 1 & 3 & -7 & 3 \\ 1 & 3 & 2 & -6 \end{pmatrix}}$$

$$P_{1,2} = P_{2,1} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} s_0 \\ s_1 \\ s_2 \end{matrix} & \begin{pmatrix} 15 & 60 & 6 \\ 3 & -6 & 3 \\ -6 & 3 & 3 \end{pmatrix} \end{matrix}.$$

Note that $A_{i,i,a}$ are symmetric for all $i \in \{1,2\}, a \in \{1,2,3,4\}$ and $A_{1,2,a}^T = A_{2,1,a}$ hold for all $a \in \{1,2,3\}$.

For $\mathcal{C}$, let $S' = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ and, we define $\Delta = \{\Delta_{i,j} \mid i,j \in \{1,2\}\}$ and T_σ, F_σ for $\sigma \in S'$ as follows:

$$\begin{aligned} \Delta_{1,1} &= \Delta_{2,2} = \{\{1\},\{2,3\},\{4\}\}, \\ \Delta_{1,2} &= \Delta_{2,1} = \{\{1,2\},\{3\}\}, \\ T_{\sigma_0} &= \{s_0\}, F_{\sigma_0} = \{1,2\}, \\ T_{\sigma_1} &= \{s_1, s_2\}, F_{\sigma_1} = \{1,2\}, \\ T_{\sigma_2} &= \{s_3\}, F_{\sigma_2} = \{1\}, \\ T_{\sigma_3} &= \{s_4\}, F_{\sigma_3} = \{2\}. \end{aligned}$$

Then $\Delta, T_\sigma, F_\sigma$ satisfy the conditions in Theorem 4.4 with $c_{i,j}^s = 1$ for all $i,j \in \{1,2\}, s \in S_{i,j}$. In this case, $O_{1,2} = O_{2,1}$ are empty. Thus a fusion configuration with the same fibers as those of $\mathcal{C}$ is obtained and its eigenmatrix $P' = (P'_{i,j})$ is

$$P'_{1,1} = \begin{matrix} & \begin{matrix} \{1\} & \{2,3\} & \{4\} \end{matrix} \\ \begin{matrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \end{matrix} & \begin{pmatrix} 1 & 50 & 30 \\ 1 & -4 & 3 \\ 1 & 5 & -6 \end{pmatrix} \end{matrix}, P'_{2,2} = \begin{matrix} & \begin{matrix} \{1\} & \{2,3\} & \{4\} \end{matrix} \\ \begin{matrix} \sigma_0 \\ \sigma_1 \\ \sigma_3 \end{matrix} & \begin{pmatrix} 1 & 50 & 30 \\ 1 & -4 & 3 \\ 1 & 5 & -6 \end{pmatrix} \end{matrix}$$

$$P'_{1,2} = P'_{2,1} = \begin{matrix} & \begin{matrix} \{1,2\} & \{3\} \end{matrix} \\ \begin{matrix} \sigma_0 \\ \sigma_1 \end{matrix} & \begin{pmatrix} 75 & 6 \\ -3 & 3 \end{pmatrix} \end{matrix}.$$

Moreover, $\mathfrak{A}$ has the following subalgebra $\mathfrak{A}''$. Since this subalgebra is not closed with respect to the transpose, this subalgebra is not an adjacency algebra of any fusion configuration. For $i = 1,2$, we construct subalgebras

$\mathfrak{A}''_{i,i} \subset \mathfrak{A}_{i,i}$ which is the same as above; i.e.

$$\begin{aligned}\mathfrak{A}''_{i,i} &= \langle A_{i,i,1}, A_{i,i,2} + A_{i,i,3}, A_{i,i,4} \rangle_{\mathbb{C}} \\ &= \langle \varepsilon^{s_0}_{i,i}, \varepsilon^{s_1}_{i,i} + \varepsilon^{s_2}_{i,i}, \varepsilon^{s_\alpha}_{i,i} \rangle_{\mathbb{C}},\end{aligned}$$

where $\alpha = 3$ if $i = 1$ and $\alpha = 4$ if $i = 2$. Thus the transition matrices with respect to these bases are $P''_{i,i} = P'_{i,i}$ for $i = 1, 2$. On the other hand, in $\mathfrak{A}_{1,2}, \mathfrak{A}_{2,1}$, we construct subspaces $\mathfrak{A}''_{1,2}, \mathfrak{A}''_{2,1}$ as follows;

$$\begin{aligned}\mathfrak{A}''_{1,2} &= \langle A_{1,2,1} + A_{1,2,3}, A_{1,2,2} \rangle_{\mathbb{C}} \\ &= \langle \varepsilon^{s_0}_{1,2}, \frac{1}{\sqrt{2}}(\varepsilon^{s_1}_{1,1} - 2\varepsilon^{s_2}_{1,2}) \rangle_{\mathbb{C}}, \\ \mathfrak{A}''_{2,1} &= \langle A_{2,1,1}, A_{2,1,2} + A_{2,1,3} \rangle_{\mathbb{C}} \\ &= \langle \varepsilon^{s_0}_{2,1}, \frac{1}{\sqrt{2}}(2\varepsilon^{s_1}_{1,1} - \varepsilon^{s_2}_{1,2}) \rangle_{\mathbb{C}}.\end{aligned}$$

Note that the transition matrices with respect to these bases are

$$P''_{1,2} = \begin{array}{c} \\ \sigma_0 \\ \sigma_1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \{1,3\} & \{2\} \end{array} \\ \begin{pmatrix} 21 & 60 \\ 6\sqrt{2} & -6\sqrt{2} \end{pmatrix} \end{array}, \quad P''_{2,1} = \begin{array}{c} \\ \sigma_0 \\ \sigma_1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \{1\} & \{2,3\} \end{array} \\ \begin{pmatrix} 15 & 66 \\ 6\sqrt{2} & -6\sqrt{2} \end{pmatrix} \end{array},$$

where σ_0, σ_1 correspond to $\{s_0\}, \{s_1, s_2\}$, respectively. Then

$$\mathfrak{A}'' = \bigoplus_{i,j=1}^{2} \mathfrak{A}''_{i,j}$$

is closed with respect to the matrix multiplication and this is a subalgebra of $\mathfrak{A}$ which is not closed with respect to the transpose.

This example shows that adjacency algebras may have subalgebras which are closed with respect to the Hadamard product and not adjacency algebras.

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