ON LOCAL MOONSHINE

LIZHONG WANG

Abstract. In this paper, we will introduce Rouquier's problems and general formulation of local moonshine conjecture. Some answers of Rouquier's problems are also included.

2000 Mathematics Subject Classification: 20C20,17B69

Keywords:local moonshine conjecture, Green correspondent, Brauer map

1. Introduction

In [FLM], an infinite-dimension graded module $V = \oplus_{n\in\mathbb{Z}} V_n$ is constructed for the monster group $\mathbb{M}$. Borcherds proved the following theorem in [B1].

Theorem 1.1. *(Borcherds 1992) The McKay-Thompson series*

$$T_g(\tau) = \sum_{n\in\mathbb{Z}} Tr(g|V_n)q^n$$

is a hauptmodul for some genus-0 congruence subgroup of $SL_2(\mathbb{R})$. In particular, the dimension of V_n is the coefficient of q^n of the elliptic modular function $j(\tau) - 744 = q - l + 196884q + ...$

The existence of such infinite-dimension graded module with properties in this theorem for the monster was known as the **Moonshine Conjecture** posed by Conway, Norton, McKay and Thompson.

Let G be a finite group with W as a G-module over some field k. We denote by W^g the subspace of fixed points of $g \in G$ in W. Let C be a conjugate class of G.

A function $f : C \longrightarrow W$ is called axis map if $f(c^g) = f(c)g$ for $c \in C$ and $g \in G$. If $c \in C$, then $f(c)$ is called the axis of c. There is a non-trivial axis map from C to W if and only if W contains a non-zero vectors fixed by the centralizer of an element of C. In [RY], Ryba gave the following suggestions: there should exists an integral form L of the moonshine module containing axis of transpositions and if we define

$$L(g) = \frac{L^g/pL^g}{L^g/pL^g \cap (L^g/pL^g)^{\perp}},$$

then $Tr(h|L_n(g) = Tr(gh|V_n)$ for a p-regular element h of $C_M(g)$, where the trace on the left is the Brauer trace. In particular, the dimensions of the homogeneous components of $L(g)$ should be the coefficients of certain hauptmoduls. This suggestion is called the **modular moonshine conjecture**.

Let $\hat{G}$ denote $\sum_{g\in G} g$. The map $N : W \to W$ is defined by $N(x) = x\hat{G}$. Then $\hat{H}^0(G, W) = W^G/N(W)$, where W^G is subspace of W consisting of fixed points of G. It was shown in [BR] that the definition of $L(g)$ is equivalent to the Tate cohomology $\hat{H}^0(g, L)$. We refer to [RY],[BR] and [B2] for details.

Supported by NSFC (11631001, 11871083).

The map N is a special case of trace map and the quotient $W^G/N(W)$ is also a special case of Brauer morphism for p-permutation modules. Based on this, Rouquier put forward his problems. These problems and some answers will be presented in the next section. Since Brauer morphism is closely related to Green correspondence, we will give a general formulation for modular moonshine in the last section.

2. Rouquier's problems

Let $\mathcal{O}$ be a commutative complete discrete valuation ring with $\mathcal{P}$ as maximal ideal. Let F be the residue field of $\mathcal{O}/\mathcal{P}$ with characteristic p.

Let W be $\mathcal{O}$-free $\mathcal{O}G$-module. Then W is called p-permutation module if, whenever P is a p-subgroup of G, there is an $\mathcal{O}$-basis of W which is stabilized by P(see [B1]). Similarly, we can define p-permutation module for FG-module. A module W is p-permutation module if and only if it is a direct summand of a permutation module.

Let H be a subgroup of G. Let W^H be the subspace of fixed points of H in W. Let K be a subgroup of H. Then the trace map Tr_K^H is a linear map from W^K to W^H defined by

$$Tr_K^H(x) = \sum_{g\in H/K} xg.$$

We denote by W_K^H the image $Tr_K^H(W^K)$ in W^H. In the case $K = 1$ and $G = H$, $Tr_K^H(x) = N(x)$, where N is defined as in section 1.

Let P be a p-subgroup of G. We assume that W is a p-permutation module of G over $\mathcal{O}$ or F. The normalizer of P in G is denoted by $N_G(P)$. The $\overline{N}_G(P) = N_G(P)/P$-module $W(P)$ is defined by

$$W(P) = W^P/(\sum_Q W_Q^P + \mathcal{P}W^P),$$

where Q runs over proper subgroups of P. The Brauer morphism Br_P^W is defined to be the natural surjection

$$Br_P^W : W^P \to W(P)(\text{see [BP]}).$$

In the view of Brauer morphism, we have $\hat{H}^0(g, L) = L(P) = Br_P^L(L^P)$ where $P =< g >$ and g is a p-element.

Keep the notation $V = \oplus_{n\in\mathbb{Z}} V_n$ as in section 1. Rouquier put forward the following problems in [R].

Problem 1. Is there an integral form L (as module) of V such that it is a p-permutation module for some subgroup U of $\mathbb{M}$ and all prime divisors of $|U|$?

Problem 2. Given L and U satisfying properties in Problem 1. Let P be a p-subgroup of U and let h be a p-regular element in $N_U(P)$ then

$$T_h(\tau) = \sum_{n\in\mathbb{Z}} Br(h|L_n(P))q^n$$

is a hauptmodul, where $Br(h|L_n(P))$ is the Brauer character of h on $L_n(P)$.

The following answer for problem 1 is from [WY].

Theorem 2.1. *(W-Y) The moonshine module is a p-permutation module for* $< g >$ $(g \in \mathbb{M})$*, where the order of* g *is* p *and* $p = 11, 17, 19, 23, 29, 31, 41, 47, 59, 71$.

Remark 2.2. *By using Brauer morphism, Ryba's modular moonshine conjecture should hold for a p-regular element* h *in* $N_M(< g >)$ *not only for p-regular element in* $C_M(g)$*. But there is no result in this direction.*

Next we will present another answer for problem 1.

Let Λ be the Leech lattice with a non-degenerate bilinear form $< .,. >$ such that $< \alpha, \alpha > \in 2\mathbb{Z}$ for any $\alpha \in \Lambda$. Let $\hat{\Lambda}$ be the central extension of Λ by $< \kappa >$, where $\kappa^2 = 1$. We denote by "-" the natural morphism form $\hat{\Lambda}$ to Λ. The group of automorphisms of Λ is denoted by Co_0 which stabilizes the form $< .,. >$ of Λ. Let C_0 be the group of automorphisms of $\hat{\Lambda}$ which stabilize κ. Set

$$K = \{a^2 \kappa^{\langle \alpha, \alpha \rangle / 2} \mid a \in \hat{\Lambda} \text{ and } \overline{a} = \alpha\}.$$

Then there is a natural morphism from C_0 to Co_0. It is also denoted by $-$. The group $\hat{\Lambda}/K$ is an extraspeical group and it has a unique (up to equivalence) irreducible faithful module T with dimension 2^{12} over F_2. The group C_0 stabilizes K and there is a morphism φ from C_0 to $Aut(\hat{\Lambda}/K)$.

Set

$$C_1 = \varphi(C_0) \subset \text{Aut } (\hat{\Lambda}/K)$$

$$C_* = \{g \in N_{\text{Aut } (T)}(\pi(\hat{\Lambda}/K)) \mid \text{int}(g) \in C_1\},$$

where π denotes the faithful representation of $\hat{\Lambda}/K$ on T and $\text{int}(g)(x) = gxg^{-1}$ for $g \in \text{Aut } (T)$, $x \in \hat{\Lambda}/K \simeq \pi(\hat{\Lambda}/K)$. Let $C_T = [C_*, C_*]$. Then $int : C_T \to C_1$ is a morphism of groups.

Defint $M_0 = \{g \in \text{Aut}(\hat{\Lambda}) | \bar{g} \in M_{24}\} \leq C_0$. Then we have

$$1 \longrightarrow \text{Hom}(\Lambda, \mathbb{Z}/2\mathbb{Z}) \longrightarrow M_0 \xrightarrow{-} M_{24} \longrightarrow 1.$$

Let $M_* := \{g \in \mathbf{N}_{\text{Aut}(T)}(\pi(\hat{\Lambda}/K)) | \text{int}(g) \in M_0\}$. Then we can prove

$$M_0 \leq M_* \leq C_*.$$

Let $M_T := M_* \cap C_T$ and let $\tilde{M} = \{(g, g_T) \in M_0 \times M_T | \varphi(g) = \text{int}(g_T)\}$. Then these groups can be presented as the following diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\pi_2} & M_T \\ \downarrow \pi_1 & & \downarrow \text{int} \\ M_0 & \xrightarrow{\varphi} & C_1. \end{array}$$

Another answer of problem 1 in [WY] is as following

Theorem 2.3. *(W-Y) There is a basis of* V *such that* $\tilde{M}$ *acts on it as permutations up to* ± 1*. Let* V_Z *be the integral form of* V *with this basis. Then* $V_Z \otimes F_p$ *is a p-permuatation module of* $\tilde{M}$*, where* p *is a prime divisor of* $|\tilde{M}|$.

3. General formulation of local moonshine

At first we will show some standard results on representation theory of finite group(see [NT]).

Let $H \leq G$. An FG-module M is said to be H-projective if there exists an FH-module W such that M is a direct summand of $\mathrm{Ind}_H^G(W)$.

Theorem 3.1. *(Higmann criterion) The following conditions are equivalent:*

1. *M is H-projective.*
2. *M is a direct summand of $Ind_H^G Res_H^G(M)$.*
3. *The $\mathbb{F}G$-morphism $\rho : Ind_H^G Res_H^G(M) \longrightarrow M(\sum_t t \otimes m_t) \mapsto \sum_t tm_t$ splits.*
4. *There is an morphism $\theta \in End_{\mathbb{F}H}(M)$ such that $Tr_H^G(\theta) = \sum_{t \in H\backslash G} \theta \circ t$ is the identity of M.*

A subgroup P of G is called a vertex of an indecomposable FG-module M if it satisfies the following conditions simultaneously: (i). M is P-projective; (ii). P is conjugate to a subgroup of H whenever M is H-projective. A vertex of an indecomposable module must a p-subgroups of G. For any indecomposable FG-module M, there exists uniquely a conjugate class of p-subgroups of G as vertices of M. Let P be a vertex of M. An indecomposable FP-module S is called a source of M over P if a vertex of S is P and M is a direct summand of $\mathrm{Ind}_P^G(S)$. Given a vertex P of M, then the source of M over P is uniquely determined up to $N_G(P)$-conjugacy, i.e. if both T and S are sources of M over P then there exists $t \in N_G(P)$ such that T^t is isomorphic to S as FP-modules. The p-permutation modules are modules with trivial module as source.

Let P be a p-subgroup of G. Assume that H is a subgroup of G containing $N_G(P)$. A subgroup Q of P is called a non-intersection subgroup of P relative to H if there exists no $x \in G - H$ such that Q is a subgroup of $P \cap P^x$. Denoted by $\mathfrak{A}(P)$ and $\mathrm{Ind}(FG|\mathfrak{A}(P))$ by the set consisting of all non-intersection subgroups of P relative to H and the set consisting of all finite dimensional indecomposable FG-modules whose vertices are contained in $\mathfrak{A}(P)$ respectively.

Theorem 3.2. *(Green) Assume P is a subgroup of G and $N_G(P) \leq H \leq G$. Then*

1. *For any $X \in Ind(FG|\mathfrak{A}(P))$, there is a unique module $Y \in Ind(FH|\mathfrak{A}(P))$ such that $X|Ind_H^G(Y)$ and $Y|Res_H^G(X)$.*
2. *Define a map f from $Ind(FG|\mathfrak{A}(P))$ to $Ind(FH|\mathfrak{A}(P))$ which sends X to Y. Then f is a bijection.*

The map f defined in above theorem is called Green correspondent with respect to (G, H, P).

Next to show the relations between Green correspondent and Brauer morphism for p-permutation modules.

Theorem 3.3. *(Broué) Let W be an indecomposable p-permutation module with vertex P. Then*

(i) *. If Q is not conjugate to a subgroup of P, then $W(Q) = 0$.*
(ii) *. $W(P)$ is the Green correspondent of W.*
(iii) *. If W' is an indecomposable module whose vertex contains (up to conjugate) P, then $W'(P)$ is a direct sum of indecomposable components whose vertices contain P.*

Remark 3.4. *The theorem does not hold true, if W is not p-permutation module. In [], Brauer morphism for general module is defined.*

Let $V = \oplus_{n\in\mathbb{Z}} V_n$ and let P be a p-subgroup of $\mathbb{M}$.

We denote $V_n(P)$ the direct summands of indecomposable components of $Res^G_{N_M(P)}(V_n)$ whose vertices contain P(up to conjugate).

General Local Moonshine Conjecture. Let h be a p-regular element in $N_M(P)$ then

$$T_h(\tau) = \sum_{n\in\mathbb{Z}} Br(h|V_n(P))q^n$$

is a hauptmodul, where $Br(h|V_n(P))$ is the Brauer character of h on $V_n(P)$.

Acknowledgement: The author is very grateful to Prof. Miyamoto and Prof.Nozaki.

REFERENCES

[B1] R. E. Borcherds, Monstrous moonshine and the monstrous Lie superalgebras, *Invent. Math.* **109** (1992), 405–444.

[B2] R. E. Borcherds, Modular moonshine III, *Duke Math. J.* **93** (1998), no. 1, 129–154.

[BP] M. Broué and L. Puig, Characters and local structure in G-algebras, *J. Alg.* **63** (1980), 306–317.

[BR] R. Borcherds and A. J. E. Ryba, Modular Moonshine. II, *Duke Math. J.* **83** (1996), no. 2, 435?459.

[Br1] M. Broué, On Scott modules and p-permutation modules: an appoach through the Brauer mophism, *Proc. Amer. Math. Soc.* **93** (1985), 401–408.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[NT] H. Nagao and Y. Tsushima, *Representation of finite groups*, Academic Press, New York, 1988.

[R] R. Rouquier, Private communications, 2012.

[RWY] R.Rouquier, Lizhong Wang and Jinwei Yang, p-indecomposable decomposition and Brauer morphisms for modules, to appear in J.Algebra

[RY] A. J. E. Ryba, Modular moonshine? in *Moonshine, the Monster, and Related Topics,* ed. by C. Dong and G. Mason, Contemp. Math. **193**, Amer. Math. Soc., Providence, 1976, 307–336.

[WY] Lizhong Wang and Jinwei Yang, On Rouquier's problems, to appear.

School of Mathematics

Peking University

Beijing 100871 P.R.China

E-mail: lwang@math.pku.edu.cn