Direct and inverse scattering problems for the local perturbation of an open periodic waveguide in the half plane

Takashi FURUYA

Graduate School of Mathematics, Nagoya University,
e-mail: takashi.furuya0101@gmail.com

1 Introduction

Let $k > 0$ be the wave number, and let $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$ be the upper half plane, and let $W := \mathbb{R} \times (0, h)$ be the waveguide in $\mathbb{R}^2_+$. We denote by $\Gamma_a := \mathbb{R} \times \{a\}$ for $a > 0$. Let $n \in L^\infty(\mathbb{R}^2_+)$ be real value, 2π-periodic with respect to x_1 (that is, $n(x_1 + 2\pi, x_2) = n(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2_+$), and equal to one for $x_2 > h$. We assume that there exists a constant $n_0 > 0$ such that $n \geq n_0$ in $\mathbb{R}^2_+$. Let $q \in L^\infty(\mathbb{R}^2_+)$ be real value with the compact support in W. We denote by $Q := \mathrm{supp} q$. We assume that $\mathbb{R}^2 \setminus \overline{Q}$ is connected.

We consider the following scattering problem: For fixed $y \in \mathbb{R}^2_+ \setminus \overline{W}$, determine the scattered field $u^s \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u^s + k^2(1+q)nu^s = -k^2 qnu^i(\cdot, y) \text{ in } \mathbb{R}^2_+, \tag{1.1}$$

$$u^s = 0 \text{ on } \Gamma_0, \tag{1.2}$$

Here, the incident field u^i is given by $u^i(x, y) = \overline{G_n(x, y)}$, where G_n is the Dirichlet Green's function in the upper half plane $\mathbb{R}^2_+$ for $\Delta + k^2 n$, that is,

$$G_n(x, y) := G(x, y) + \tilde{u}^s(x, y), \tag{1.3}$$

where $G(x, y) := \Phi_k(x, y) - \Phi_k(x, y^*)$ is the Dirichlet Green's function in $\mathbb{R}^2_+$ for $\Delta + k^2$, and $y^* = (y_1, -y_2)$ is the reflected point of y at $\mathbb{R} \times \{0\}$. Here, $\Phi_k(x, y)$ is the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, that is,

$$\Phi_k(x, y) := \frac{i}{4} H^{(1)}_0(k|x-y|), \ x \neq y, \tag{1.4}$$

and $\tilde{u}^s$ is the scattered field of the unperturbed problem by the incident field $G(x, y)$, that is, $\tilde{u}^s$ vanishes for $x_2 = 0$ and solves

$$\Delta \tilde{u}^s + k^2 n \tilde{u}^s = k^2(1-n)G(\cdot, y) \text{ in } \mathbb{R}^2_+. \tag{1.5}$$

If we impose a suitable radiation condition (see Definition 2.4), the unperturbed solution $\tilde{u}^s$ is uniquely determined.

In order to show the well-posedness of the perturbed scattering problem (1.1)–(1.2), we make the following assumption.

Assumption 1.1. *We assume that k^2 is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ in $H^1_0(\mathbb{R}^2_+)$, that is, evey $v \in H^1(\mathbb{R}^2_+)$ which satisfies*

$$\Delta v + k^2(1+q)nv = 0 \text{ in } \mathbb{R}^2_+, \tag{1.6}$$

$$v = 0 \text{ on } \Gamma_0, \tag{1.7}$$

has to vanish for $x_2 > 0$.

The following theorem was shown in Theorem 1.2 of [1].

Theorem 1.2. *Let Assumptions 1.1 hold and let* $f \in L^2(\mathbb{R}^2_+)$ *such that* $\mathrm{supp} f = Q$*. Then, there exists a unique solution* $u \in H^1_{loc}(\mathbb{R}^2_+)$ *such that*

$$\Delta u + k^2(1+q)nu = f \text{ in } \mathbb{R}^2_+, \tag{1.8}$$

$$u = 0 \text{ on } \Gamma_0, \tag{1.9}$$

and u *satisfies the radiation condition in the sense of Definition 2.4.*

Roughly speaking, this radiation condition requires that we have a decomposition of the solution u into $u^{(1)}$ which decays in the direction of x_1, and a finite combination $u^{(2)}$ of *propagative modes* which does not decay in x_1, but it exponentially decays in x_2. In Section 2, we will study details of the radiation condition.

By Theorem 1.2, the well-posedness of this perturbed scattering problem (1.1)–(1.2) holds. Then, we are now able to consider the inverse problem of determining the support of q from measured scattered field u^s by the incident field u^i. Let $M := \{(x_1, m) : a < x_1 < b\}$ for $a < b$ and $m > h$, and $Q := \mathrm{supp} q$. With the scattered field u^s, we define the *near field operator* $N : L^2(M) \to L^2(M)$ by

$$Ng(x) := \int_M u^s(x,y)g(y)ds(y), \; x \in M. \tag{1.10}$$

The inverse problem we consider here is to determine support Q of q from the scattered field $u^s(x,y)$ for all x and y in M with one fixed $k > 0$. In other words, given the near field operator N, determine Q.

The following theorem was shown in Theorem 1.1 of [2].

Theorem 1.3. *Let* $B \subset \mathbb{R}^2$ *be a bounded open set. We assume that there exists* $q_{min} > 0$ *such that* $q \geq q_{min}$ *a.e. in* Q*. Then for* $0 < \alpha < k^2 n_{min} q_{min}$*,*

$$B \subset Q \quad \Longleftrightarrow \quad \alpha H^*_B H_B \leq_{\mathrm{fin}} \mathrm{Re} N, \tag{1.11}$$

where the operator $H_B : L^2(M) \to L^2(B)$ *is given by*

$$H_B g(x) := \int_M \overline{G_n(x,y)} g(y)ds(y), \; x \in B, \tag{1.12}$$

and the inequality on the right hand side in (1.11) denotes that $\mathrm{Re}N - \alpha H^*_B H_B$ *has only finitely many negative eigenvalues, and the real part of an operator* A *is self-adjoint operators given by* $\mathrm{Re}(A) := \frac{1}{2}(A + A^*)$.

By Theorem 1.3, we can understand whether an artificial domain B is contained in Q or not. Then, by dispersing a lot of balls B in $\mathbb{R}^2_+$ and for each B checking (1.11) we can reconstruct the shape and location of unknown Q.

This paper is organized as follows. In Section 2, we recall a radiation condition introduced in [4]. In Section 3, we study several factorizations of the near field operator N, which prepare for the proof of Theorem 1.3. Finally in Sections 4, we prove Theorems 1.3.

2 A radiation condition

In Section 2, we recall a radiation condition introduced in [4]. Let $f \in L^2(\mathbb{R}^2_+)$ have the compact support in W. First, we consider the following direct problem: Determine the scattered field $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u + k^2 nu = f \text{ in } \mathbb{R}^2_+, \tag{2.1}$$

$$u = 0 \text{ on } \Gamma_0. \tag{2.2}$$

(2.1) is understood in the variational sense, that is,

$$\int_{\mathbb{R}^2_+} \left[\nabla u \cdot \nabla \overline{\varphi} - k^2 nu\overline{\varphi}\right] dx = -\int_W f\overline{\varphi} dx, \tag{2.3}$$

for all $\varphi \in H^1(\mathbb{R}^2_+)$, with compact support. In such a problem, it is natural to impose the *upward propagating radiation condition*, that is, $u(\cdot, h) \in L^\infty(\mathbb{R})$ and

$$u(x) = 2\int_{\Gamma_h} u(y)\frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y) = 0, \ x_2 > h. \tag{2.4}$$

However, even with this condition we can not expect the uniqueness of this problem. (see Example 2.3 of [4].) In order to introduce a *suitable radiation condition*, Kirsch and Lechleiter discussed limiting absorption solution of this problem, that is, the limit of the solution u_ϵ of $\Delta u_\epsilon + (k + i\epsilon)^2 nu_\epsilon = f$ as $\epsilon \to 0$. For the details of an introduction of this radiation condition, we refer to [4].

Let us prepare for the exact definition of the radiation condition. We denote by $C_R := (0, 2\pi) \times (0, R)$ for $R \in (0, \infty]$. The function $u \in H^1(C_R)$ is called α-quasi periodic if $u(2\pi, x_2) = e^{2\pi i\alpha}u(0, x_2)$. We denote by $H^1_\alpha(C_R)$ the subspace of the α-quasi periodic function in $H^1(C_R)$, and $H^1_{\alpha,loc}(C_\infty) := \{u \in H^1_{loc}(C_\infty) : u\big|_{C_R} \in H^1_\alpha(C_R) \text{ for all R} > 0\}$. Then, we consider the following problem, which arises from taking the quasi-periodic Floquet Bloch transform in (2.1)–(2.2): For $\alpha \in [-1/2, 1/2]$, determine $u_\alpha \in H^1_{\alpha,loc}(C_\infty)$ such that

$$\Delta u_\alpha + k^2 nu_\alpha = f_\alpha \text{ in } C_\infty. \tag{2.5}$$

$$u_\alpha = 0 \text{ on } (0, 2\pi) \times \{0\}. \tag{2.6}$$

Here, it is a natural to impose the *Rayleigh expansion* of the form

$$u_\alpha(x) = \sum_{n\in\mathbb{Z}} u_n(\alpha)e^{inx_1 + i\sqrt{k^2-(n+\alpha)^2}(x_2-h)}, \ x_2 > h, \tag{2.7}$$

where $u_n(\alpha) := (2\pi)^{-1}\int_0^{2\pi} u_\alpha(x_1, h)e^{-inx_1}dx_1$ are the Fourier coefficients of $u_\alpha(\cdot, h)$, and $\sqrt{k^2 - (n + \alpha)^2} = i\sqrt{(n + \alpha)^2 - k^2}$ if $n + \alpha > k$. But even with this expansion the uniqueness of this problem fails for some $\alpha \in [-1/2, 1/2]$. We call α *exceptional values* if there exists non-trivial solutions $u_\alpha \in H^1_{\alpha,loc}(C_\infty)$ of (2.5)–(2.7). We set $A_k := \{\alpha \in [-1/2, 1/2] : \exists l \in \mathbb{Z} \ s.t. \ |\alpha + l| = k\}$, and make the following assumption:

Assumption 2.1. *For every $\alpha \in A_k$ the solution of $u_\alpha \in H^1_{\alpha,loc}(C_\infty)$ of (2.5)–(2.7) has to be zero.*

The following properties of exceptional values was shown in [4].

Lemma 2.2. *Let Assumption 2.1 hold. Then, there exists only finitely many exceptional values $\alpha \in [-1/2, 1/2]$. Furthermore, if α is an exceptional value, then so is $-\alpha$. Therefore, the set of exceptional values can be described by $\{\alpha_j : j \in J\}$ where some $J \subset \mathbb{Z}$ is finite and $\alpha_{-j} = -\alpha_j$ for $j \in J$. For each exceptional value α_j we define*

$$X_j := \left\{ \phi \in H^1_{\alpha_j, loc}(C_\infty) : \begin{array}{c} \Delta\phi + k^2 n\phi = 0 \text{ in } C_\infty, \quad \phi = 0 \text{ for } x_2 = 0, \\ \phi \text{ satisfies the Rayleigh expansion (2.7)} \end{array} \right\}$$

Then, X_j are finite dimensional. We set $m_j = \dim X_j$. Furthermore, $\phi \in X_j$ is evanescent, that is, there exists $c > 0$ and $\delta > 0$ such that $|\phi(x)|, \ |\nabla\phi(x)| \leq ce^{-\delta|x_2|}$ for all $x \in C_\infty$.

Next, we consider the following eigenvalue problem in X_j: Determine $d \in \mathbb{R}$ and $\phi \in X_j$ such that

$$-i\int_{C_\infty} \frac{\partial\phi}{\partial x_1}\overline{\psi}dx = dk\int_{C_\infty} n\phi\overline{\psi}dx, \tag{2.8}$$

for all $\psi \in X_j$. We denote by the eigenvalues $d_{l,j}$ and eigenfunction $\phi_{l,j}$ of this problem, that is,

$$-i\int_{C_\infty} \frac{\partial\phi_{l,j}}{\partial x_1}\overline{\psi}dx = d_{l,j}k\int_{C_\infty} n\phi_{l,j}\overline{\psi}dx, \tag{2.9}$$

for every $l = 1, ..., m_j$ and $j \in J$. We normalize the eigenfunction $\{\phi_{l,j} : l = 1, ..., m_j\}$ such that

$$k\int_{C_\infty} n\phi_{l,j}\overline{\phi_{l',j}}dx = \delta_{l,l'}, \tag{2.10}$$

for all l, l'. We will assume that the wave number $k > 0$ is *regular* in the following sense.

Definition 2.3. $k > 0$ is *regular* if $d_{l,j} \neq 0$ for all $l = 1, ...m_j$ and $j \in J$.

Now we are ready to define the radiation condition.

Definition 2.4. Let Assumptions 2.1 hold, and let $k > 0$ be regular in the sense of Definition 2.3. We set

$$\psi^{\pm}(x_1) := \frac{1}{2}\left[1 \pm \frac{2}{\pi}\int_0^{x_1/2} \frac{sint}{t}dt\right], \ x_1 \in \mathbb{R}. \tag{2.11}$$

Then, $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfies the *radiation condition* if u satisfies the upward propagating radiation condition (2.4), and has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)}|_{\mathbb{R}\times(0,R)} \in H^1(\mathbb{R} \times (0, R))$ for all $R > 0$, and $u^{(2)} \in L^\infty(\mathbb{R}^2_+)$ has the following form

$$u^{(2)}(x) = \psi^+(x_1)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \psi^-(x_1)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x) \tag{2.12}$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, ..., m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (2.8).

Remark 2.5. It is obvious that we can replace ψ^+ by any smooth functions $\tilde{\psi}^{\pm}$ with $\tilde{\psi}^+(x_1) = 1 + \mathcal{O}(1/x_1)$ as $x_1 \to \infty$ and $\tilde{\psi}^+(x_1) = \mathcal{O}(1/x_1)$ as $x_1 \to -\infty$ and $\frac{d}{dx_1}\tilde{\psi}^+(x_1) \to 0$ as $|x_1| \to \infty$ (and analogously for ψ^-).

The following was shown in Theorems 2.2, 6.6, and 6.8 of [4].

Theorem 2.6. *For every $f \in L^2(\mathbb{R}^2_+)$ with the compact support in W, there exists a unique solution $u_{k+i\epsilon} \in H^1(\mathbb{R}^2_+)$ of the problem (2.1)–(2.2) replacing k by $k + i\epsilon$. Furthermore, $u_{k+i\epsilon}$ converge as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to some $u \in H^1_{loc}(\mathbb{R}^2_+)$ which satisfy (2.1)–(2.2) and the radiation condition in the sense of Definition 2.4. Furthermore, the solution u of this problem is uniquely determined.*

3 A factorization of the near field operator

In Section 3, we discuss a factorization of the near field operator N. We define the operator $L : L^2(Q) \to L^2(M)$ by $Lf := v\big|_M$ where v satisfies the radiation condition in the sense of Definition 2.4 and

$$\Delta v + k^2(1+q)nv = -k^2 \frac{nq}{\sqrt{|nq|}} f, \text{ in } \mathbb{R}^2_+, \tag{3.1}$$

$$v = 0 \text{ on } \mathbb{R} \times \{0\}. \tag{3.2}$$

We define $H : L^2(M) \to L^2(Q)$ by

$$Hg(x) := \sqrt{|n(x)q(x)|} \int_M \overline{G_n(x,y)} g(y) ds(y), \ x \in Q. \tag{3.3}$$

Then, by these definition we have $N = LH$. In order to make a symmetricity of the factorization of the near field operator N, we will show the following symmetricity of the Green function G_n.

Lemma 3.1.

$$G_n(x,y) = G_n(y,x), \ x \neq y. \tag{3.4}$$

Proof of Lemma 3.1. We take a small $\eta > 0$ such that $B_{2\eta}(x) \cap B_{2\eta}(y) = \emptyset$ where $B_\epsilon(z) \subset \mathbb{R}^2$ is some open ball with center z and radius $\epsilon > 0$. We recall that $G_n(z,y) = G(z,y) + \tilde{u}^s(z,y)$ where $G(z,y) = \Phi_k(z,y) - \Phi_k(z,y^*)$ and $\tilde{u}^s(z,y)$ is a radiating solution of the problem (1.5) such that $\tilde{u}^s(z,y) = 0$ for $z_2 = 0$. In Introduction of [4] $\tilde{u}^s$ is given by $\tilde{u}^s(z,y) = u(z,y) - \chi(|z-y|)G(z,y)$ where $\chi \in C^\infty(\mathbb{R}_+)$ satisfying $\chi(t) = 0$ for $0 \leq t \leq \eta/2$ and $\chi(t) = 1$ for $t \geq \eta$, and u is a radiating solution such that $u = 0$ on $\mathbb{R} \times \{0\}$ and

$$\Delta u + k^2 nu = f(\cdot, y) \text{ in } \mathbb{R}^2_+, \tag{3.5}$$

$$u = 0 \text{ on } \mathbb{R} \times \{0\}, \tag{3.6}$$

where

$$f(\cdot,y) := \Big[k^2(1-n)\big(1-\chi(|\cdot - y|)\big) + \Delta\chi(|\cdot - y|)\Big] G(\cdot,y) + 2\nabla\chi(|\cdot - y|) \cdot \nabla G(\cdot,y). \tag{3.7}$$

Then, we have $G_n(z,y) = u(z,y) + (1-\chi(|z-y|))G(z,y)$. By Theorem 2.6 we can take an solution $u_\epsilon \in H^1(\mathbb{R}^2_+)$ of the problem (3.5)–(3.6) replacing k by $(k+i\epsilon)$ satisfying u_ϵ converges as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to u. We set $G_{n,\epsilon}(z,y) := u_\epsilon(z,y) + (1-\chi(|z-y|))G(z,y)$, and $G_{n,\epsilon}(z,y)$ converges as $\epsilon \to +0$ to $G(z,y)$ pointwise for $z \in \mathbb{R}^2_+$. By the simple calculation, we have

$$\big[\Delta_z + (k+i\epsilon)^2 n(z)\big] G_{n,\epsilon}(z,y) = -\delta(z,y) + (2k\epsilon i - \epsilon^2) n(z)\big(1-\chi(|z-y|)\big)G(z,y). \tag{3.8}$$

Let $r > 0$ be large enough such that $x, y \in B_r(0)$. By Green's second theorem in $B_r(0) \cap \mathbb{R}^2_+$ we have

$$
\begin{aligned}
&-G_{n,\epsilon}(y,x) + (2k\epsilon i - \epsilon^2)\int_{B_{2\eta}(y)} u_\epsilon(z,x)n(z)(1-\chi(|z-y|))G(z,y)dz \\
&+ \quad G_{n,\epsilon}(x,y) - (2k\epsilon i - \epsilon^2)\int_{B_{2\eta}(x)} u_\epsilon(z,y)n(z)(1-\chi(|z-x|))G(z,x)dz \\
&= \int_{B_r(0)\cap\mathbb{R}^2_+} G_{n,\epsilon}(z,x)\big[\Delta_z + (k+i\epsilon)^2 n(z)\big]G_{n,\epsilon}(z,y)dz \\
&- \int_{B_r(0)\cap\mathbb{R}^2_+} G_{n,\epsilon}(z,y)\big[\Delta_z + (k+i\epsilon)^2 n(z)\big]G_{n,\epsilon}(z,x)dz \\
&= \int_{\partial B_r(0)\cap\mathbb{R}^2_+} u_\epsilon(z,x)\frac{\partial u_\epsilon(z,y)}{\partial \nu_z} - u_\epsilon(z,y)\frac{\partial u_\epsilon(z,x)}{\partial \nu_z} ds(z).
\end{aligned} \tag{3.9}
$$

Since $u_\epsilon \in H^1(\mathbb{R}^2_+)$, the right hand side of (3.9) converges as $r \to \infty$ to zero. Then, as $r \to \infty$ in (3.9) we have

$$
\begin{aligned}
&G_{n,\epsilon}(x,y) - G_{n,\epsilon}(y,x) \\
&\quad = (2k\epsilon i - \epsilon^2)\int_{B_{2\eta}(x)} u_\epsilon(z,y)n(z)(1-\chi(|z-x|))G(z,x)dz \\
&\quad - (2k\epsilon i - \epsilon^2)\int_{B_{2\eta}(y)} u_\epsilon(z,x)n(z)(1-\chi(|z-y|))G(z,y)dz
\end{aligned} \tag{3.10}
$$

Since u_ϵ converges as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to u, the right hand side of (3.10) converges to zero as $\epsilon \to +0$. Therefore, we conclude that $G_n(x,y) = G_n(y,x)$ for $x \neq y$. □

By the symmetricity of G_n,

$$
\begin{aligned}
\langle Hg, f\rangle &= \int_Q \big\{\sqrt{|n(x)q(x)|}\int_M \overline{G_n(x,y)}g(y)ds(y)\big\}\overline{f(x)}dx \\
&= \int_M g(y)\overline{\big\{\int_Q \sqrt{|n(x)q(x)|}G_n(x,y)f(x)ds(x)\big\}}ds(y) \\
&= \int_M g(y)\overline{\big\{\int_Q \sqrt{|n(x)q(x)|}G_n(y,x)f(x)ds(x)\big\}}ds(y),
\end{aligned} \tag{3.11}
$$

which implies that

$$
H^* f(x) = \int_Q \sqrt{|n(y)q(y)|}G_n(x,y)f(y)ds(y), \ x \in M. \tag{3.12}
$$

We define $T : L^2(Q) \to L^2(Q)$ by $Tf := \frac{|nq|}{k^2 nq}f - \sqrt{|nq|}w$ where w satisfies the radiation condition and

$$
\Delta w + k^2 n w = -\sqrt{|nq|}f, \ \text{in } \mathbb{R}^2_+, \tag{3.13}
$$

$$
v = 0 \text{ on } \mathbb{R}\times\{0\}. \tag{3.14}
$$

We will show the following integral representation of w.

Lemma 3.2.

$$w(x) = \int_Q \sqrt{|n(y)q(y)|} G_n(x,y) f(y) dy, \;\; x \in \mathbb{R}^2_+. \tag{3.15}$$

Proof of Lemma 3.2. Let $w_\epsilon \in H^1_{loc}(\mathbb{R}^2_+)$ be a solution of the problem (3.13)–(3.14) replacing k by $(k + i\epsilon)$ satisfying w_ϵ converges as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to w. Let $G_{n,\epsilon}(y,x)$ be an approximation of the Green's function $G_n(y,x)$ as same as in Lemma 3.1. Let $r > 0$ be large enough such that $x \in B_r(0)$. By Green's second theorem in $B_r(0) \cap \mathbb{R}^2_+$ we have

$$\begin{aligned}
&-w_\epsilon(x) + (2k\epsilon i - \epsilon^2) \int_{B_{2\eta}(x)} w_\epsilon(y) n(y)(1 - \chi(|y-x|)) G(y,x) dy \\
&+ \int_Q \sqrt{|n(y)q(y)|} G_{n,\epsilon}(y,x) f(y) dy \\
&= \int_{B_r(0)\cap \mathbb{R}^2_+} w_\epsilon(y) \big[\Delta_y + (k+i\epsilon)^2 n(y)\big] G_{n,\epsilon}(y,x) dy \\
&- \int_{B_r(0)\cap \mathbb{R}^2_+} G_{n,\epsilon}(y,x) \big[\Delta_y + (k+i\epsilon)^2 n(y)\big] w_\epsilon(y) dz \\
&= \int_{\partial B_r(0)\cap \mathbb{R}^2_+} w_\epsilon(y) \frac{\partial u_\epsilon(y,x)}{\partial \nu_y} - u_\epsilon(y,x) \frac{\partial w_\epsilon(y)}{\partial \nu_y} ds(y).
\end{aligned} \tag{3.16}$$

Since u_ϵ, $w_\epsilon \in H^1(\mathbb{R}^2_+)$, the right hand side of (3.16) converges as $r \to \infty$ to zero. Then, as $r \to \infty$ in (3.16) we have

$$\begin{aligned}
w_\epsilon(x) &= (2k\epsilon i - \epsilon^2) \int_{B_{2\eta}(x)} w_\epsilon(y) n(y)(1 - \chi(|y-x|)) G(y,x) dy \\
&+ \int_Q \sqrt{|n(y)q(y)|} G_{n,\epsilon}(y,x) f(y) dy
\end{aligned} \tag{3.17}$$

The first term of right hand side in (3.17) converges to zero as $\epsilon \to +0$, and the second term converges to $\int_Q \sqrt{|n(y)q(y)|} G_n(y,x) f(y) dy$ as $\epsilon \to +0$. As $\epsilon \to +0$ in (3.17) and by the symmetricity of G_n (Lemma 3.1) we conclude (3.15). □

Since w satisfies

$$\begin{aligned}
\Delta w + k^2(1+q)nw &= -k^2 \frac{nq}{\sqrt{|nq|}} \Big\{ \frac{|nq|}{k^2 nq} f - \sqrt{|nq|} w \Big\} \text{ in } \mathbb{R}^2_+ \\
&= -k^2 \frac{nq}{\sqrt{|nq|}} T f,
\end{aligned} \tag{3.18}$$

we have $w\big|_M = LTf$. Therefore, by (3.12) and (3.15) we have $H^* = LT$. Then, we have the following symmetric factorization:

$$N = LT^* L^*. \tag{3.19}$$

We will show the following lemma.

Lemma 3.3. **(a)** *L is compact with dense range in $L^2(M)$.*

(b) *If there exists the constant* $q_{min} > 0$ *such that* $q_{min} \leq q$ *a.e. in* Q*, then* $\mathrm{Re}T$ *has the form* $\mathrm{Re}T = C + K$ *with some self-adjoint and positive coercive operator* C *and some compact operator* K *on* $L^2(Q)$*.*

(c) $\mathrm{Im}\langle f, Tf\rangle \geq 0$ *for all* $f \in L^2(Q)$*.*

(d) T *is injective.*

Proof of Lemma 3.3. **(d)** Let $f \in L^2(Q)$ and $Tf = 0$, i.e., $\frac{|nq|}{k^2 nq} f = \sqrt{|nq|}w$ where w satisfies (3.13)–(3.14). Then, $\Delta w + k^2 n(1+q)w = 0$. By the uniqueness, $w = 0$ in $\mathbb{R}^2_+$ which implies that $f = 0$. Therefore T is injective.

(b) Since n and q are bounded below (that is, $n \geq n_{min} > 0$ and $q \geq q_{min} > 0$), T has the form $T = C + K$ where K is some compact operator and C is some self-adjoint and positive coercive operator. Furthermore, from the injectivity of T we obtain that T is bijective.

(a) By the trace theorem and $v \in H^1_{loc}(\mathbb{R}^2_+)$, $Lf = v\big|_M \in H^{1/2}(M)$, which implies that $L : L^2(Q) \to L^2(M)$ is compact.

By the bijectivity of T and $H = T^* L^*$, it is sufficient to show the injectivity of H. Let $g \in L^2(M)$ and $Hg(x) = \sqrt{|n(x)q(x)|}\int_M \overline{G_n(x,y)}g(y)ds(y) = 0$ for $x \in Q$. We set $v(x) := \int_M \overline{G_n(x,y)}g(y)ds(y)$. By the definition of v we have

$$\Delta v + k^2 nv = 0, \text{ in } \mathbb{R}^2_+ \setminus M, \tag{3.20}$$

and since q are bounded below, $v = 0$ in Q. By unique continuation principle we have $v = 0$ in $\mathbb{R}^2_+ \setminus M$. By the jump relation, we have $0 = \frac{\partial v_+}{\partial \nu} - \frac{\partial v_-}{\partial \nu} = g$, which conclude that the operator H is injective.

(c) For the proof of (c) we refer to Theorem 3.1 in [1]. By the definition of T we have

$$\mathrm{Im}\langle f, Tf\rangle = -\mathrm{Im}\int_Q f\sqrt{|nq|}\overline{w}dx = \mathrm{Im}\int_Q \overline{w}[\Delta + k^2 n]wdx, \tag{3.21}$$

where w is a radiating solution of the problem (3.13)–(3.14). We set $\Omega_N := (-N, N) \times (0, N^s)$ where $s > 0$ is small enough and $N > 0$ is large enough. By the same argument in Theorem 3.1 of [1] we have

$$\begin{aligned} \mathrm{Im}\langle f, Tf\rangle &= \mathrm{Im}\int_{\Omega_N} \overline{w}[\Delta + k^2 n]wdx = \mathrm{Im}\int_{\Omega_N} \overline{w}\Delta wdx \\ &\geq \left[\frac{1}{2\pi}\sum_{j\in J}\sum_{d_{l,j},d_{l',j}>0} \overline{a_{l,j}}a_{l',j}\int_{C_{\phi(N)}} \overline{\phi_{l,j}}\frac{\partial\phi_{l',j}}{\partial x_1}dx\right] \\ &- \mathrm{Im}\left[\frac{1}{2\pi}\sum_{j\in J}\sum_{d_{l,j},d_{l',j}<0} \overline{a_{l,j}}a_{l',j}\int_{C_{\phi(N)}} \overline{\phi_{l,j}}\frac{\partial\phi_{l',j}}{\partial x_1}dx\right] + o(1), \end{aligned} \tag{3.22}$$

where where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, ..., m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (2.8). By Lemmas 6.3 and 6.4 of [4], as $N \to \infty$ in (3.22) we have

$$\mathrm{Im}\langle f, Tf\rangle \geq \frac{k}{2\pi}\sum_{j\in J}\left[\sum_{d_{l,j}>0}|a_{l,j}|^2 d_{l,j} - \sum_{d_{l,j}<0}|a_{l,j}|^2 d_{l,j}\right] \geq 0, \tag{3.23}$$

which concludes (c). □

In order to show Theorems 1.1 and 1.2, we consider another factorization of the near field operator N. We define $\tilde{T} : L^2(Q) \to L^2(Q)$ by $\tilde{T}v := k^2 \frac{nq}{|nq|} g - k^2 \frac{nq}{\sqrt{|nq|}} v$ where v satisfies the radiation condition and

$$\Delta v + k^2(1+q)nv = -k^2 \frac{nq}{\sqrt{|nq|}} g, \text{ in } \mathbb{R}^2_+, \tag{3.24}$$

$$v = 0 \text{ on } \mathbb{R} \times \{0\}. \tag{3.25}$$

Then, by the definition of T and $\tilde{T}$ we can show that $\tilde{T}T = I$ and $T\tilde{T} = I$, which implies that $T^{-1} = \tilde{T}$. Therefore, we have by $L = H^* T^{-1}$

$$N = LT^*L^* = H^*T^{-1}H = H^*\tilde{T}H = H_Q^* \hat{T} H_Q, \tag{3.26}$$

where $H_Q : L^2(M) \to L^2(Q)$ is defined by

$$H_Q g(x) := \int_M \overline{G_n(x,y)} g(y) ds(y), \ x \in Q, \tag{3.27}$$

and $\hat{T} : L^2(Q) \to L^2(Q)$ is defined by $\hat{T}f = k^2 nqf + k^2 nqw$ where w satisfies the radiation condition and

$$\Delta w + k^2(1+q)nw = -k^2 nqf, \text{ in } \mathbb{R}^2_+, \tag{3.28}$$

$$w = 0 \text{ on } \mathbb{R} \times \{0\}. \tag{3.29}$$

We will show the following lemma.

Lemma 3.4. *Let B and Q be a bounded open set in $\mathbb{R}^2_+$.*

(a) $\dim(\mathrm{Ran}(H_B^*)) = \infty$.

(b) *If $B \cap Q = \emptyset$, then* $\mathrm{Ran}(H_B^*) \cap \mathrm{Ran}(H_Q^*) = \{0\}$.

Proof of Lemma 3.4. (a) By the same argument of the injectivity of H in (a) of Lemma 4.3, we can show that H_B is injective. Therefore, H_B^* has dense range.

(b) Let $h \in \mathrm{Ran}(H_B^*) \cap \mathrm{Ran}(H_Q^*)$. Then, there exists f_B, f_Q suct that $h = H_B^* f_B = H_Q^* f_Q$. We set

$$v_B(x) := \int_B G_n(x,y) f_B(y) dy, \ x \in \mathbb{R}^2_+ \tag{3.30}$$

$$v_Q(x) := \int_Q G_n(x,y) f_Q(y) dy, \ x \in \mathbb{R}^2_+ \tag{3.31}$$

then, v_B and v_Q satisfies $\Delta v_B + k^2 n v_B = -f_B$, and $\Delta v_Q + k^2 n v_Q = -f_Q$, respectively, and $v_B = v_Q$ on M. By Rellich lemma and unique continuation we have $v_B = v_Q$ in $\mathbb{R}^2_+ \setminus (\overline{B \cap Q})$. Hence, we can define $v \in H^1_{loc}(\mathbb{R}^2)$ by

$$v := \begin{cases} v_B = v_Q & \text{in } \mathbb{R}^2_+ \setminus (\overline{B \cap Q}) \\ v_B & \text{in } Q \\ v_Q & \text{in } B \end{cases} \tag{3.32}$$

and v is a radiating solution such that $v = 0$ for $x_2 = 0$ and

$$\Delta v + k^2 nv = 0 \text{ in } \mathbb{R}^2_+. \tag{3.33}$$

By the uniqueness, we have $v = 0$ in $\mathbb{R}^2$, which implies that $h = 0$. □

4 Proof of Theorem 1.1

In Section 4, we will show Theorem 1.3. Let $B \subset Q$. We define $K : L^2(Q) \to L^2(Q)$ by $Kf := k^2 nqw$ where w is a radiating solution of the problem (3.28)–(3.29). Since $w\big|_Q \in H^1(Q)$, K is a compact operator. Let V be the sum of eigenspaces of ReK associated to eigenvalues less than $\alpha - k^2 n_{min} q_{min}$. Since $\alpha - k^2 n_{min} q_{min} < 0$, then V is a finite dimensional and for $H_Q g \in V^{\perp}$

$$\begin{aligned} \langle \mathrm{Re}Ng, g \rangle &= \int_Q k^2 nq |H_Q g|^2 dx + \langle (\mathrm{Re}K) H_Q g, H_Q g \rangle \\ &\geq k^2 n_{min} q_{min} \|H_Q g\|^2 + (\alpha - k^2 n_{min} q_{min}) \|H_Q g\|^2 \\ &\geq \alpha \|H_Q g\|^2 \geq \alpha \|H_B g\|^2 \end{aligned} \tag{4.1}$$

Since for $g \in L^2(M)$

$$H_Q g \in V^{\perp} \quad \Longleftrightarrow \quad g \in (H_Q^* V)^{\perp}, \tag{4.2}$$

and $\dim(H_Q^* V) \leq \dim(V) < \infty$, we have by Corollary 3.3 of [3] that $\alpha H_B^* H_B \leq_{\mathrm{fin}} \mathrm{Re}N$.

Let now $B \not\subset Q$ and assume on the contrary $\alpha H_B^* H_B \leq_{\mathrm{fin}} \mathrm{Re}N$, that is, by Corollary 3.3 of [3] there exists a finite dimensional subspace W in $L^2(M)$ such that

$$\langle (\mathrm{Re}N - \alpha H_B^* H_B) w, w \rangle \geq 0, \tag{4.3}$$

for all $w \in W^{\perp}$. Since $B \not\subset Q$, we can take a small open domain $B_0 \subset B$ such that $B_0 \cap Q = \emptyset$, which implies that for all $w \in W^{\perp}$

$$\begin{aligned} \alpha \|H_{B_0} w\|^2 &\leq \alpha \|H_B w\|^2 \\ &\leq \langle (\mathrm{Re}N) w, w \rangle \\ &= \langle (\mathrm{Re}\hat{T}) H_Q w, H_Q w \rangle \\ &\leq \left\| \mathrm{Re}\hat{T} \right\| \|H_Q w\|^2 . \end{aligned} \tag{4.4}$$

By (a) of Lemma 4.7 in [3], we have

$$\mathrm{Ran}(H_{B_0}^*) \not\subseteq \mathrm{Ran}(H_Q^*) + W = \mathrm{Ran}(H_Q^*, \ P_W), \tag{4.5}$$

where $P_W : L^2(M) \to L^2(M)$ is the orthognal projection on W. Lemma 4.6 of [3] implies that for any $C > 0$ there exists a w_c such that

$$\|H_{B_0} w_c\|^2 > C^2 \left\| \begin{pmatrix} H_Q \\ P_V \end{pmatrix} w_c \right\|^2 = C^2 (\|H_Q w_c\|^2 + \|P_W w_c\|^2). \tag{4.6}$$

Hence, there exists a sequence $(w_m)_{m \in \mathbb{N}} \subset L^2(\mathbb{S}^1)$ such that $\|H_{B_0} w_m\| \to \infty$ and $\|H_Q w_m\| + \|P_V w_m\| \to 0$ as $m \to \infty$. Setting $\tilde{w}_m := w_m - P_W w_m \in W^{\perp}$ we have as $m \to \infty$,

$$\|H_{B_0} \tilde{w}_m\| \geq \|H_{B_0} w_m\| - \|H_{B_0}\| \|P_W w_m\| \to \infty, \tag{4.7}$$

$$\|H_Q \tilde{w}_m\| \leq \|H_Q w_m\| + \|H_Q\| \|P_W w_m\| \to 0. \tag{4.8}$$

This contradicts (4.4). Therefore, we have $\alpha H_B^* H_B \not\leq_{\mathrm{fin}} \mathrm{Re}N$. Theorem 1.3 has been shown. □

By the same argument in Theorem 1.3 we can show the following.

Corollary 4.1. *Let $B \subset \mathbb{R}^2$ be a bounded open set. Let Assumption hold, and assume that there exists $q_{max} < 0$ such that $q \leq q_{max}$ a.e. in Q. Then for $0 < \alpha < k^2 n_{min} |q_{max}|$,*

$$B \subset Q \quad \Longleftrightarrow \quad \alpha H_B^* H_B \leq_{\mathrm{fin}} -\mathrm{Re}N, \tag{4.9}$$

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