Zeros of some random polynomials and connectedness locus of family of fractals

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Abstract

We consider the closure of the set of zeros of polynomials with complex coefficients randomly chosen from a "good" compact subset of complex plane. We prove that the closure of the set of zeros is connected. Furthermore, we apply this result to the study of connectedness locus($\mathcal{M}_n$) of fractal n-gons and the remarkable subset $\mathcal{M}_n^0$. Fractal n-gons and $\mathcal{M}_n$ are introduced by C. Bandt and N. V. Hung(2008). It is already known that $\mathcal{M}_2$, $\mathcal{M}_2^0$ and $\mathcal{M}_3$ are connected(T. Bousch(1988), Y. Himeki under supervision Y. Ishii(2018)). We prove that for each $n = 2, 3, 4, 5, 6$, $\mathcal{M}_n$ is connected and for all n $\mathcal{M}_n^0$ is connected.

1 Introduction

In this paper, we consider the following self-similar sets which is called "*fractal n-gons*" and introduced by Bandt and Hung in 2008([2]).

Definition 1.1 (fractal n-gons)**.** Let $n \in \mathbb{N}_{\geq 2}$. Let $\mathbb{D}^{\times} := \{\lambda \in \mathbb{C} | 0 < |\lambda| < 1\}$. Let $\lambda \in \mathbb{D}^{\times}$. We set $\xi_n = \exp(2\pi\sqrt{-1}/n)$. For each $i \in \{0, 1, ..., n-1\}$, we set $\phi_i^{n,\lambda} : \mathbb{C} \to \mathbb{C}$ by $\phi_i^{n,\lambda}(z) = \lambda z + {\xi_n}^i$. Then there uniquely exists a non-empty compact subset $A_n(\lambda)$ such that

$$\bigcup_{i=0}^{n-1} \phi_i^{n,\lambda}(A_n(\lambda)) = A_n(\lambda)$$

(See [8], [12]).

For examples of fractal n-gons, see Figures 1, 2, 3. As can be seen from those figures, fractal n-gons have the rotational symmetry of order n(See [2]). For each n, we define connectedness locus $\mathcal{M}_n$ of fractal n-gons and the remarkable subset $\mathcal{M}_n^0$ as the following.

$$\mathcal{M}_n = \{\lambda \in \mathbb{D}^{\times} | A_n(\lambda) \text{ is connected}\},$$

$$\mathcal{M}_n^0 = \{\lambda \in \mathbb{D}^{\times} | 0 \in A_n(\lambda)\}.$$

We give a short history of $\mathcal{M}_n$ and $\mathcal{M}_n^0$ as we can see in [7]. In 1985, Barnsley and Harrington defined $\mathcal{M}_2$ as the analogue of well-known *Mandelbrot set*([3]). They proved $\mathcal{M}_2$ has "whiskers" as in the following theorem.

Theorem 1.2 (Barnsley and Harrington, 1985 [3])**.** There is a neighborhood of the points $\{0.5, -0.5\}$ in which $\mathcal{M}_2$ is contained in $\mathbb{R}$.

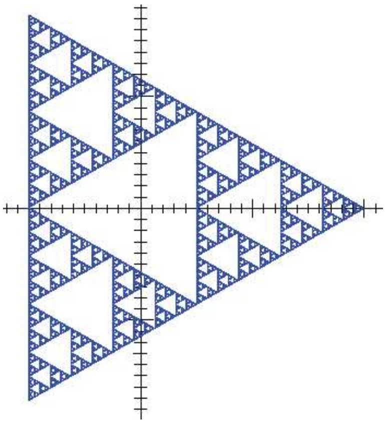

Figure 1. fractal 3-gon

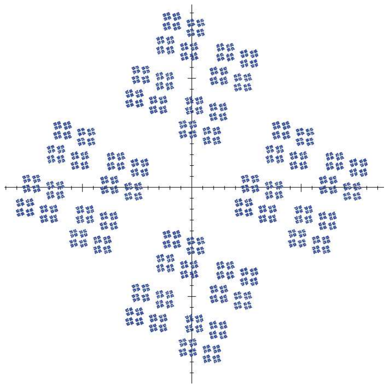

Figure 2. fractal 4-gon

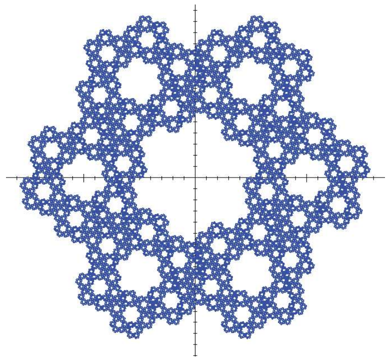

Figure 3. fractal 6-gon

They also conjectured the existence of a non-apparent hole of $\mathcal{M}_2$. In 1988, Bousch investigated the connectedness about $\mathcal{M}_2$ and $\mathcal{M}_2^0$ as in the following.

Theorem 1.3 (Bousch, 1988, 1992 [5], [6])**.** $\mathcal{M}_2$ and $\mathcal{M}_2^0$ are connected and locally connected.

In 2002, Bandt developed some fast algorithms to draw accurate pictures of $\mathcal{M}_2$, and managed to rigorously prove the existence of a hole in $\mathcal{M}_2$, thus positively answering the conjecture of Barnsley and Harrington. Bandt first realized the importance of understanding the set of interior points in $\mathcal{M}_2$, and made the following conjecture.

Conjecture 1.4 (Bandt, interior almost dense)**.** $\overline{\mathrm{int}(\mathcal{M}_2)} \cup (\mathcal{M}_2 \cap \mathbb{R}) = \mathcal{M}_2$.

In 2008, Bandt and Hung introduced $\mathcal{M}_n$([2]). They proved many remarkable theorems about $\mathcal{M}_n$, including the following result.

Theorem 1.5 (Bandt and Hung, 2008 [2])**.** If $n \neq 2, 4$, $\overline{\mathrm{int}(\mathcal{M}_n)} = \mathcal{M}_n$.

In 2016, Calegari, Koch and Walker positively answering the conjecture of Bandt as in the following theorem.

Theorem 1.6 (Calegari, Koch and Walker, 2016 [7])**.** $\overline{\mathrm{int}(\mathcal{M}_2)} \cup (\mathcal{M}_2 \cap \mathbb{R}) = \mathcal{M}_2$.

In 2018, Himeki and Ishii proved the following theorem.

Theorem 1.7 (Himeki and Ishii, 2018 [11])**.** If $n \geq 4$, $\overline{\mathrm{int}(\mathcal{M}_n)} = \mathcal{M}_n$.

In particular, in the case $n = 4$, they showed the new result and the regular closedness of $\mathcal{M}_n$ is completely understood. However, the study of the connectedness of $\mathcal{M}_n$ is insufficient. We can see the results about the connectedness of $\mathcal{M}_n$ as in the following theorem.

Theorem 1.8 (Himeki under supervision Ishii, 2018 [10])**.** $\mathcal{M}_3$ is connected.

In this paper, we consider the connectedness of $\mathcal{M}_n$ and $\mathcal{M}_n^0$ which is the natural analogue of $\mathcal{M}_2^0$. In order to investigate the connectedness, we define some random polynomials as the following.

Definition 1.9 (random polynomials and the set of zeros)**.** Let G be a subset of $\mathbb{C}$. Let $N \in \mathbb{N}_{\geq 2}$. Let $\mathbb{D}$ be the unit disk. We set

$$P_N^G = \{1 + \sum_{i=1}^{N-1} a_i z^i | a_i \in G\},$$
$$Z_N^G = \{z \in \mathbb{C} | \text{there exists } f \in P_N^G \text{ such that } f(z) = 0\},$$
$$Z^G = \bigcup_{N \geq 2} Z_N^G.$$

We give one of main results in this paper as in the following. In this paper, for a set $A \subset \mathbb{C}, \mathrm{cl}(A)$ denotes the closure of A with respect to Euclidean topology on $\mathbb{C}$.

Main result A. Suppose that G satisfies the following conditions.

1. $0 \in G$.
2. For all $a \in G, -a \in G$.
3. For all $a, b \in G$, there exists $c \in G$ such that $ab + c \in G$.
4. G is compact.

Suppose that there exists $0 < r < 1$ such that $\{z \in \mathbb{C} | r < |z| < 1\} \subset \mathrm{cl}(Z^G)$. Then we have $\mathrm{cl}(Z^G) \cap \mathbb{D}$ is connected.

We give a result which is similar to Main result A as in the following.

Main result B. Suppose that G satisfies the following conditions.

1. $1 \in G$
2. For all $a, b, c \in G$, there exists $d \in G$ such that $(a - b)c + d \in G$.
3. G is finite.

Suppose that there exists $0 < r < 1$ such that $\{z \in \mathbb{C} | r < |z| < 1\} \subset \mathrm{cl}(Z^G)$. Then we have $\mathrm{cl}(Z^G) \cap \mathbb{D}$ is connected.

We showed results above by using the methods of Bousch ([5]). We can get the following results as the corollaries of main results A and B.

Main result C. If $2 \leq n \leq 6$, $\mathcal{M}_n$ is connected.

Main result D. For each n, $\mathcal{M}_n^0$ is connected.

Remark 1.10. By main result C, we can give new results about the connectedness of $\mathcal{M}_n$ in the case $4 \leq n \leq 6$. On the other hand, in the case $\mathcal{M}_n^0$ for each $n \geq 3$, we can give new results about the connectedness of $\mathcal{M}_n^0$. We see the difference of results about $\mathcal{M}_n$ and $\mathcal{M}_n^0$ in the complexity of coefficients G which correspond to $\mathcal{M}_n$ or $\mathcal{M}_n^0$.

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2 Preliminaries

Let $\{\varphi_1, ..., \varphi_k\}$ be a general IFS on a complete metric space X. We define the address map as follows. Let $I = \{1, 2, ..., k\}$. For each $\omega = \omega_1\omega_2\omega_3 \cdots \in I^\infty$, we set $\omega|_n := \omega_1\omega_2 \cdots \omega_n \in I^n$ and $\varphi_{\omega|_n} := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n}$. Then it is well known that for each $\omega \in I^\infty$, $\lim_{n\to\infty} \varphi_{\omega|_n}(x) \in X$ exists, where $x \in X$. Note that this limit does not depend on the choice of x. It is denoted by v_ω. The address map $p\colon I^\infty \to X$ is defined by $\omega \mapsto v_\omega$. Note that $p(I^\infty) = A$, where A is the limit set generated by $\{\varphi_1, ..., \varphi_k\}$. If $p(\omega) = v$, then ω is called an address of v. In the following, for each finite word $\omega = \omega_1 \cdots \omega_n$, we set $\overline{\omega} = \omega_1 \cdots \omega_n\omega_1 \cdots \omega_n\omega_1 \cdots \omega_n \cdots \in I^\infty$. In our case(See Definition 1.1), the address map $\pi^{n,\lambda}$ has a particularly simple form as we see in the following.

Lemma 2.1 (Bandt and Hung, 2008 [2]). If $\omega = \omega_1\omega_2 \cdots \omega_i \cdots \in I^\infty$, $\pi^{n,\lambda}(\omega) = {\xi_n}^{\omega_1} + \sum_{i=1}^\infty {\xi_n}^{\omega_{i+1}}\lambda^i$.

We give some remarks about $\mathcal{M}_n$ and $\mathcal{M}_n^0$.

Remark 2.2. (1) $\mathcal{M}_n^0 \subset \mathcal{M}_n$.

(2) $\mathcal{M}_n$ and $\mathcal{M}_n^0$ are closed subset of $\mathbb{D}^\times$.

(3) $A_n(\lambda)$ is connected $\Leftrightarrow \phi_0^{n,\lambda}(A_n(\lambda)) \cap \phi_1^{n,\lambda}(A_n(\lambda)) \neq \emptyset$.

See [9], [13] for the general theory about connectedness of self-similar sets.

We set $I := \{0, 1, ..., n-1\}$, $V_n := \{{\xi_n}^0, {\xi_n}^1, ..., {\xi_n}^{n-1}\}$ and $\Delta_n := \{({\xi_n}^j - {\xi_n}^k)/(1 - \xi_n) | j, k \in I\}$.

Lemma 2.3. $\mathcal{M}_n = \{\lambda \in \mathbb{D} | \exists \{a_i\}_{i=1}^\infty\ (a_i \in \Delta_n) \text{ s.t. } 1 + \sum_{i=1}^\infty a_i\lambda^i = 0\}$ and $\mathcal{M}_n^0 = \{\lambda \in \mathbb{D} | \exists \omega = \omega_1\omega_2 \cdots \text{ s.t. } 1 + \sum_{i=1}^\infty {\xi_n}^{\omega_i}\lambda^i = 0\}$.

Proof. First, we prove the equation in the case $\mathcal{M}_n^0$.

$$\begin{aligned}
&0 \in A_n(\lambda)\\
&\Leftrightarrow \exists \omega = \omega_1\omega_2 \cdots \in I^\infty \text{s.t. } \pi^{n,\lambda}(\omega) = 0\\
&\Leftrightarrow \exists \omega \in I^\infty \text{ s.t. } {\xi_n}^{\omega_1} + \sum_{i=1}^\infty {\xi_n}^{\omega_{i+1}}\lambda^i = 0 \text{ (By Lemma 2.1)}\\
&\Leftrightarrow \exists \omega \in I^\infty \text{ s.t. } 1 + \sum_{i=1}^\infty {\xi_n}^{\omega_i}\lambda^i = 0
\end{aligned}$$

Hence we have that $\mathcal{M}_n^0 = \{\lambda \in \mathbb{D} | \exists \omega = \omega_1\omega_2 \cdots \text{ s.t. } 1 + \sum_{i=1}^\infty {\xi_n}^{\omega_i}\lambda^i = 0\}$. Next, we prove the equation in the case $\mathcal{M}_n$.

$$\begin{aligned}
&A_n(\lambda) \text{ is connected}\\
&\Leftrightarrow \phi_0^{n,\lambda}(A_n(\lambda)) \cap \phi_1^{n,\lambda}(A_n(\lambda)) \neq \emptyset \text{ (By Remark 2.2(3))}\\
&\Leftrightarrow \exists \omega = \omega_1\omega_2\cdots, \mu = \mu_1\mu_2\cdots \in I^\infty \text{s.t. } \phi_0^{n,\lambda}(\pi^{n,\lambda}(\omega)) = \phi_1^{n,\lambda}(\pi^{n,\lambda}(\mu))\\
&\Leftrightarrow \exists \omega, \mu \in I^\infty \text{ s.t. } {\xi_n}^0 + \sum_{i=1}^{\infty} {\xi_n}^{\omega_i}\lambda^i = {\xi_n}^1 + \sum_{i=1}^{\infty} {\xi_n}^{\mu_i}\lambda^i\\
&\Leftrightarrow \exists \omega, \mu \in I^\infty \text{ s.t. } 1 - \xi_n + \sum_{i=1}^{\infty} ({\xi_n}^{\omega_i} - {\xi_n}^{\mu_i})\lambda^i = 0\\
&\Leftrightarrow \exists \omega, \mu \in I^\infty \text{ s.t. } 1 + \sum_{i=1}^{\infty} ({\xi_n}^{\omega_i} - {\xi_n}^{\mu_i})/(1-\xi_n)\lambda^i = 0\\
&\Leftrightarrow \exists (a_i)_{i=1}^\infty \text{ with } a_i \in \Delta_n \text{ s.t. } 1 + \sum_{i=1}^{\infty} a_i\lambda^i = 0
\end{aligned}$$

Hence we have that $\mathcal{M}_n = \{\lambda \in \mathbb{D} | \exists \{a_i\}_{i=1}^\infty\ (a_i \in \Delta_n) \text{ s.t. } 1 + \sum_{i=1}^\infty a_i\lambda^i = 0\}$. □

By Lemma 2.3, $\mathcal{M}_n$ and $\mathcal{M}_n^0$ are the sets of all zeros of modulus < 1 of some power series. We describe $\mathcal{M}_n$ and $\mathcal{M}_n^0$ in terms of some random polynomials as in the following.

Lemma 2.4. $\mathcal{M}_n = \mathrm{cl}(Z^{\Delta_n}) \cap \mathbb{D}$ and $\mathcal{M}_n^0 = \mathrm{cl}(Z^{V_n}) \cap \mathbb{D}$.

Here, for a set $G \subset \mathbb{C}$, Z^G is defined as in Definition 1.9. The proof of this lemma is partially found in [1], [4]. The key tools in the proof is theorem of Rouché.

3 The proofs of main results

First, we prove the main results A and B. We fix G in the assumption of main results A or B. In order to prove those results, we define some terminologies as in the following.

Definition 3.1. We set $L := \sup\{|a|, |ab|, |(a-b)c| \big| a, b, c \in G\} < \infty$. and

$$W := \{1 + \sum_{i=1}^{\infty} a_i z^i | |a_i| \leq L\}.$$

We set for each natural number N,

$$W_N := \{1 + \sum_{i=1}^{N-1} a_i z^i | |a_i| \leq L\}.$$

Remark 3.2. W is a compact subset of the space of all holomorphic functions on $\mathbb{D}$ endowed with compact open topology and $P_N^G \subset W_N$.

Definition 3.3 (the value of power series)**.** Let $f, g \in W$. We often denote by $(a_0, a_1, a_2, ...)$ the power series with coefficients $a_0, a_1, a_2, ...$ If $f - g = (0, 0, ..., a_j)$, where $a_j \neq 0$, we define $\mathrm{Val}(f - g) := a_j$, that is $\mathrm{Val}(f-g) = \inf\{k \in \mathbb{N} | a_k \neq 0\}$, where $a_0, a_1, a_2, ..., a_k, ...$ are the coefficients of $f - g$. If $f = g$, we define $\mathrm{Val}(f - g) = \infty$.

Definition 3.4 (Cutting map)**.** Let $N \in \mathbb{N}_{\geq 2}$. We define the map $C_N : W \to W_N$ by $C_N(1 + \sum_{i=1}^\infty a_i z^i) = 1 + \sum_{i=1}^{N-1} a_i z^i$.

We use the following lemma which is a strong version of theorem of Rouché and found in [5].

Lemma 3.5. Let $R > 0$ and $\epsilon > 0$ with $R+\epsilon < 1$. Then there exists $N \in \mathbb{N}_{\geq 2}$ such that for all $(f, s) \in F := \{(f, s) \in W \times \overline{B(0, R)} | f(s) = 0\}$ and for all $g \in W$ with $\mathrm{Val(f - g)} \geq N$, there exists $s' \in B(s, \epsilon)$ such that $g(s') = 0$.

The following is the key lemma which is found in [5].

Lemma 3.6. Let $N \in \mathbb{N}_{\geq 2}$. Let $A, B \in P_N^G$ with $A \neq B$. Then there exists a sequence of functions on $\mathbb{D}$ $p_0, q_0, p_1, q_1, ..., p_{m-1}, q_{m-1}, p_m$ which satisfies the following.

(1) for each i, $p_i \in P_N^G$.

(2) for each i, $q_i \in W$.

(3) for each i, there exists f: holomorphic on $\mathbb{D}$ such that $q_i(z) = f(z) \cdot p_i(z)$ for all $z \in \mathbb{D}$.

(4) for each i, $C_N(q_i) = p_{i+1}$.

(5) $p_0 = A, p_m = B$.

We give the proofs of main results A and B.

(Proofs of main results A and B)**.** Since there exists $0 < r < 1$ such that $\{z \in \mathbb{C} | r < |z| < 1\} \subset \mathrm{cl}(Z^G)$, it suffices to show that $(\mathrm{cl}(Z^G) \cap \mathbb{D}) \cup \partial\mathbb{D}$ is connected. We fix $\epsilon > 0$ with $\epsilon + r < 1$. Since $(\mathrm{cl}(Z^G) \cap \mathbb{D}) \cup \partial\mathbb{D}$ is compact, we prove that $(\mathrm{cl}(Z^G) \cap \mathbb{D}) \cup \partial\mathbb{D}$ is ϵ-connected. We take $s \in \mathrm{cl}(Z^G) \cap \mathbb{D}$. Let N be a natural number determined by r and ϵ in lemma 3.5. By Lemma 3.6, there exist $p \in P_N^G$ and s_0 with $p(s_0) = 0$ such that $|s_0| \geq r$ and s and s_0 are ϵ-connected. Hence we have proved $\mathrm{cl}(Z^G) \cap \mathbb{D}$ is connected.

Next, we give the proof of main result C.

(Proof of main result C)**.** By lemma 2.4 and Main result A, it suffices to show that the set Δ_n satisfies the assumption of Main result A. It is easy to show that Δ_n contains 0, for all a, $-a \in \Delta_n$ and Δ_n is compact. We take $a, b \in \Delta_n$. In the case $n = 4$,

$$\Delta_4 = \{0, 1, \sqrt{-1}, -1, -\sqrt{-1}, \sqrt{2}\exp(\sqrt{-1}\pi/4), \sqrt{2}\exp(\sqrt{-1}3\pi/4),\\ \sqrt{2}\exp(\sqrt{-1}5\pi/4), \sqrt{2}\exp(\sqrt{-1}7\pi/4)\}.$$

Hence we have that

$$\Delta_4 \times \Delta_4 = \{0, 1, \sqrt{-1}, -1, -\sqrt{-1}, \sqrt{2}\exp(\sqrt{-1}\pi/4), \sqrt{2}\exp(\sqrt{-1}3\pi/4),\\ \sqrt{2}\exp(\sqrt{-1}5\pi/4), \sqrt{2}\exp(\sqrt{-1}7\pi/4), 2, 2\sqrt{-1}, -2, -2\sqrt{-1}\}.$$

If $ab \in \Delta_4$, we take 0 so that $ab + c \in \Delta_4$. If $ab \in \Delta_4 \times \Delta_4 \backslash \Delta_4$, we take $c = -ab/2 \in \{1, \sqrt{-1}, -1, -\sqrt{-1}\}$ so that $ab+c \in \Delta_4$. Hence we show that for all $a, b \in \Delta_4$, there exists $c \in \Delta_4$ such that $ab + c \in \Delta_4$. In other cases, we prove that Δ_n satisfies the condition similarly.

By [2] Proposition 3,

$$\{\lambda \in \mathbb{C} | \frac{1}{\sqrt{n}} < |\lambda| < 1\} \subset \mathrm{cl}(Z^{\Delta_n}).$$

Hence we have proved Main result C.

Finally, we give the proof of main result D.

(Proof of main result D). By lemma 2.4 and Main result B, it suffices to show that the set V_n satisfies the assumption of Main result B. It is easy to show that V_n contains 1 and V_n is finite. We take $a, b, c \in V_n$. Since V_n is a group, we take $d := bc \in V_n$ so that $(a-b)c + d \in V_n$. Hence we have proved that for all $a, b, c \in V_n$, there exists $d \in V_n$ such that $(a-b)c + d \in V_n$.

By [5],

$$\{\lambda \in \mathbb{C} | \frac{1}{\sqrt[4]{2}} < |\lambda| < 1\} \subset \mathrm{cl}(Z^{V_2}).$$

By [4], if $n \geq 3$, it follows that

$$\{\lambda \in \mathbb{C} | \tilde{r}_n < |\lambda| < 1\} \subset \mathrm{cl}(Z^{V_n}),$$

where $\tilde{r}_n := \frac{\sqrt{5-4\cos\frac{\pi}{n}}}{2}$.

Hence we have proved Main result D.

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