On an overdetermined problem for composite materials

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain of class $\mathcal{C}^2$ and let $D \subset \overline{D} \subset \Omega$ be an open set. Let $\sigma_c \neq 1$ be a positive constant and let σ denote the following piece-wise constant function:

$$\sigma := \sigma_c \, \mathcal{X}_D + \mathcal{X}_{\Omega \setminus D}, \tag{1.1}$$

where $\mathcal{X}_A$ is the characteristic function of the set A (i.e., $\mathcal{X}_A(x)$ is 1 if $x \in A$ and 0 otherwise). We consider the following overdetermined problem:

Problem 1. *Find the pairs (D, Ω) for which the solution of*

$$-\operatorname{div}(\sigma \nabla u) = 1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

also satisfies the overdetermined condition

$$\partial_n u \equiv \text{const.} \quad \text{on } \partial\Omega, \tag{1.3}$$

where ∂_n denotes the (outward) normal derivative on $\partial\Omega$.

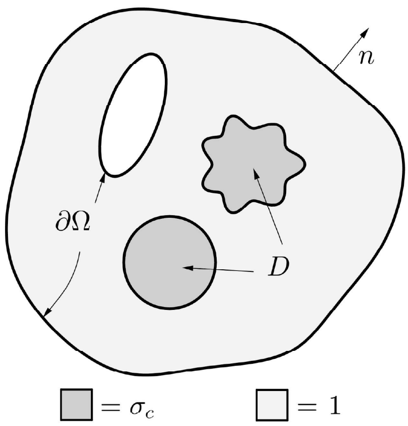

Figure 1: Problem setting.

Remark 1.1. *Any pair of concentric balls (D_0, Ω_0) is a solution of Problem 1 (trivial solution).*

Remark 1.2. *Serrin [Se] showed that, when $D = \emptyset$, the only solution of class $\mathcal{C}^2$ of Problem 1 is given by $\Omega =$ball.*

Remark 1.3. *Sakaguchi [Sa] showed that, when Ω is a ball and D is an open set of class $\mathcal{C}^2$ with finitely many connected components and such that $\Omega \setminus \overline{D}$ is connected, then Problem 1 is solvable if and only if D and Ω are concentric balls.*

Remark 1.4. *By the (local) result of [KN], if (D, Ω) is a classical solution of Problem 1, then $\partial\Omega$ is an analytic surface.*

2 Variational interpretation of Problem 1

For a fixed bounded open set D, let

$$E_D(\Omega) = \int_\Omega \sigma |\nabla u|^2, \tag{2.4}$$

where σ is the piece-wise constant function (1.1) and u denotes the solution to (1.2). Now, for some constant $V_0 > |D|$ consider the following constrained maximization problem:

Problem 2.

$$\max \left\{ E_D(\Omega) \ : \ \Omega \supset \overline{D}, \quad |\Omega| = V_0 \right\}. \tag{2.5}$$

Proposition 2.1. *Let Ω be a bounded domain of class $\mathcal{C}^2$. If Ω is a critical shape for Problem 2, then u satisfies the overdetermined condition* (1.3)*.*

Proof. By hypothesis Ω is a critical shape for the following Lagrangian

$$\mathcal{L}(\Omega) := E_D(\Omega) - \mu|\Omega|$$

for some suitable Lagrange multiplier μ. Computing the shape derivative of $\mathcal{L}$ with respect to some perturbation field $h : \mathbb{R}^N \to \mathbb{R}^N$ yields (see [Ca2, Theorem 4.2]):

$$\mathcal{L}'(\Omega)[h] = \int_{\partial\Omega} |\partial_n u|^2 \ h \cdot n - \mu \int_{\partial\Omega} h \cdot n.$$

Now, since by hypothesis $\mathcal{L}'(\Omega)[h] = 0$ for all perturbation fields h, we must have $|\partial_n u|^2 \equiv \mu$ on $\partial\Omega$. In other words, u satisfies (1.3) as claimed. □

Definition 2.2. *We say that a solution (D, Ω) of Problem 1 is a* variational solution *if it is a local extremizer of Problem 2. Otherwise, we say that (D, Ω) is a* saddle-type solution.

Remark 2.3. *Critical shapes for Problem 2 (that is solutions to Problem 1) are not necessarily variational solutions. Indeed, as shown in [Ca1], the trivial solution (D_0, Ω_0) is of* saddle-type *for $\sigma_c \in (0, 1)$ and a* variational solution *(local maximizer) for $\sigma_c \in (1, \infty)$.*

3 Known results (local behavior near trivial solutions)

Let (D_0, Ω_0) denote the trivial solution given by the concentric balls centered at the origin with radii R and 1 respectively ($0 < R < 1$). Moreover, for $k \in \mathbb{N}$, let

$$s(k) := \frac{k(N+k-1) - (N+k-2)(k-1)R^{2-N-2k}}{k(N+k-1) + k(k-1)R^{2-N-2k}},$$
$$\Sigma := \{ s \in (0, \infty) \ : \ s = s(k) \ \text{ for some } k \in \mathbb{N} \}.$$

Depending on whether σ_c belongs to Σ or not, the local behavior of solutions near (D_0, Ω_0) changes drastically.

Theorem 3.1 (Local existence for $\sigma_c \notin \Sigma$, [CY1])**.** *If $\sigma_c \notin \Sigma$, then for every domain D of class $\mathcal{C}^{2,\alpha}$ sufficiently close to D_0, there exists a domain Ω of class $\mathcal{C}^{2,\alpha}$ sufficiently close to Ω_0 (and with the same volume of Ω_0) such that the pair (D, Ω) solves Problem 1.*

Theorem 3.2 (Bifurcation phenomenon around $\sigma_c = s(k)$, [CY2])**.** *The values $\sigma_c = s(k)$ are bifurcation points for Problem 1 in the following sense. There exists a function $t \mapsto \lambda(t) \in \mathbb{R}$ and a continuous branch of the form (D_0, Ω_t) that solves Problem 1 for $\sigma_c = s(k) + \lambda(t)$ for small $|t|$. Moreover, Ω_t is a ball only for $t = 0$.*

Remark 3.3. *A simple calculation yields that $s(k) < 1$. As a result, for $\sigma_c > 1$ we always have local existence for Problem 1 near trivial solutions. Moreover, by Remark 2.3 we know that such solutions are of variational type in a small enough neighborhood. Similarly, we know that the symmetry-breaking solutions given by Theorem 3.2 are of saddle type in a neighborhood of $\sigma_c = s(k)$.*

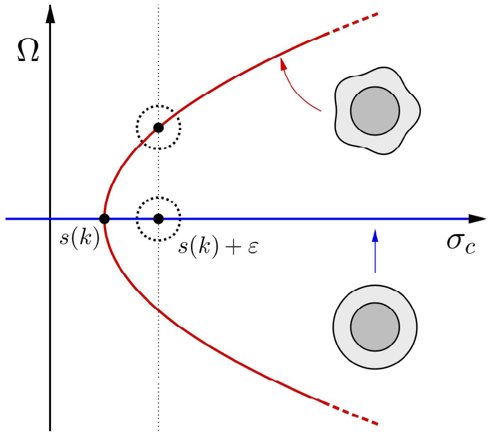

Figure 2: Bifurcation diagram for Problem 1 (Theorem 3.2).

Remark 3.4. *The result of Theorem 3.1 can be extended to Lipschitz continuous perturbations of D_0 in a similar way (see [Ca3]). This yields the existence of nontrivial solutions of the form (D,Ω), where ∂D is Lipschitz continuous and $\partial\Omega$ is an analytic surface.*

Remark 3.5. *There are only a finite number of $k\in\mathbb{N}$ such that $s(k)>0$. In other words, for any given radius $R\in(0,1)$ there is only a finite number of bifurcation points in the sense of Theorem 3.2.*

4 Numerical computation of the solutions

The study of solutions of Problem 1 has also been treated numerically ([CY1]), employing a steepest-descent algorithm based on the following Kohn–Vogelius functional. For given D, let

$$\mathcal{F}(\Omega) := \int_\Omega \sigma|\nabla v - \nabla w|^2,$$

where v is the unique solution of (1.2) and w is the unique solution of the following Neumann boundary value problem:

$$-\operatorname{div}(\sigma\nabla w) = 1 \quad \text{in } \Omega, \qquad \partial_n w = -|\Omega|/|\partial\Omega| \quad \text{on } \partial\Omega, \qquad \int_{\partial\Omega} w = 0.$$

Remark 4.1. *By construction, $\mathcal{F}(\Omega)\geq 0$ for all domains $\Omega\supset\overline{D}$ and $\mathcal{F}(\Omega)=0$ if and only if (D,Ω) solves Problem 1.*

In what follows, let D be fixed. By Remark 4.1, it is clear (D,Ω) is a solution of Problem 1 with $|\Omega|=V_0$ if and only if Ω is a solution of the following minimization problem.

Problem 3. *Minimize the following augmented Lagrangian:*

$$\mathcal{L}(\Omega) := \mathcal{F}(\Omega) - \mu G(\Omega) + \frac{b}{2}G(\Omega)^2, \qquad G(\Omega) := \frac{|\Omega|-V_0}{V_0},$$

where μ is a Lagrange multiplier and $b>0$ is a large parameter.

In order to solve Problem 3 (and hence Problem 1) numerically, we first need to find the steepest descent direction of $\mathcal{L}$, which we obtain by computing the shape derivative of $\mathcal{L}$ with respect to a smooth perturbation field $h:\mathbb{R}^N\to\mathbb{R}^N$. We get:

$$\mathcal{L}'(\Omega)(h) = \int_{\partial\Omega} \phi\, h\cdot n,$$

where $\phi := \left(-|\nabla w|^2 + 2w + 2cHw - |\nabla v|^2 + 2c^2 - \mu + b\,\frac{|\Omega|-V_0}{V_0^2}\right)$. In particular, notice that $h^\star = -\phi n$ is a descent direction, because $\mathcal{L}'(\Omega)(h^\star) = -\int_{\partial\Omega}\phi^2 < 0$. By the above, we obtain the following steepest descent algorithm:

Fix an initial shape Ω_0. For $k = 0, 1, \ldots,$ until convergence:

1. Compute the descent direction $h^\star := -\phi n$ corresponding to the domain Ω_k.
2. Update the shape according to $\Omega_{k+1} := (\mathrm{Id} + \varepsilon \mathrm{h}^\star)(\Omega_k)$ for some small parameter $\varepsilon > 0$.
3. Repeat

In what follows we can see that the numerical results are in line with the expected results (Figure 4 shows the numerical approximation computed by the algorithm above, while Figure 5 shows the first-order approximation of the solution as given by the corollary of Theorem 3.1.)

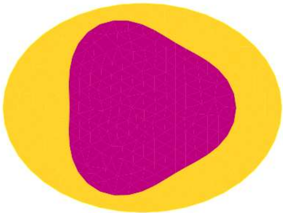

Figure 3: Initial shape

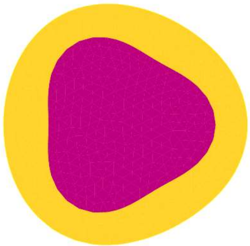

Figure 4: Final shape

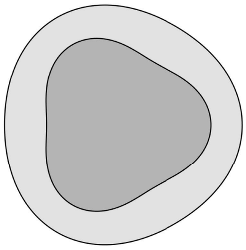

Figure 5: Analytical result

Figure 4, in particular, suggests that the solution Ω "inherits the geometry" of D. This is indeed the case. Nevertheless, it is worth mentioning that the way the geometry of D is inherited also depends on the coefficient σ_c, as shown in the following figures.

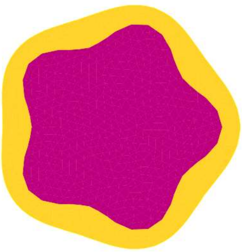

Figure 6: Final shape for $\sigma_c = 10$

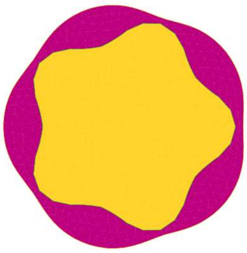

Figure 7: Final shape for $\sigma_c = 0.1$

Finally, we will consider the cases when the effect of D is negligible, that is when D is either small or σ_c is close to 1. The numerical results below suggest that, in both cases, the solution Ω is close to being a ball. This result has been made precise in a quantitative sense and proven rigorously in [CPY].

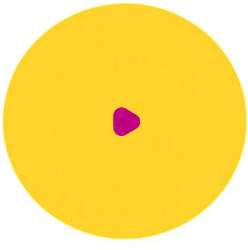

Figure 8: When D is small

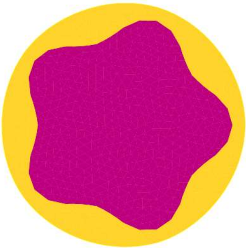

Figure 9: When σ_c is close to 1

5 What is left to do: a peek into global existence

We are left with one big open problem, that is, the global existence of solutions for Problem 1.

Conjecture 5.1. *Let $D \subset \mathbb{R}^N$ be a bounded open set and let $\sigma_c > 0$. Then there exists some bounded domain $\Omega \supset \overline{D}$ such that the pair (D, Ω) is a solution to Problem 1.*

We can think of two possible approaches:

- **Variational approach.** Find a solution of Problem 2 in the class of quasi-open sets by the variational method of Buttazzo–Dal Maso ([BD]) and then bootstrap the regularity of the solution obtained. **Downside:** by this method, we cannot find saddle-type solutions.
- **Perturbation approach.** Take a very large ball $\Omega_0 \supset \overline{D}$. Since D is very small in comparison, notice that the pair (D, Ω_0) is close to being a solution to Problem 1 (see [CPY] for the precise result). Then, construct the solution Ω as a suitable perturbation of Ω_0 by the implicit function theorem. **Downside:** by this method, we can only find solutions with $|\Omega| \gg |D|$.

Acknowledgment

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