The scaling limit of eigenfunctions for 1d random Schrödinger operator

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Abstract

We report our results on the scaling limit of the eigenvalues and the corresponding eigenfunctions for the 1-d random Schrödinger operator with random decaying potential. The formulation of the problem is based on the paper by Rifkind-Virag [9].

1 Introduction

In this note we consider the following one-dimensional Schrödinger operator with random decaying potential :

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t)$$

where $a \in C^\infty(\mathbf{R})$, $a(-t) = a(t)$, $a(t)$ is monotone decreasing for $t > 0$ and

$$a(t) = t^{-\alpha}(1 + o(1)), \quad t \to \infty$$

for some $\alpha > 0$. $F \in C^\infty(M)$ is a smooth function on a torus M such that

$$\langle F \rangle := \int_M F(x)dx = 0$$

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and $\{X_t\}_{t\in\mathbf{R}}$ is the Brownian motion on M. Since $a(t)F(X_t)$ is a compact perturbation with respect to $(-\triangle)$, the spectrum $\sigma(H)\cap(-\infty,0)$ on the negative real axis is discrete. The spectrum $\sigma(H)\cap[0,\infty)$ on the positive real axis is [4] :

$$\sigma(H)\cap[0,\infty) \text{ is } \begin{cases} \text{a.c.} & (\alpha>1/2) \\ \text{p.p. on } [0,E_c] \text{ and s.c. on } [E_c,\infty) & (\alpha=1/2) \\ \text{p.p.} & (\alpha<1/2) \end{cases}$$

where E_c is a deterministic constant. For the level statistics problem, we consider the point process ξ_{L,E_0} composed of the rescaling eigenvalues $\{L(\sqrt{E_j(L)}-\sqrt{E_0})\}_j$ of the finite box Dirichlet Hamiltonian $H_L := H|_{[0,L]}$ whose behavior as $L\to\infty$ is given by [3, 6, 8]

$$\xi_{L,E_0} \stackrel{d}{\to} \begin{cases} Clock(\theta(E_0)) & (\alpha>1/2) \\ Sine(\beta(E_0)) & (\alpha=1/2) \\ Poisson(d\lambda/\pi) & (\alpha<1/2) \end{cases}$$

where $Clock(\theta) := \sum_{n\in\mathbf{Z}}\delta_{n\pi+\theta}$, is the clock process for some random variable θ on $[0,\pi)$, and $Sine(\beta)$ is the Sine$_\beta$-process which is the bulk scaling limit of the Gaussian beta emsemble [10]. For $\alpha=1/2$, $\beta(E_0)=\tau(E_0)^{-1}$ is equal to the reciprocal of the Lyapunov exponent $\tau(E_0)$ such that the solution to the Schrödinger equation $H\varphi=E\varphi$ has the power-law decay : $\varphi(x)\simeq|x|^{-\tau(E)}$, $|x|\to\infty$. Since $\lim_{E_0\downarrow 0}\beta(E_0)=0$ and $\lim_{E_0\uparrow\infty}\beta(E_0)=\infty$, small (resp. large) E_0 corresponds to small (resp. large) repulsion of eigenvalues, which is consistent to the following fact [1, 7] :

$$Sine(\beta) \stackrel{d}{\to} \begin{cases} Poisson(d\lambda/\pi) & (\beta\downarrow 0) \\ Clock(unif[0,\pi)) & (\beta\uparrow\infty) \end{cases}$$

In this note, we consider the scaling limit of the measure corresponding to the eigenfunction of H_L under the formulation studied by Rifkind-Virag [9]. To formulate the problem, we need some notations. Let $\{E_j(L)\}_j$ be the positive eigenvalues of H_L, and $\{\psi^{(L)}_{E_j(L)}\}$ be the corresponding eigenfunctions. We consider the associated random probability measure $\mu^{(L)}_{E_j(L)}$ on $[0,1]$.

$$\mu^{(L)}_{E_j(L)}(dt) := C\left(|\psi^{(L)}_{E_j(L)}(Lt)|^2 + \frac{1}{E_j(L)}\left|\frac{d}{dt}\psi^{(L)}_{E_j(L)}(Lt)\right|^2\right)dt.$$

Let $J := [a, b](\subset (0, \infty))$ be an interval, $\mathcal{E}_J^{(L)} := \{E_j(L)\}_j \cap J$ be the eigenvalues of H_L on J, and $E_J^{(L)}$ be the uniform distribution on $\mathcal{E}_J^{(L)}$. Our aim is to consider the large L limit of the eigenvalue-eigenvector pairs :

$$\mathbf{Q} : \left(E_J^{(L)}, \mu^{(L)}_{E_J^{(L)}} \right) \stackrel{d}{\to} ?$$

For d-dimensional discrete random Schrödinger operator, if J is in the localized region, we have [2, 5]

$$\left(E_J^{(L)}, \mu^{(L)}_{E_J^{(L)}} \right) \stackrel{d}{\to} \left(E_J, \delta_{unif[0,1]^d} \right)$$

where E_J is the random variable obeying $\frac{1_J(E)}{N(J)} dN(E)$, where dN is the density of states measure. Rifkind-Virag studied the 1-d discrete Schrödinger operator with critical decaying coupling constant, and obtained that the limit of $\mu^{(L)}_{E_J^{(L)}}$ is given by an exponential Brownian motion [9] :

$$\left(E_J^{(L)}, \mu^{(L)}_{E_j(L)} \right) \stackrel{d}{\to} \left(E_J, \frac{\exp\Big(2\mathcal{Z}_{\tau(E_J)(t-U)} - 2\tau(E_J)\,|t-U| \Big) dt}{\int_0^1 \exp\Big(2\mathcal{Z}_{\tau(E_J)(s-U)} - 2\tau(E_J)\,|s-U| \Big) ds} \right).$$

To state our result, we need notations further. Let $N(E) := \pi^{-1}\sqrt{E}$ be the integrated density of states, $N(J) := N(b) - N(a)$, and

$$\tau(E) := \frac{1}{8E} \int_M |\nabla (L + 2i\sqrt{E})^{-1} F|^2 dx.$$

where L is the generator of (X_t). Moreover, let E_J be the random variable whose distribution is equal to $N(J)^{-1} 1_J(E) dN(E)$, let U be the uniform distribution on $[0, 1]$, and let $\mathcal{Z}$ be the 2-sided Brownian motion, where E_J, U, and $\mathcal{Z}$ are independent.

Theorem 1.1

$$\left(E_J^{(L)}, \mu^{(L)}_{E_j(L)} \right) \stackrel{d}{\to} \begin{cases} \left(E_J, 1_{[0,1]}(t) dt \right) & (\alpha > 1/2) \\ \left(E_J, \dfrac{\exp\left(2\mathcal{Z}_{\tau(E_J)\log\frac{t}{U}} - 2\tau(E_J)\left|\log\frac{t}{U}\right| \right) dt}{\int_0^1 \exp\left(2\mathcal{Z}_{\tau(E_J)\log\frac{s}{U}} - 2\tau(E_J)\left|\log\frac{s}{U}\right| \right) ds} \right) & (\alpha = 1/2) \\ \left(E_J, \delta_{unif[0,1]}(dt) \right) & (\alpha < 1/2) \end{cases}$$

When $\alpha < 1/2$, this result is the same as that in [2, 5], while for $\alpha > 1/2$ ths result is natural. For $\alpha = 1/2$, this result implies that, the localization center U of the eigenfunction ψ is uniformly distributed and ψ has the power law decay around U with Brownian fluctuation. Since $\lim_{E\downarrow 0} \tau(E) = \infty$ and $\lim_{E\uparrow\infty} \tau(E) = 0$, ψ is localized (resp. delocalized) for $E \downarrow 0$ (resp. $E \uparrow \infty$) which is consistent with the previous picture.

2 Sketch of Proof

For the proof, we mostly follow the strategy in [2, 5, 9], except for some technical points.

2.1 Step 1 : renormalize the radial coordinate

In what follows, we describe the solution x_t to the equation $Hx_t = \kappa^2 x_t$ in terms of the Prüfer variales :

$$\begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = r_t(\kappa) \begin{pmatrix} \sin\theta_t(\kappa) \\ \cos\theta_t(\kappa) \end{pmatrix}$$

Introducing $\rho_t(\kappa)$ defined by $r_t(\kappa) := \exp(\rho_t(\kappa))$, we have

$$\rho_t(\kappa) = \frac{1}{2\kappa} Im \int_0^t e^{2i\theta_s(\kappa)} a(s) F(X_s) ds$$

Let $\kappa_\lambda := \kappa_0 + \frac{\lambda}{n}$, $\kappa_0 := \sqrt{E_0}$, $\widetilde{\rho}_t^{(n)}(\kappa) := \rho_{nt}(\kappa) - \langle F g_\kappa \rangle \int_0^n a(s)^2 ds$, $g_\kappa := (L + 2i\kappa)^{-1} F$, $t \in [0, 1]$. We then have

Lemma 2.1 *If $\alpha = 1/2$, then*

$$\widetilde{\rho}_t^{(n)}(\kappa_\lambda) \stackrel{d}{\to} \widetilde{\rho}_t(\lambda), \quad t \in [0,1], \ \textit{locally uniformly}$$
$$d\widetilde{\rho}_t^{(n)}(\kappa_\lambda) = \frac{\tau(\kappa_0^2)}{t} dt + \sqrt{\frac{\tau(\kappa_0^2)}{t}} dB_t^\lambda, \quad t > 0$$

where $\{B_t^\lambda\}$ is a family of Brownian motion.

2.2 Step 2 : limit of the local version

Let $\Xi^{(n)}$ be the local version of our problem :

$$\Xi^{(n)} := \sum_j \delta_{\left(n\left(\sqrt{E_j(n)}-\sqrt{E_0}\right), \mu^{(n)}_{E_j(n)}\right)}$$

It then follows that

Lemma 2.2 $\Xi^{(n)} \stackrel{d}{\to} \Xi$, *where*

$$\Xi = \begin{cases} \sum_{j\in\mathbf{Z}} \delta_{j\pi+\theta} \otimes \delta_{1_{[0,1]}(t)dt} & (\alpha > 1/2) \\ \sum_{\lambda:Sine_\beta} \delta_\lambda \otimes \delta\left(\frac{\exp(2\widetilde{\rho}_t(\lambda))dt}{\int_0^1 \exp(2\widetilde{\rho}_s(\lambda))ds}\right) & (\alpha = 1/2) \\ \sum_{j\in\mathbf{Z}} \delta_{P_j} \otimes \delta_{\widetilde{P}_j}, & (\alpha < 1/2) \end{cases}$$

where $\{P_j\}$: $Poisson(d\lambda/\pi)$, $\{\widetilde{P}_j\}$: $Poisson(1_{[0,1]}(t)dt)$. *The intensity measure of* Ξ *is given by*

$$\begin{aligned} &\mathbf{E}\left[G(\lambda,\nu)d\Xi(\lambda,\nu)\right] \\ &= \frac{1}{\pi}\begin{cases} \int d\lambda \mathbf{E}\left[G\left(\lambda, 1_{[0,1]}(t)dt\right)\right] & (\alpha > 1/2) \\ \int d\lambda \mathbf{E}\left[G\left(\lambda, \frac{\exp\left(2\mathcal{Z}_{\tau(E_0)\log\frac{t}{U}} - 2\tau(E_0)\log\left|\frac{t}{U}\right|\right)dt}{\int_0^1 \exp\left(2\mathcal{Z}_{\tau(E_0)\log\frac{s}{U}} - 2\tau(E_0)\log\left|\frac{s}{U}\right|\right)ds}\right)\right] & (\alpha = 1/2) \\ \int d\lambda \mathbf{E}\left[G\left(\lambda, \delta_U\right)\right] & (\alpha > 1/2) \end{cases} \end{aligned}$$

where $U := unif[0,1]$.

2.3 Step 3 : averaging over the reference energy

Following [9], we introduce

$$\begin{aligned} g_1(x) &:= (1-|x|)1(|x|\le 1) \\ G_L(E) &:= \sum_{E_j(L)\in J} g_1\left(L\left(\sqrt{E_j(L)}-\sqrt{E_0}\right)\right)\cdot g_2\left(E_j(L), \mu^{(L)}_{E_j(L)}\right) \end{aligned}$$

where $g_2 \in C_b(\mathbf{R}\times\mathcal{P}(0,1))$. We compute $\int \frac{dN(E)}{N(J)}\mathbf{E}[G_L(E)]$ by the following two ways, and then equate them by the Fubini theorem, which leads to the conclusion.

(1) Since $\int \frac{dN(E)}{N(J)} g_1 = 1/(L\pi)$, we have

$$
\begin{aligned}
&\mathbf{E}\left[\int \frac{dN(E)}{N(J)} G_L(E)\right] \\
&= \frac{1}{N(J)} \frac{1}{\pi L} \mathbf{E}\left[\sum_{E_j(L)\in J} g_2\left(E_j(L), \mu^{(L)}_{E_j(L)}\right)\right] \\
&= \mathbf{E}\left[\frac{1}{\sharp\{\text{ eigenvalues of } H_L \text{ on } J\ \}(1+o(1))} \cdot \frac{1}{\pi} \cdot \sum_{E_j(L)\in J} g_2\left(E_j(L), \mu^{(L)}_{E_j(L)}\right)\right]
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\int \frac{dN(E)}{N(J)} \mathbf{E}[G_L(E)] \\
&= \int \frac{dN(E)}{N(J)} \mathbf{E}\left[\sum_{E_j(L)\in J} g_1\left(L\left(\sqrt{E_j(L)} - \sqrt{E_0}\right)\right) \cdot g_2\left(E_j(L), \mu^{(L)}_{E_j(L)}\right)\right] \\
&\sim \int \frac{dN(E)}{N(J)} \mathbf{E}\left[\int g_1(\lambda) g_2(E,\mu) d\Xi^{(L)}(\lambda,\mu)\right] \\
&\to \int \frac{dN(E)}{N(J)} \mathbf{E}\left[\int g_1(\lambda) g_2(E,\mu) d\Xi(\lambda,\mu)\right] \\
&= \int \frac{dN(E)}{N(J)} \frac{1}{\pi} \begin{cases} \int d\lambda \mathbf{E}\left[g_2\left(E, 1_{[0,1]}(t)dt\right)\right] & (\alpha > 1/2) \\ \int d\lambda \mathbf{E}\left[g_2\left(E, \frac{\exp\left(2\mathcal{Z}_{\tau(E_0)\log\frac{t}{U}} - 2\tau(E_0)\log\left|\frac{t}{U}\right|\right)dt}{\int_0^1 \exp\left(2\mathcal{Z}_{\tau(E_0)\log\frac{s}{U}} - 2\tau(E_0)\log\left|\frac{s}{U}\right|\right)ds}\right)\right] & (\alpha = 1/2) \\ \int d\lambda \mathbf{E}\left[g_2\left(E, \delta_U\right)\right] & (\alpha > 1/2) \end{cases}
\end{aligned}
$$

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